Maximizing the Ratio of Two Linear Functions Subject to A Set of Linear Inequalities

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ABSTRACT

This paper explains a step-by-step method algorithm to solve a special type of mathematical problem called a linear fractional functional programming problem. This problem helps solve economic situations where different activities use fixed resources based on their value or importance. Charnes and Cooper found a way to simplify these fractional problems which involve ratios by turning them into two regular linear programming problems, which are easier to solve. The method repeats a limited number of steps to find the solution. By solving a group of inequality equations, we can get the final answer to the problem.

Keywords: fractional, optimization, allocations, feasible solution, optimality, optimum.

INTRODUCTION

This type of problem is useful for solving economic situations where different activities use fixed amounts of resources according to their value. The main goal of optimization here is not just to find how to get the highest profit or lowest cost like in normal linear programming. Instead, the aim is to find the best possible ratio, for example, the most favourable balance between revenue and resources used. Such problems are very important in the financial analysis of a company because they help measure how efficiently the company uses its resources to earn moneychanger and Cooper developed a method to solve these fractional (ratio-based) problems by changing them into two regular linear programming problems, which are easier to handle. This chapter also introduces five modern techniques of mathematical programming: Linear Fractional Programming, Goal Programming, Geometric Programming, Stochastic Programming, Extreme Point Mathematical Programming. Most of the recent research in mathematical programming focuses on these advanced methods.

This problem involves maximizing the ratio of two linear functions while following a set of linear inequalities and keeping all variables non-negative. The problem is solved directly, starting from an initial feasible solution and then finding the conditions under which this solution can be improved. Conditions for deciding when the solution is optimal are also explained. The method used is similar to the simplex technique that is applied in linear programming. Such types of programming problems have recently gained a lot of attention in the field of non-linear programming. An example of fractional programming was studied by J. R. Isbell and W. H. Marlow.

Mathematical formulation, assumptions and algorithm

Maximize $R(\mathbf{x}) = (c'\mathbf{x} + \alpha)/(d'\mathbf{x} + \beta)$

subject to the constrains:

$$\mathrm{Ax} = \mathrm{b}$$
$$\mathrm{x} \geqslant 0$$

1. $\mathbf{x}$, c, and $\mathbf{d}$ are $n \times 1$ vectorts,
2. $\mathbf{A}$ is an $m \times n$ matrix, $\quad \mathbf{A} = \|a_{ij}\| (i = 1, \ldots, m; j = 1, \ldots, n)$
3. b is an $m \times 1$ vector,
4. $c', \mathbf{d}'$ denote transpose of vectors $\mathbf{c}$ and $\mathbf{d}$, and (v) α, β are scalar constants It is assumed that the constraints set

$$S = \{x \mid Ax = b, x > 0\}$$

Is non empty and bounded.

Consider the linear fractional programming problem Maximize

$$(\mathbf{c}'\mathbf{x} + \alpha)/(\mathbf{d}'\mathbf{x} + \boldsymbol{\beta}) = \mathrm{R}(\mathbf{x})$$

subject to the constraints:

$$\mathrm{Ax} = \mathrm{b}$$
$$\mathrm{x} \geqslant 0$$

with the additional assumption that the denominator is positive for all feasible solutions. a new nonsingular matrix obtained from B by removing b_r and replacing it by a. The columns of the new matrix $\hat{\mathbf{B}}$ are given by

$$\hat{b}_i = b_i$$
$$\hat{b}_q = a_j$$

We obtain the value of the new basic variables in terms of the original ones and the u_{ij}, i.e., where

$$\hat{x}_{B_i} = x_{Bi} - x_{Br}(u_{iJ}/u_{rj}) \ (i \neq r)$$
$$\hat{x}_{Br} = x_{B_r}/u_{r_j} = \theta(\text{ say })$$
$$\mathbf{a}_j = \sum_{i=1}^{m} u_{ij}\, b_i.$$
$$\bar{z} > z_j$$
$$\left[z^2 + \theta(c_j - z_j\, d)\right]/\left[z^3 - \theta(d_j - z_j a)\right] > z^3/z^2$$
$$\left[z^2 + \theta(c_j - z_j a)\right]/\left[z^3 + \theta(d_j - z_j a)\right] - z^1/z^2 > 0,$$
$$z^2\left[z^1 + \theta(c_j - z_j l)\right] - z^1\left[z^2 + \theta(d_j - z_i a)\right] > 0$$
$$z^2(c_j - z_j{}^1) - z^1(d_j - z_j{}^2) > 0$$
$$\Delta_j < 0 \text{ where } z^2(z_j^1 - c_j) - z'(z_j^2 - d_j)$$

