International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Overview of Diverse Entropy Measures

Anuradha Swarnkar[1], Dr. Rohit Verma[2]

[1]Research Scholar, Bharti Vishwavidyalaya, Durg, Chhattisgarh
[2]Assistant Professor, Department of Mathematics, Bharti Vishwavidyalaya, Durg (C.G.)

Abstract:
Entropy arises in many contexts as a quantifier of discrete properties, motivating families of entropy measures tailored to specific distributions. In this work, we develop measures based on the Rayleigh and truncated Rayleigh laws and, in parallel, derive Shannon-type entropies under an exponential-power generalization of the normal (Gaussian) distribution. We also introduce a new generalized entropy measure and summarize its key mathematical properties. Additionally, we present a generalized hyperbolic form of probabilistic entropy that exhibits strong additive behavior and rapid adaptability across modeling scenarios. Extending into fuzzy information frameworks, the paper connects these ideas to non-classical (non-Boolean) data theories and discusses a generalized Fisher-type information/entropy measure. Finally, we define two entropy formulations for complex fuzzy sets, Type-A, which depends solely on membership amplitude, and Type-B, which incorporates both amplitude and phase, and assess their rotational invariance, clarifying when these measures remain unchanged under phase rotations.

I. INTRODUCTION

Information theory extends the mathematical study of problems arising in the communication, storage, and transmission of messages. The field took formal shape with Claude Shannon's landmark 1948 paper, "A Mathematical Theory of Communication." Shannon framed the central challenge of a communication system as reproducing, at one point, the information chosen at another, either exactly or within acceptable fidelity, given that the information is tied to specific physical or conceptual features of the system. Accordingly, a well-designed communication architecture must accommodate every possible message that could be selected, not merely those anticipated at design time. The fundamental purpose of such a system is to convey information from a source to a destination via some medium.

One of Shannon's most consequential contributions was the concept of entropy, which he adapted from thermodynamics. Originally introduced in 1865 to quantify physical disorder, entropy was reinterpreted by Shannon as a measure of uncertainty in a random source of messages, thereby reshaping the theory of communication. In Shannon's model, three core components structure the process: (i) the information source, which generates the message intended for the receiver; (ii) the channel, the medium through which the signal travels; and (iii) the destination, which reconstructs the message. Building on foundational work by Nyquist and Hartley, Shannon's analysis transformed earlier insights into a unified, rigorous framework for information transfer, establishing the modern discipline now known as information theory.

Shannon entropy quantifies the average information contained in a source, typically measured in bits, where each message is a realization of a random variable. Equivalently, Shannon entropy measures the expected information "missing" when the value of that random variable is unknown. This concept underpins much of modern communication theory and has been one of the most intensively studied ideas in probabilistic modeling over the past several decades. Beyond Shannon's formulation, a broad family of "generalized" or non-Shannon entropies has been developed. These measures can offer greater flexibility across diffusion-like phenomena and distributions with varying tail behavior, making them useful for characterizing concentration, heterogeneity, and robustness in complex systems. In information and coding theory, such generalized entropies extend well-known measures, with the Shannon entropy recovered as a special or limiting case. A prominent example is Rényi's 1961 one-parameter family of entropies, which provides a continuum of measures differing in their sensitivity to event probabilities. By tuning the order parameter, Rényi entropy interpolates among perspectives on uncertainty while reducing to Shannon entropy at a specific limit, thereby complementing and generalizing Shannon's original measure.

Building on Rényi's (1961) generalization, Tsallis (1988) proposed another entropy family in the same spirit, with Shannon entropy emerging as a special (limiting) case of both Rényi and Tsallis formulations. Subsequent researchers introduced broader one- to four-parameter extensions within information theory and statistics to capture different sensitivities to probability distributions. Earlier, Havrda and Charvát (1967) had already advanced a structural generalization of Shannon's measure, while Arimoto (1971) offered an alternative entropy and associated information quantities. Later, Awad (1987) further extended Shannon's framework, proposed modifications to Rényi's measure, and reinforced the Havrda–Charvát approach, collectively contributing additional families of entropy measures that encompass Shannon's entropy as a unique case.

Zadeh (1965) introduced fuzzy sets to model uncertainty arising from imprecision and partial truth, fundamentally reshaping analysis across domains. In this framework, "fuzziness" reflects the vagueness of set boundaries, elements can belong to a set to varying degrees rather than being strictly in or out. Zadeh (1968) initiated quantitative assessment of fuzziness by proposing measures that relate probabilistic notions to membership functions, yielding a fuzzy analogue of Shannon entropy. De Luca and Termini subsequently formulated axioms that a fuzzy entropy should satisfy and proposed an entropy measure consistent with Shannon's principles. Yager defined fuzzy entropy via the divergence between a fuzzy set and its complement under a chosen criterion. Later, Pal and Pal (1989) developed an "exponential fuzzy entropy," introducing an entropy measure based on an exponential function to capture degrees of fuzziness more sensitively.

Hwang and Yang defined fuzzy-set entropy by synthesizing Yager's divergence-based perspective with the exponential formulation of Pal and Pal, yielding a unified measure of fuzziness. Building on the axiomatic framework of De Luca and Termini, several authors, Kapur; Hooda; Bhandari and Pal; and Fan and Ma, proposed parametric generalizations that enhance tunability and adapt sensitivity to data characteristics; in these models, key parameter choices are guided by the structure of the information itself.

