Recent Developments in Bicomplex Operator Theory

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Abstract:
Bicomplex operator theory has evolved from the idempotent reduction of bicomplex modules into paired complex spaces. However, recent research has focused on semigroups, compact operators, frames, reproducing kernels, tensor products, ergodic theory, and quantum-information structures. A critical mathematical review with theorem-level exposition is provided in this paper. Bicomplex-linear operators are decomposed as $T=T_1e_1+T_2e_2$, and their complex components determine their boundedness, compactness, adjoints, normality, self-adjointness, and unitarity. The reduced paired spectrum is distinguished from the full algebraic spectrum. The full spectrum is a union of cylinders and is typically unbounded, as $\lambda I-T$ is invertible only when both component resolvents exist. Consequently, the null cone alters the interpretation of spectral assertions. The following are derived: polynomial functional calculus, point spectra, componentwise resolvent formulas, and compact-normal spectral structure. The review subsequently examines the following developments from 2020–2025: bicomplex operator semigroups, frames, Bergman and Bloch spaces, mean ergodic theorems, matrix canonical forms, Wiener and Herglotz theories, neural operators, and a bicomplex Choi theorem. Although these advancements are substantial, a significant number of them are reductions to two complex theories that are coupled by bicomplex notation. The paper identifies the areas in which this reduction is complete and the areas in which genuinely bicomplex questions remain, particularly in the context of null-cone spectral geometry, C*-structures, unbounded operators, perturbation theory, and physically constrained quantum models.

Keywords: bicomplex operator; idempotent decomposition; compact operator; adjoint; spectrum; resolvent; semigroup; bicomplex Hilbert space

Mathematical Notation and Symbols

Symbol	Meaning
$\mathscr{B}\mathbb{C}$	Bicomplex algebra generated by commuting $\mathbf{i}^2=\mathbf{j}^2=-1$.
$\mathbf{k}=\mathbf{ij}$	Hyperbolic unit with $\mathbf{k}^2=1$.
$\mathbf{e}_1,\mathbf{e}_2$	Idempotents $(1\pm\mathbf{k})/2$.
$Z=Z_1\mathbf{e}_1+Z_2\mathbf{e}_2$	Idempotent form with $Z_j\in\mathbb{C}(\mathbf{i})$.

Symbol	Meaning
$X=\mathbf{e}_1X_1\oplus\mathbf{e}_2X_2$	Bicomplex Banach module.
$T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$	Bicomplex-linear operator decomposition.
$\sigma\mathscr{B}\mathbb{C}(T)$	Full spectrum defined by failure of bounded invertibility.
σred(T)	Reduced spectrum $\sigma(T_1)\mathbf{e}_1+\sigma(T_2)\mathbf{e}_2$.
T*	Bicomplex Hilbert-space adjoint.

1. Introduction

The bicomplex algebra is a commutative, finite-dimensional algebra over $\mathbb{R}$ but not a field. Its idempotent representation identifies it with $\mathbb{C}\times\mathbb{C}$, and most bounded-operator questions can therefore be tested on two complex spaces. This reduction has supported a functional-analysis literature on bicomplex Banach modules, Hilbert spaces, linear functionals, and operators (Alpay et al., 2014; Kumar & Saini, 2016). From 2020 onward, the scope has broadened to operator semigroups, frame theory, function spaces, ergodic theorems, matrix canonical forms, tensor products, and Choi-type positivity results (Sharma & Kunga, 2020; Elgourari et al., 2021; Koumantos, 2023; Alpay et al., 2024).

A critical review must separate three layers. First, some theorems are direct equivalences with a pair of classical complex theorems. Second, bicomplex notation exposes null-cone and module-theoretic phenomena that have no analogue over a field, especially in invertibility and spectrum. Third, applications may impose coupling or positivity conditions that prevent the two channels from being treated independently. The product decomposition is thus both a solution method and a test of whether a purportedly bicomplex result contains additional content.

This paper develops the common operator core and then interprets recent contributions. Statements labelled "established result" summarize standard or published componentwise theory; statements labelled "derived summary" follow from the definitions used here. No priority claim is made for the elementary recombination arguments.

2. Algebraic and Topological Foundations

Definition 2.1. The bicomplex algebra is

$$\mathscr{B}\mathbb{C}=\{z_1+z_2\mathbf{j}:z_1,z_2\in\mathbb{C}(\mathbf{i}),\ \mathbf{i}^2=\mathbf{j}^2=-1,\ \mathbf{ij}=\mathbf{ji}\}. \qquad (2.1)$$

With $\mathbf{k}=\mathbf{ij}$ and $\mathbf{e}_1=(1+\mathbf{k})/2$, $\mathbf{e}_2=(1-\mathbf{k})/2$, every Z has the unique form $Z=Z_1\mathbf{e}_1+Z_2\mathbf{e}_2$. The three standard conjugations change **i**, **j**, or both signs; the involution relevant to Hilbert positivity acts as Z^{†3}$=\bar{Z}_1\mathbf{e}_1+\bar{Z}_2\mathbf{e}_2$. The hyperbolic modulus is $|Z|_{\mathbb{D}}=|Z_1|\mathbf{e}_1+|Z_2|\mathbf{e}_2$.