Now Δj is less than zero if

$$z_j^2 - d_j > 0,$$
$$(z_j^1 - c_j)/(z_j^2 - d_j) < z^1/z^2.$$
$$z_j^2 - d_j < 0,$$
and
$$(z_j^1 - c_j)/(z_j^2 - d_j) > z^1/z^2.$$
$$z_j^2 - d_j = 0,$$
$$z_j^1 - c_j > 0.$$

In general, a linear fractional functionals programming problem may be stated as:

$$\text{maximize } z = \frac{c_1x_1 + c_2x_2 + \cdots \quad + c_nx_n + \alpha}{d_1x_1 + d_2x_2 + \cdots + d_nx_n + \beta}$$

subject to the constrains:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$
$$x_j \geq 0, i = 1,2, \ldots, m; \; j = 1,2, \ldots, n$$

maximize z

subject to the constrains:

$$(zd_1 - c_1)x_1 + (zd_2 - c_2)x_2 + \cdots + (zd_n - c_n)x_n + z\beta - \alpha \leq 0$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i \leq 0$$
$$x_j \geq 0, i = 1,2, \ldots, m; j = 1,2, \ldots, n$$

We are supposed to find the maximum value of z and non-negative x_j 's which satisfy all the constraints given. From the first constraint of (4), we have the several cases to consider according to the signs of the coefficients of $x_1, x_2, \ldots, x_n$.

Let P be the set of all z 's whose inequalities look like $z \geq \frac{c_j}{d_j}$, and let z_M be the maximum of all those z values, i.e. $z \geq z_M$. Similarly let Q be the set of all z 's whose inequalities look like $z \leq \frac{c_j}{d_j}$, and let z_m be the minimum of those z values, i.e. $z \leq z_M$. All the z values for which $zd_j - c_j = 0$ for $j = 1,2, \ldots, n$ are called critical z-values. We arrange these z values on the number line in ascending order, z_m being the smallest value

greatest value and start searching the number line in a systemic way for a solution of the problem. Solve the system of linear inequalities of problem with a value of $z > z_M$.

In case we find a feasible solution $(x_1^*, x_2^*, \ldots, x_n^*)$ with $z = z^*$, we are done; otherwise go to next step

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Solve the system of linear inequalities of problem with a value of $z > z_{M-1}$ and $z < z_M$. In case we find a feasible solution $(x_1^*, x_2^*, \ldots, x_n^*)$ with $z = z^*$, we are done otherwise repeat above steps until a solution is reached.

Sample problem for linear fractional functional programming problem

Maximize

$$Z = \frac{5x_1 + 3x}{5x_1 + 2x_2 + 1}$$

Subject to the condition

$$3x_1 + 5x \leq 15$$
$$5x_1 + 2x \leq 10$$
$$x_1, x_2 \geqslant 0$$

After adding slack variables then the problem change in the standard form is

maximize

$$Z = \frac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1}$$

subject to condition

$3x_1 + 5x_2 + 73 + 0x_4$ =15

$$x_1 + 2x_2 + 0x_3 + x_4 = 10$$

After Calculating for optimality for above given problem
$(z_1 - c_1), (z_2 - c_2), (3 - c_3)(z_1 - c_2)$ and $(z_1^2 - d_1)(z_2^2 - d_2)(z_2^2 - d_3)(z_4^2 - d_4)$
Usual Matrix X
Initial baric feasible solution is given in the following by starting simplex table

$$z_1 - c_1 = 0, z_2 - c_2 = 0, z_3 - 5 = 5$$
$$\Delta_j = -1, -1, -5. -3$$

then first iteration introducing $y1$, and drop $y_y \Delta_j = -,1, -, -11$
second iteration introducing y_2 drop y_3 Δ_j are $\frac{3}{5}, -, \frac{59}{5}, -$
Same proceeding all the Δ_j are positive then the maximize $z = 9/7$ and
$x_1 = 0 \quad x_2 = 3, x_3 = 0, x_4 = 4$, in the optimum table all Δ_j are all positive, thus we have reached maximum Z and the optimum solution as above in the final iteration.

CONCLUSION

The proposed algorithm introduces a new approach since it does not rely on any variation of the simplex method to obtain the solution. In the modern computational era, developing an efficient computer program based on this algorithm is quite feasible. Furthermore, the method can be easily adapted for minimization problems, with only a minor modification required in the first constraint of equation.

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