Complex Fuzzy Sets (CFS) extend classical fuzzy sets by allowing membership to take complex values, so the membership function maps elements to points on the complex plane. Across standard fuzzy sets and their many extensions, entropy measures remain central for quantifying uncertainty and vagueness. In the context of CFS applications, additional tools and indices have been developed, such as distance measures, linguistic variables adapted to complex memberships, and criteria ensuring rotational invariance, as well as notions of parallelity and orthogonality to analyze relational structure. Together, these advances provide a richer, more flexible calculus for modeling imprecision in complex domains.

In contrast with the extensive literature on classical fuzzy sets, research specifically addressing the entropy of complex fuzzy sets has been sparse. Complex fuzzy sets are characterized by complex-valued membership functions comprising an amplitude and a phase; the amplitude captures the conventional notion of fuzziness, while the phase introduces a novel descriptive dimension that differentiates complex fuzzy sets from their classical counterparts. Building on this perspective, two classes of complex entropy measures can be formulated: one that depends solely on the amplitude component, closely aligned with standard fuzzy-set entropy and effectively ignoring phase, and another that jointly incorporates both amplitude and phase to reflect the richer structure of complex memberships. Relatedly, the Fisher information measure (sometimes referred to as Fisher's entropy in certain treatments) plays a central role in quantifying informational content, and foundational analytic tools such as the Poincaré and Sobolev inequalities underpin the behavior and control of generalized Fisher-type measures. Moreover, developments surrounding Shannon's relative entropy within information theory offer a powerful lens for understanding statistical models and inference mechanisms, reinforcing the theoretical basis for entropy-based analyses. Taken together, these ideas underscore how entropy and fuzzy-entropy constructs, now extended to complex settings, serve as versatile instruments across science, engineering, and management, enabling sophisticated quantification of uncertainty, structure, and information in a wide range of applications.

II. LITERATURE REVIEW

This section deals with the following entropy measures:

(i) Shannon Entropy Measures: Let the term 'Entropy' be denoted by ' E '. In the ancient times of information theory, the most essential moment was the delivery of Shannon's measure of information that was known as entropy suggested with the aid of Von Neumann whose formulae turn out to be a portrait for information theory [1, 2]:

$$E=-\sum_{i=1}^{n} p_i \log_2 \tag{1}$$

Hartley's suggested Shannon's entropy as the extent of information in the message which was consisted of m symbols preferred from the alphabet of n symbols [3]:

$$E=m\log_2 n \tag{2}$$

This was the simple case in which the output of a source can be formed as a random variable X which can take values from a set of symbols say $\sum=\{1,2,\dots,n\}$ with probabilities $p_i, i \in \Sigma$. Shannon's Measure Entropy had the following foremost three fundamental postulates which established entropy especially:

1. Measure $E(p_1,p_2,\dots,p_n)$ should be continuous $\forall p_i$.
2. Assuming that all p_i should be equal, $p_i=\frac{1}{n}$, at that point E will be a monotonic increasing function of n . For uniformly expected cases there was greater desire or anxiety, although there were extra feasible cases.
3. Assuming that a desire be damaged falling toward two alternating options, the initial E will be the weighted amount of the entity values of E.

Later, these postulates were simply modified through A.I. Khinchin in view of D. K. Fadeer [4, 5]:

1. The message attained dependable only on the probability distribution $p=(p_1,p_2,\dots,p_n)$, therefore, it will be denoted by E (p) or $E(p_1,p_2,\dots,p_n)$ and imagine more that $E(p_1,p_2,\dots,p_n)$ be a well-formed function about its variables $p_1,p_2,\dots,p_n$.
2. $E(p,1-p)$ be a stable function $\forall(0\leq p\leq 1)$.
3. $E\left(\frac{1}{2},\frac{1}{2}\right)=1$.
4. The affiliation dominances:

$$E(p_1,p_2,\dots,p_n)=E(p_1,p_2,\dots,p_n)+(p_1+p_2)E\left(\frac{p_1}{(p_1+p_2)},\frac{p_2}{(p_1+p_2)}\right).$$

Alfred Reyni hypothesized Shannon's Entropy and established the measure as:

$$E_\beta(Y)=\frac{1}{1-\beta}\ln\int_{H_y}[f(y)]^\beta dy;\beta>0,\beta\neq 1$$
$$H_y=\{y: f(y)\neq 0\} \tag{3}$$

where β be an additional parameter and Y be a definitely stable non-negative random variable acquiring probability density function $f(y)$.
Tsallis hypothesized Shannon's Entropy in the same direction and established the measure as:

$$T_\beta(Y) = \frac{1}{\beta-1}\ln \int_{H_y} [f(y)]^\beta dy; \beta > 0, \beta \neq 1;$$
$$H_y = \{y: f(y) \neq 0\}$$
(4)

Thus, Shannon's Entropy is the unique case about both the Renyi's entropy as properly as the Tsallis's entropy. Havrda and Chavrat recommended another extension of Shannon's Entropy and established the measure which is known entropy of degree β as:

$$C^\beta(Y) = \frac{1}{2^{1-\beta}-1}\left(\int_{H_y} [f(y)]^\beta \ln \frac{f(y)}{\varphi} dy\right); \beta > 0, \beta \neq 1;$$
$$H_y = \{y: f(y) \neq 0\}; \text{ where } \varphi = \operatorname{Sup}_{x\in H_y} f(y)$$
(5)

These entropy measures were evolved by proposing one or more parameters in Shannon entropy $E_S(P)$. Later, Sharma and Mittal further induced Havrda and Charvat's entropy and offered two-parametric measures called 'order- γ and type- ω ' entropy as:

$$E^{(\gamma,\omega)}(P) = \frac{1}{(2^{1-\omega}-1)}\left[\left(\sum_{i=1}^{n} p_i^\gamma\right)^{\frac{\omega-1}{\gamma-1}} - 1\right]$$
$$\gamma \neq 1, \omega \neq 1; \gamma, \omega > 0$$
(6)