Definition 2.2. The null cone is $\mathcal{NC}=\{Z:Z_1Z_2=0\}$. A bicomplex number is invertible exactly when $Z_1\neq0$ and $Z_2\neq0$, in which case $Z^{-1}=Z_1^{-1}\mathbf{e}_1+Z_2^{-1}\mathbf{e}_2$.

Definition 2.3. A bicomplex Banach module X is a module $X=\mathbf{e}_1X_1\oplus\mathbf{e}_2X_2$ in which X_1 and X_2 are complex Banach spaces. Its hyperbolic norm is $\|x\|_{\mathbb{D}}=\|x_1\|_1\mathbf{e}_1+\|x_2\|_2\mathbf{e}_2$.

Proposition 2.4. A sequence $x^{(n)}$ converges, is Cauchy, or is bounded in X if and only if both component sequences have the corresponding property.

Proof. Every defining hyperbolic inequality $\|x^{(n)}-x\|_{\mathbb{D}}\prec_{\mathbb{D}}\varepsilon$ is the pair $\|x_1^{(n)}-x_1\|<\varepsilon_1$ and $\|x_2^{(n)}-x_2\|<\varepsilon_2$. The Cauchy and bounded cases are identical projections. □

3. Bounded and Compact Bicomplex Operators

Proposition 3.1. Every bicomplex-linear $T:X\to Y$ has the unique representation

$$T(x_1\mathbf{e}_1+x_2\mathbf{e}_2)=T_1x_1\mathbf{e}_1+T_2x_2\mathbf{e}_2, \quad (3.1)$$

where $T_j:X_j\to Y_j$ is complex-linear.

Proof. Bicomplex linearity gives $T(\mathbf{e}_jx)=\mathbf{e}_jT(x)$, so each idempotent summand is invariant. The coefficient maps are complex-linear and unique. Conversely, any pair defines a bicomplex-linear map by Equation (3.1). □

Theorem 3.2. Boundedness and compactness criterion (established component principle). T is bounded if and only if T_1 and T_2 are bounded. T is compact if and only if both T_1 and T_2 are compact.

Proof. For boundedness, $\|Tx\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}(C_1\mathbf{e}_1+C_2\mathbf{e}_2)\|x\|_{\mathbb{D}}$ is exactly the pair of complex bounds. For compactness, let B be a bounded sequence in X. Its components are bounded; if both T_j are compact, successive subsequence extraction gives a common subsequence for which both image components converge, hence $Tx^{(n)}$ converges. Conversely, apply compactness of T to bounded pure-component sequences $x_1^{(n)}\mathbf{e}_1$ and $x_2^{(n)}\mathbf{e}_2$. □

Corollary 3.3. Finite-rank bicomplex operators are compact. Moreover, rank should be recorded as the pair ($\dim_{\mathbb{C}}$ ran T_1, $\dim_{\mathbb{C}}$ ran T_2); a single integer can conceal unequal idempotent dimensions.

Proof. Finite-dimensional ranges in both components make T_1 and T_2 finite-rank and compact. Theorem 3.2 recombines them. □

Proposition 3.4. The bounded-operator module $\mathcal{L}_{\mathcal{BC}}(X)$ is the product algebra $\mathcal{L}(X_1)\mathbf{e}_1\oplus\mathcal{L}(X_2)\mathbf{e}_2$. With the hyperbolic operator norm $\|T\|_{\mathbb{D}}=\|T_1\|\mathbf{e}_1+\|T_2\|\mathbf{e}_2$ it is complete when X is complete.

Proof. Addition, scalar multiplication, and composition act componentwise. A Cauchy sequence of operators is a pair of Cauchy sequences in the Banach algebras $\mathcal{L}(X_j)$, which converge to bounded operators. □

4. Adjoint Classes and Spectral Theorems

Definition 4.1. A bicomplex Hilbert space $H=\mathbf{e}_1H_1\oplus\mathbf{e}_2H_2$ has inner product $\langle x,y\rangle=\langle x_1,y_1\rangle_1\mathbf{e}_1+\langle x_2,y_2\rangle_2\mathbf{e}_2$, positive in $\mathbb{D}^+$. The adjoint T *satisfies* $\langle Tx,y\rangle=\langle x,Ty\rangle$.