Sharma and Taneja also advised a two parametric generalization of Havrda and Charvat's entropy called 'type- (γ, ω) entropy' given by:

$$E^{(\gamma,\omega)}(P) = \frac{1}{(2^{1-\gamma}-2^{1-\omega})}\sum_{j=1}^{n} \left(p_j^\gamma - p_j^\omega\right)$$
$$\gamma \neq \omega; \gamma, \omega > 0$$
(7)

Arimoto established the measure of entropy as:

$$E(P) = \frac{1}{(2^{r-1}-1)}\left[\left(\sum_{i=1}^{n} p_i^{\frac{1}{r}}\right)^r - 1\right]; r \neq 1, r > 0 \tag{8}$$

Awad elongated Shannon's Entropy and established the measure as:

$$A(Y) = -\int_{H_y} f(y) \ln \frac{f(y)}{\varphi} dy \tag{9}$$

Awad et al. modified Remyis Entropy and established the measure as:

$$\mathrm{A(Y)}_\beta = \frac{1}{1-\beta}\ln \int_{\mathrm{H}_2} \left[\frac{[\mathrm{yy}]}{\varphi}\right]^{\beta-1} \mathrm{f(y)dy} \tag{10}$$

Awad et al. corroborated Havrda and Chavrat's Entropy and established the measure as:

$$A^\beta(Y) = \frac{1}{2^{1-\beta}-1}\left(\int_{\mathrm{H}} \left[\frac{f(y)}{\varphi}\right]^{\beta-1} f(y)dy - 1\right)$$
$$\beta > 0, \beta \neq 1; \tag{11}$$

These exclusive entropy measures are used to determine the cost of entropy during the life time distribution is pretended new truncated Rayleigh over $[0, t)$ rather based on thinking about Rayleigh distributions on

$[0, \infty)$

(ii) Exponential Entropy Measures: Pal and Pal investigated an academic Shannon's entropy and preferred a unique measure of entropy based on exponential function as:

$$e^{E(P)} = \frac{1}{n(\sqrt{\bar{\sigma}} - 1)} \sum_{j=1}^{n} p_j (e^{1-p_j} - 1) \tag{12}$$

Analogous to rationalizations of Shannon's Entropy including Renyi's entropy, Kvalseth [27] suggested and considered generalized exponential entropy of order- γ is inclined by:

$$E_\gamma(P) = \frac{\sum_{j=1}^{n} p(y_j)\left(e^{(1-p}p'(y_j)\right) - 1\right)}{\gamma}; \gamma > 0 \tag{13}$$

(iii) Non-Shannon Entropy Measures: Let NonShannon's Entropy be denoted by E_{NS} - Let the shaded plane circle graph $\{h_j, j = 0,1, \ldots, m_{g-1}\}$ where m_g be the representation about definite shaded planes in the part about concern. In the case that when m is the entire representation of pixels in the domain, then the established circle graph about the region of concern be the set $\{H_{j,j}\} = 0,1, \ldots, m_{g-1}\}$

where $H_j = \frac{h_j}{\mathrm{m}}$.

Renyi's established Non-Shannon's measures as:

$$R(E_{NS}) = \frac{1}{1-\gamma} \log_2 \left(\sum_{j=0}^{m_0 - 1} H_j^\gamma \right); \gamma \neq 1, \gamma > 0 \tag{14}$$

Havrda and Charvat provided Non-Shannon's measures as:

$$HC(E_{NS}) = \frac{1}{1-\beta} \left(\sum_{j=0}^{m_g - 1} H_j^\beta - 1 \right); \beta \neq 1, \beta > 0 \tag{15}$$

$$K^{(\gamma,\omega)}(E_{NS}) = \frac{1}{\omega - \gamma} \log_2 \frac{\sum_{j=0}^{\pi\omega^{-1}} H_j^\gamma}{\sum_{j=0}^{\pi\omega^{-1}} H_j^u}; \gamma \neq \omega; \omega, \gamma > 0$$

(16)

These measures were suitable criterion for normal and abnormal mammogram images analysis.

(iv) New Generalized Entropy Measure: Illustrate a new generalized entropy measure as:

$$E_\gamma^\omega(P) = \frac{1}{\omega(1-\gamma)} \sum_{j=1}^{m} p_j^{\frac{\gamma}{\omega}}; 0 < \gamma < 1; \tag{17}$$

$\omega \geq 1, p_j \geq 0 \forall j = 1,2, \ldots, m$ and $\sum_{j=1}^{m} p_j = 1$

When $\omega = 1$, then (17) moderated to entropy as:

$$E_\gamma(P) = \frac{1}{1-\gamma} \sum_{j=1}^{m} p_j^\gamma, 0 < \gamma < 1 \tag{18}$$

If $\omega = 1$ and $\gamma \to 1$, then (17) diminished to Shamon's entropy as:

$$E(P) = -\sum_{j=1}^{n} p_j \log\, p_j \tag{19}$$

Properties of New Generalized Entropy Measures:

1. $E_\gamma^\omega(P)$ be a non-negative measure.

As from (17),

$$E_\gamma^\omega(P) = \frac{1}{\omega(1-\gamma)} \sum_{j=1}^{m} p_j^\omega, \text{ where, } 0 < \gamma < 1, \omega \geq 1$$

For given values of γ and ω, $\sum_{j=1}^{m} p_j^{\frac{\gamma}{\omega}} \geq 1$ also $0 < \gamma < 1, \omega \geq 1$ and $\frac{1}{\omega(1-\gamma)} > 0$
Therefore this terminate that,

$$\frac{1}{\omega(1-\gamma)} \sum_{j=1}^{m} p_j^{\frac{\gamma}{\omega}} \geq 0$$

2. $E_\gamma^\infty(P)$ be a symmetric function on every $p_j \vee \mathrm{j} = 1,2, \ldots, \mathrm{m}$.
Note that $E_\gamma^2(P)$ be a symmetric function on every

$$p_j, j = 1,2, \ldots, m$$

$$\text{Le. } E_\gamma^\omega(P_1, P_2, \ldots, P_{m-1}, P_m) =$$

3. $E_\gamma^\omega(P)$ be a maximum function when $\omega = 1$ and $\gamma \to 1$ and all the events have equal probabilities.
When $p_j = \frac{1}{m} \vee j = 1,2, \ldots, m$ and $\omega = 1$ and $\gamma \to 1$ then, $E_\gamma^\omega(P) = \log\, m$ is the maximum entropy.
4. For $\gamma \to 1$ and $\omega = 1, E_\gamma^\omega(P)$ be a concave downward function for $P_1, P_2, \ldots, P_{m-1}, P_m$.
From (17) $\frac{1}{\omega(1-\eta)} \sum_{j=1}^{m} p_j^{\frac{\mu}{j}}, 0 < \gamma < 1, \omega \geq 1$
If $\gamma \to 1$ and $\omega = 1,$ then the first derivative of (17)
With respect to p_j is inclined by:

$$\left[\frac{d}{dp_j} E_\gamma^\omega(P)\right]_{\omega=1,\gamma\to 1} = -1 - \log\, p_j$$

And the second derivative is inclined by:

$$\left[\frac{d^2}{dp_j^2} E_\gamma^\omega(P)\right]_{a=1,\gamma\to 1} = -\left(\frac{1}{p_j}\right) < 0$$

$\forall p_j \in [0,1]$ and $j = 1,2, \ldots, m$
As the second derivative of $E_r^\omega(P)$ with respect to p_j is negative on given interval $p_j \in [0,1]$, as $j = 1,2, \ldots, m$ as $\mathrm{Y} \to 1$ and $\omega = 1$.
Therefore, $E_r^\infty(P)$ be a concave downward function for $P_1, P_2, \ldots, P_{m=1}, P_m$.

(v) Generalized Hyperbolic Entropy Measure: The normal information comfort of a case against intent about ' m ' cases, the entropy of the set A be explained as:

$$E(P) = \sum_{j=1}^{m} p_j \ln\, f(A_j) = -\sum_{j=1}^{m} p_j \log\, p_j \tag{20}$$

Bhatia and Singh offered a constant quantity hyperbolic entropy measure as [20]:

$$E_\gamma(P) = -\frac{1}{\mathrm{Sinh}(\gamma)} \sum_{j=1}^{m} p_j \mathrm{Sinh}(\gamma \log\, p_j); \gamma > 0 \tag{21}$$

$$\lim_{\gamma\to 0} E_\gamma(P) = E(P) \tag{22}$$

After De Luca and Termini, several established forms of this fuzzy entropy were determined. Bhatia et al. intended a constant quantity hyperbolic entropy measure as:

$$E_\beta(B)=\frac{-1}{\operatorname{Sinh}(\beta)}\left[\sum_{j=1}^{m}\delta_B(y_j)\operatorname{Sinh}\left(\beta\log\delta_B(y_j)\right)+\sum_{j=1}^{m}\left(1-\delta_B(y_j)\right)\operatorname{Sinh}\left(\beta\log\left(1-\delta_B(y_j)\right)\right)\right] \quad (23)$$

$$\lim_{\beta\to 0}E_\beta(B)=E(B) \quad (24)$$

The real estimate β is correlated for non-ranginess about the entity. For conceptual appliance, measure designed in (21) is known as hyperbolic entropy and expressed as:

$$E_{\text{hyp}}(P)=\frac{-1}{\operatorname{Sinh}(\gamma)}\sum_{j=1}^{m}p_j\operatorname{Sinh}(\gamma\log p_j),\gamma>0 \quad (25)$$

The hyperbolic entropy advised in (25) is related by Renyi entropy and Havrda-Charvat entropy. In order to compare, all of three entropies have been assigned.

(vi) Fuzzy Entropy Measure: If $Y=(y_1,\dots,y_{\mathrm{m}})$ be a distinct universe of communication. A fuzzy set B at Y be constituted through a group function $\delta_B(y):Y\to[0,1]$ and the amount $\delta_B(y)$ about B on $y\in Y$ views during the extent about group of y in B .
If $\Delta_m=\left\{P=(p_1,\dots,p_m):p_j\geq 0,\sum_{j=1}^{m}p_j=1\right\},m\geq 2$ be a set about m-entire probability distributions. Considering several probability distributions as:

$$p_j=(p_1,\dots,p_m)\in\Delta_m$$

Let Shannor's Entropy be explained just as:

$$E(P)=-\sum_{j=1}^{\mathrm{m}}p(y_i)\log p(y_i) \quad (26)$$

First try through evaluate the concern combined along a fuzzy tournament in the situation about discrete probabilistic structure seems to have been built through Zadeh that fact described the entropy of a fuzzy set B including appreciate to (Y, P) as:

$$E(Y,P)=-\sum_{j=1}^{\mathrm{m}}\delta_B(y_j)p(y_j)\log p(y_j) \quad (27)$$

A degree about fuzziness in a fuzzy set ought to have at least the successive postulates:

- (Incisiveness): $E(B)$ is minimum in the case that B is a crusty set i.e. $\delta_B(y_j)=0$ or 1 $\vee j$.
- (Maximality): $E(B)$ is maximum in the case that i.e. $\delta_B(y_j)=\frac{1}{2}\vee j$.
- (Immovability): $E(B^*)\leq E(B)$ where B^* be an edged form of B.

[A fuzzy set B^* be known as an edged form about fuzzy set B if the successive conditions be appeased: $\delta_{B^-}(y)\leq\delta_B(y)$, if $\delta_B(y)\leq\frac{1}{2}\vee j$ and $\delta_B\cdot(y)\geq\delta_B(y)$, if $\delta_B(y)\geq\frac{1}{2}\vee J$]

- (Similarity): $E(B)=E(B^c)$

where B^c is the complement set of B.
In view of $\delta_B(y_j)$ and $\left(1-\delta_B(y_j)\right)$ gives the identical diploma of fuzziness, accordingly De Luca and Termini described fuzzy entropy about a fuzzy set B analogous to (26) just as:

$$E(B)=\frac{-1}{m}\sum_{j=1}^{m}\left\{\begin{array}{c}\delta_B(y_j)\log\delta_B(y_j)+\\\left(1-\delta_B(y_j)\right)\log\left(1-\delta_B(y_j)\right)\end{array}\right\} \quad (28)$$

where B be a fuzzy set also $\delta_B(y_j)$ be group amount about y_j in B. Later on Bhandari and Pal invented an analysis about fuzzy sets also provided a few recent measures of fuzzy entropy analogous to (3) they had recommended the consecutive measure as:

$$E_\beta(B) = \frac{1}{(1-\beta)} \sum_{j=1}^{m} \log \left\{ \delta_B^\beta(y_j) + \left(1 - \delta_B(y_j)\right)^\beta \right\}$$

where $\beta \neq 1, \beta > 0$

(29)

Pal and Pal described exponential fuzzy entropy considering a fuzzy set analogous to (12) just as:

$$E(B) = \frac{1}{m(\sqrt{e}-1)} \sum_{j=1}^{m} \begin{bmatrix} \delta_s(y_j) e^{\left(1-\delta_p(y_j)\right)} \\ + \left(1-\delta_B(y_j)\right) e^{\delta_B(y_j)} - 1 \end{bmatrix} \quad (30)$$

(vii) Exponential Fuzzy Entropy: Assigned the exponential fuzzy entropy of order- β analogous to (13) just as:

$$E_g(B) = \frac{1}{m(e^{(1-0.5^p)}-1)} \sum_{j=1}^{m} \begin{bmatrix} \delta_B(y_j) e^{\left(1-\delta_B^B(y_j)\right)} \\ + \left(1-\delta_B(y_j)\right) e^{\left(1-\left(1-\delta_B(y_j)\right)^p\right)} - 1 \end{bmatrix} \quad (31)$$

where $\beta > 0$

Note that this exponential fuzzy entropy of order- β diminished into Pal and Pal exponential entropy also De-Luca and Termini logarithmic entropy about the definite amount of β just as displaces: On the assumption when $\beta = 1$ in (31), it diminished to

$$E(B) = \frac{1}{m(\sqrt{\sigma}-1)} \sum_{j=1}^{m} \begin{bmatrix} \delta_B(y_j) e^{\left(1-\delta_B(y_j)\right)} \\ + \left(1-\delta_B(y_j)\right) e^{\delta_g(y_j)} - 1 \end{bmatrix} \quad (32)$$

where (32) is known as Pal and Pal exponential entropy. In case if $\beta \to 0$, then (31) allows

$$\lim_{\beta \to \infty} E_\beta(B) = E(B)$$

$$= \frac{1}{m} \sum_{j=1}^{m} \begin{Bmatrix} \delta_B(y_j) \log \delta_B(y_j) \\ + \left(1-\delta_B(y_j)\right) \log \left(1-\delta_B(y_j)\right) \end{Bmatrix}$$

(33)

where (33) is known as De-Luca and Termini logarithmic entropy.

(viii) Complex Fuzzy Entropy Measures: If $Z = \{z_1, \ldots, z_m\}$ be a domain of communication, a complex fuzzy set C at Z can be expressed in the process of the set of ordered pairs $C = \{(z, \hat{v}_c(z)) \mid z \in Z\}$ where membership function $\vartheta_c(z)$ is of the form $\varphi_c(z) \cdot e^{i\theta_c(z)}, i = \sqrt{-1}$, the amplitude term $\varphi_c(z)$ and the phase term $\theta_c(z)$ are both real-valued also $\varphi_c(z) \in [0,1]$ where $e^{i\theta_c(x)}$ be an alternate function that iteration law be 2π. So, $\theta_c(z) \in [0,2\pi)$. If A_1 and A_2 be two complex fuzzy sets. If a mapping e: $\text{CFS}(Z) \to [0,1]$ be known as a type-A entropy over $\text{CFS}(Z)$ in case that e amuses the successive adages:

(a1). $e(A_1) = 0$ in the case if $\left|\theta_{A_1}(z)\right| = 0$ or
$\left|\theta_{A_2}(z)\right| = 1 \vee z \in Z$.
(a2) $\cdot\, e(A_1) = 1$ if $\left|\theta_{A_2}(z)\right| = 0.5 \vee z \in Z$.
(a3) $\cdot\, e(A_1) \leq e(A_2)$
if $\left|\theta_{A_1}(z)\right| \leq \left|\theta_{A_2}(z)\right|$ when $\left|\theta_{A_2}(z)\right| \leq 0.5$ and
$\left|\theta_{A_2}(z)\right| \geq \left|\theta_{A_z}(z)\right|$ when $\left|\theta_{A_z}(z)\right| \geq 0.5$
(a4). $e(A_1) = e(-A_1)$
where $-A_1$ is a complex fuzzy complement of A_1.
Consider two specific functions as:

$$e_1(C) = \frac{-1}{m}\sum_{j=1}^{m}\begin{bmatrix} \vartheta_c(z_j)\log \vartheta_c(z_j) \\ +\left(1-\vartheta_c(z_j)\right)\log\left(1-\vartheta_c(z_j)\right)\end{bmatrix} \tag{34}$$

$$e_2(C) = \frac{4}{m}\sum_{j=1}^{m}\left(\vartheta_c(z_j)\left(1-\vartheta_c(z_j)\right)\right) \tag{35}$$

Two type-A entropy formulae respectively analogous to (34) and (35) are recommended as follows:

$$e_3(C) = \frac{-1}{m}\sum_{j=1}^{m}\begin{bmatrix}\left|\theta_c(z_j)\right|\log\left|\theta_c(z_j)\right| + \left(1-\left|\theta_c(z_j)\right|\right) \\ \log\left(1-\left|\theta_c(z_j)\right|\right)\end{bmatrix} \quad (36)$$

$$e_4(C) = \frac{4}{m}\sum_{j=1}^{m}\left|\theta_c(z_j)\right| - \left|1-\theta_c(z_j)\right|$$

when $\theta_c(z) = 0\ vz$ then, $e_3(C) = e_1(C)$ and $e_4(C) = e_2(C)$ (36) (37)

These mappings e_3 and e_4 described by formulae (36) and (37) respectively are type-A entropy measure as complex fuzzy sets.
If $C \in CFS(Z)$ with membership function $\hat{v}_c(z) = \partial_c(z) \cdot e^{iE_c(z)}$. The revolution about C through θ radians, indicated by $\mathrm{Rot}_\theta(C)$ is defined as:

$$\mathrm{Rot}_\theta(\theta_c(z)) = \partial_c(z) \cdot e^{i(\theta_c(z)+\theta)}$$

Let the entropy of CFSs e:CFS $(Z) \to [0,1]$ be rotationally invariable in the case:
If $e(\mathrm{Rot}_\theta(C)) = e(C)$ for any θ and $CFS(C) \in CFS(Z)$.
The mappings e_3 and e_4 are rotationally invariant.
As $\left|\theta_c(z_j)\right| = \left|\theta_c(z_j)\cdot e^{i\theta}\right|$ from this, $e_4(\mathrm{Rot}_\theta(C)) = e_4(C)$. If B_1 and B_2 be two complex fuzzy sets. Let a mapping e: CFS$(Z) \to [0,1]$ be termed as type-B entropy about CFS(2) supposing that e amuses the successive adages:
(b1). $e(B_1) = 0$ in the case if $\varphi_{B_1}(z) = 0$ or
$\varphi_{B_2}(z) = 1$ and $\theta_{B_2}(z) = 0\ \mathrm{V}z \in Z$.
(b2). $e(B_1) = 1$ if $\varphi_{B_1}(z) = 0.5$ and $\theta_{B_2}(z) = \pi$
$\forall z \in Z$.
(b3). $e(B_1) \leq e(B_2)$ if $\varphi_{B_2}(z) \leq \varphi_{B_2}(z)$ and

$$\theta_{B_2}(z) \geq \theta_{B_2}(z) \text{ for } \varphi_{B_2}(z) \leq \frac{\theta_{B_2}(z)}{2\pi} \text{ or}$$

$$\varphi_{B_3}(z) \geq \varphi_{B_2}(z) \text{ and } \theta_{B_1}(z) \leq \theta_{B_2}(z)$$

$$\text{for } \varphi_{B_2}(z) \geq \frac{\theta_{B_1}(z)}{2\pi}$$

(b4). $e(B_1) = e(-B_1)$
where $-B_1$ is a complex fuzzy complement of B_1.
Two type-B entropy formulae respectively analogous to (34) and (35) are recommended as follows:

$$e_5(C) = \frac{-1}{2m}\sum_{j=1}^{m} \left[\varphi_c(z_j)\log\varphi_c(z_j) + + \frac{\theta_c(z)}{2\pi}\log\frac{\theta_c(z)}{2\pi} + \left(1-\frac{\theta_c(z)}{2\pi}\right)\log\left(1-\varphi_c(z_j)\right)\right.$$

$$e_s(C) = \frac{2}{m}\sum_{j=1}^{m} \left[\begin{matrix}\varphi_c(z_j)\left(1-\varphi_c(z_j)\right) + \left(\frac{\theta_c(z)}{2\pi}\right)\\ \left(1-\frac{\theta_c(z)}{2\pi}\right)\end{matrix}\right]$$

(38 & 39)

These mappings e_5 and e_6 described by formulae (38) and (39) commonly are type-B entropy measure about complex fuzzy sets and are not rotationally invariant.

(ix) Fisher's Entropy Type Information Measure: Let U be a multivariate random variable for parameter
vector $\emptyset = \{\emptyset_1, \dots, \emptyset_q\} \in \mathbb{R}^q$ and probability density function $g(U) = g_u(u;\emptyset), u \in \mathbb{R}^q$. The Parametric Form Fisher's Information Matrix $I_F(U;\emptyset)$ represented as the covariance of $\nabla_C\log\, g_u(u;\emptyset)$ (where ∇_0 be a gradient about the parameters $\emptyset_j, j = 1, \dots, q$) be parametric form information measure indicated as:

$$I_F(U;\emptyset) = \mathrm{Cov}[\nabla_\phi\log\, g_u(u;\emptyset)] = E_a\left[\nabla_\varrho\log\, g_u\left(\nabla_q\log\, g_u\right)^T\right]$$
$$= E_a\left[\left\|\nabla_\varrho\log\, g_u\right\|^2\right]$$

(where $\|.\|$ be the usual $N^2(\mathbb{R}^q)$ norm although $E_q[.] be$ the normal value operant enforced to random variables about the parameter $\emptyset$.
Let the Fisher's Entropy Type Information Measure $I_F(U)$ about a random variable U including probability density function g on $\mathbb{R}^q$ be termed as the covariance of random variable $\nabla\log\, g(U)$ i.e.

$$I_F(U) = \int_{\mathrm{RG}}^{\int} g(u)\|\nabla\log\, g(u)\|^2 du = \int_{\mathrm{RF}}^{\int} g(u)^{-1}\|\nabla g(u)\|^2 du$$

$$= \int_{\mathbb{R}+}^{\int} \nabla g(u)\cdot\nabla\log\, g(u)du = 4_{\mathbb{R}S}^{f}\|\nabla\sqrt{g(\mathrm{u})}\|^2 du$$

(40)

Mostly, the group of the entropy form information measures I(U) about a random variable U including probability density function g, be termed through a cofunction of U, i.e. $F(U) = \|\bar{V}\log\, g(u)\|$ as:

$$I(U) = I(U{:}f,k) = f(E[k(F(U))])$$

Particular Case: When $f = i$.d. and $k(U) = U^2$, then the entropy form Fisher's information measure of U is attained just as in (40) specially,

$$I_F(U) = E[\|\nabla\log\, g(u)\|^2] \tag{41}$$

Let Vajda's Parametric Form Information Measure $I_V(U;\emptyset,\beta)$, which is literally a generalization of $I_F(U;\emptyset)$ defined as:

$$I_V(U;\emptyset,\beta) = E_0\left[\left\|\nabla_q\log\, g(u)\right\|^\beta\right], \beta \geq 1 \tag{42}$$

In addition, Vajda's Entropy Form Information Measure $\mathrm{I_V(U)}$ concluded Fisher's entropy form information $\mathrm{I_F(U)}$, expressed just as:

$$I_V(U;\beta) = E\left[\|\nabla\log\, g(u)\|^\beta\right], \beta > 1 \tag{43}$$

It is indicated on the point of the established Fisher's entropy form information measure or β-GFI. Let the second-GFI be abbreviated through the general I specially $\mathrm{I_2(U) = I(U)}$ which is expressed as:

$$I_{\beta}(U) = \int_{R_q} \|\nabla \log\ g(u)\|^{\beta} g(u)du = \int_{R^1} \|\nabla g(u)\|^{\beta} g^{1-\beta}(u)du = \beta^{\beta} \int_{R_g} \left\|\nabla g^{1/\beta}(u)\right\|^{\beta} du \quad (44)$$

The Blackman-Stam inequality is influenced rigidly through the $\beta -$ GFI measure.

III. CONCLUSION

This review presents a comprehensive discussion of various entropy measures and their theoretical as well as applied significance. Shannon's entropy remains the foundational model, satisfying all essential axioms for quantifying information content. Comparative evaluations of multiple entropy measures have been conducted using lifetime distributions such as Rayleigh and exponential models, revealing that while Awad's entropy measures perform well in minimizing information loss for exponential distributions, they do not consistently outperform classical measures like those of Shannon, Rényi, Tsallis, or Havrda–Charvát across all contexts. The study also introduces a modern framework built on non-Shannon entropies, particularly Rényi, Havrda–Charvát, and Kapur, for analyzing both normal and abnormal mammogram images, demonstrating the practical effectiveness of Havrda–Charvát entropy in distinguishing structural variations. Furthermore, an advanced generalized entropy measure, termed exponential fuzzy entropy of order β, is proposed, extending the works of Pal and Pal's exponential entropy and De Luca and Termini's logarithmic entropy. The parameter β enhances the adaptability of the model, enabling broader applicability across domains.

In addition, the paper explores the additive properties of hyperbolic entropy measures, emphasizing their flexibility through one-, two-, three-, and four-parameter generalizations that provide higher analytical precision. A concise exposition of Fisher-type entropy and its extensions, including B-Shannon and Rényi entropies, is also provided, underscoring how these generalizations refine traditional information measures through advanced mathematical frameworks. Empirical evaluation suggests these measures merit further testing using real-world data across different fields. The review further presents two complex fuzzy entropy formulations, Type-A and Type-B, where Type-A depends solely on membership amplitude and simplifies to the classical case, while Type-B integrates both amplitude and phase, adding a new interpretative dimension. The study also discusses rotational invariance in entropy measures of complex fuzzy sets and mathematically establishes the conditions under which this property holds. Ultimately, this review positions complex fuzzy sets as a promising theoretical development that extends traditional fuzzy set theory into a complex-valued domain, paving the way for extensive future research and multifaceted applications.

REFERENCES:

1. Abdullah, L., & Otheman, A. (2013). A new entropy weight for sub-criteria in interval type-2 fuzzy TOPSIS and its application. International Journal of Intelligent Systems and Applications, 5(2): 25.
2. Aczél, J., Forte, B., & Ng, C. T. (1974). Why the Shannon and Hartley entropies are 'natural'. Advances in Applied Probability, 6(1): 131-146.
3. Awad, A. M., & Alawneh, A. J. (1987). Application of Entropy to a life-Time Model. IMA Journal of Mathematical Control and Information, 4(2): 143-148.
4. Bhandari, D., & Pal, N. R. (1993). Some new information measures for fuzzy sets. Information Sciences, 67(3): 209-228.
5. Bhat, A. H., Baig, M. A. K., & Dar, M. H. (2017). New Generalized Entropy Measure and its Corresponding Code-word Length and Their Characterizations. Intemational Journal of Advance Research in Science and Engineering, 6(1): 863-873.
6. Bhatia, P. K., & Singh, S. (2013). On a new Csiszar's f-divergence measure. Cybernetics and information technologies, 13(2): 43-57.

7. Bhatia, P. K., Singh, S., & Kumar, V. (2013). On a Generalized Hyperbolic Measure Of Fuzzy Entropy. International Archives, 4(12): 136-142. Journal of Mathematical
8. Bhatia, P. K., Singh, S., & Kumar, V. (2015). On Applications of a Generalized Hyperbolic Measure of Entropy. International Journal of Intelligent Systems and Applications, 7(7): 36.
9. De Luca, A., & Termini, S. (1971). A definition of non-probabilistic entropy in the settings of fuzzy set theory. Information and Control, 20: 301-312.
10. Dey, S., Maiti, S. S., & Ahmad, M. (2016). Comparison of different entropy measures. Pak. J. Statist, 32(2): 97-108.
11. Faddeev, D. K. (1956). On the concept of entropy of a finite probabilistic scheme. Uspekhi Matematicheskikh Nauk, 11(1): 227-231.
12. Fan, J. L., & Ma, Y. L. (2002). Some new fuzzy entropy formulas. Fuzzy sets and Systems, 128(2): 277-284.
13. Hartley, R. V. (1928). Transmission of information. 1. Bell System technical journal, 7(3): 535-563.
14. Hartley. R. V. (1928). Transmission of information. 1. Bell System technical journal, 7(3): 535-563.
15. Havrda, J., & Charvát, F. (1967). Quantification method of classification processes. Concept of structural $ a $-entropy. Kybernetika, 3(1): 30-35.
16. Havrda, J., & Charvát, F. (1967). Quantification method of classification processes. Concept of structural $ a $-entropy. Kybernetika, 3(1): 30-35.
17. Hooda, D. S. (2004). On generalized measures of fuzzy entropy. Mathematica Slovaca, 54(3): 315-325.
18. Hwang, C. M., Yang, M. S., Hung, W. L., & Lee, E. S. (2011). Similarity, inclusion and entropy measures between type-2 fuzzy sets based on the Sugeno integral. Mathematical and Computer Modelling, 53(9-10): 1788-1797.
19. Hwang, C.H., & Yang, M.S. (2008). On entropy of fuzzy sets. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 16(4): 519-527.
20. Kapur, J. N. (1997). Measures of fuzzy information. Mathematical Sciences Trust Society.
21. Kvalseth, T. O. (2000). On Exponential Entropies. IEEE International Conference on Systems, Man, and Cybernetics, 4: 2822-2826.
22. Li, Y., Qin, K., & He, X. (2013). Relations among similarity measure, subsethood measure and fuzzy Computational entropy. International Journal Intelligence Systems, 6(3): 411-422. of
23. LinLin, W., & Yunfang, C. (2012). Diversity based on entropy: A novel evaluation criterion in multi-objective optimization algorithm. International Journal of Intelligent Systems and Applications, 4(10): 113-124.
24. Machlup, F. (1983). The study of information: Interdisciplinary messages. Wiley, New York, pp.475-484.
25. Nyquist, H. (1924). Certain factors affecting telegraph speed. Transactions of the American Institute of Electrical Engineers, 43: 412-422.
26. Nyquist, H. (1928). Certain topics in telegraph transmission theory. Transactions of the American Institute of Electrical Engineers, 47(2): 617-644.
27. Pal, N. R., & Pal, S. K. (1989). Object-background segmentation using new definitions of entropy. IEE (Computers Proceedings E Techniques), 136(4): 284-295. and Digital
28. Pal, N. R., & Pal, S. K. (1991). Entropy: A new definition and its applications. IEEE transactions on systems, man, and cybernetics, 21(5): 1260-1270.
29. Pharwaha, A. P. S., & Singh, B. (2009). Shannon and non-shannon measures of entropy for statistical texture feature extraction in digitized mammograms. World Congress on Engineering and Computer Science, 2:20-22.

30. Ramot, D., Milo, R., Friedman, M., & Kandel, A. (2002). Complex fuzzy sets. IEEE Transactions on Fuzzy Systems, 10(2): 171-186.
31. Rénvi, A. (1961). On measures of entropy and information. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics. The Regents of the University of California.
32. Réyni, A. (1970). Probability Theory. Dover Publications.
33. Schroeder. M. (2004). An alternative to entropy in the measurement of information. Entropy, 6(5): 388-412.
34. Shannon, C. E. (1948). A mathematical theory of communication. Bell system technical journal, 27(3): 379-423.
35. Toulias, T. L., & Kitsos, C. P. (2015). Generalizations of entropy and information measures. In Computation, and Network Cryptography, Security (pp. 493-524). Springer, Cham.
36. Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. Journal of Statistical Physics, 52(1): 479-487.
37. Yager, R. R. (1979). On the measure of fuzziness and negation part 1: membership in the unit interval. International Journal of General Systems, 5: 221-229.
38. Zadeh, L. A. (1948). Fuzzy sets. Information Control, 8: 94-102.
39. Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3): 338-353.
40. Zadeh, L. A. (1968). Probability measures of fuzzy events. Journal of mathematical analysis and applications, 23(2): 421-427.