Theorem 4.2. Adjoint decomposition. Every bounded $T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$ on H has the unique adjoint

$$T^*=T_1^*\mathbf{e}_1+T_2^*\mathbf{e}_2. \quad (4.1)$$

Proof. The component Riesz representation theorem gives T_j. *Then* $\langle Tx,y\rangle=\langle T_1x_1,y_1\rangle\mathbf{e}_1+\langle T_2x_2,y_2\rangle\mathbf{e}_2=\langle x_1,T_1y_1\rangle\mathbf{e}_1+\langle x_2,T_2^*y_2\rangle\mathbf{e}_2$. Uniqueness follows from nondegeneracy in each component. □

Corollary 4.3. T is self-adjoint, normal, an isometry, or unitary if and only if T_1 and T_2 have the corresponding property. Positivity $T\succcurlyeq0$ is equivalent to $T_1\succcurlyeq0$ and $T_2\succcurlyeq0$.

Proof. Each defining equation—$T=T$, $TT=TT$, $TT=I$, or $\langle Tx,x\rangle\in\mathbb{D}^+$—splits into its two complex equations or inequalities. □

Theorem 4.4. Bicomplex normal spectral theorem (derived summary of the established complex theorem). If T is bounded and normal on H, there are complex spectral measures E_1 and E_2 such that

$$T=\left(\int_{\sigma(T_1)} z\, dE_1(z)\right)\mathbf{e}_1+\left(\int_{\sigma(T_2)} z\, dE_2(z)\right)\mathbf{e}_2. \quad (4.2)$$

Proof. Apply the complex spectral theorem to T_1 and T_2. Define a bicomplex projection-valued measure only after specifying whether sets are paired or cylindrical; the unambiguous statement is Equation (4.2). Algebraic identities, norm estimates, and Borel functional calculus then hold componentwise. □

Proposition 4.5. If A and B are bounded self-adjoint operators and $A\preccurlyeq B$ componentwise, then $\langle Ax,x\rangle\preccurlyeq_{\mathbb{D}}\langle Bx,x\rangle$ for all x. Conversely, this quadratic-form inequality for all x implies $A\preccurlyeq B$.
Proof. The statement is equivalent to positivity of $B-A$. By Corollary 4.3, positivity is exactly component positivity, where the classical quadratic-form characterization applies. □

5. Spectrum, Resolvent, and Functional Calculus

Definition 5.1. The full bicomplex resolvent is $\rho_{\mathscr{B}\mathbb{C}}(T)=\{\lambda\in\mathscr{B}\mathbb{C}:\lambda I-T$ has a bounded bicomplex inverse$\}$; $\sigma_{\mathscr{B}\mathbb{C}}(T)=\mathscr{B}\mathbb{C}\setminus\rho_{\mathscr{B}\mathbb{C}}(T)$. The reduced spectrum is $\sigma_{red}(T)=\sigma(T_1)\mathbf{e}_1+\sigma(T_2)\mathbf{e}_2$.
Theorem 5.2. Full spectral-cylinder theorem. For $\lambda=\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2$,

$$\lambda\in\rho_{\mathscr{B}\mathbb{C}}(T) \Leftrightarrow \lambda_1\in\rho(T_1) \text{ and } \lambda_2\in\rho(T_2), \tag{5.1}$$

$$\sigma_{\mathscr{B}\mathbb{C}}(T)=[\sigma(T_1)\mathbf{e}_1+\mathbb{C}\mathbf{e}_2]\cup[\mathbb{C}\mathbf{e}_1+\sigma(T_2)\mathbf{e}_2]. \tag{5.2}$$

Proof. The operator $\lambda I-T$ has components λ_jI-T_j. A bicomplex inverse exists exactly when both component inverses exist, and then $R(\lambda,T)=R(\lambda_1,T_1)\mathbf{e}_1+R(\lambda_2,T_2)\mathbf{e}_2$. Taking the complement of $\rho(T_1)\times\rho(T_2)$ gives the union in Equation (5.2). □
Remark 5.3. Unlike a complex Banach-algebra spectrum, $\sigma_{\mathscr{B}\mathbb{C}}(T)$ is generally unbounded: once λ_1 lies in $\sigma(T_1)$, the second coefficient is arbitrary. Statements about compactness of "the spectrum" must therefore specify the reduced spectrum or a component spectrum. This is one of the clearest effects of the null cone.
Proposition 5.4. The full point spectrum is also cylindrical:

$$\sigma_{p,\mathscr{B}\mathbb{C}}(T)=[\sigma_p(T_1)\mathbf{e}_1+\mathbb{C}\mathbf{e}_2]\cup[\mathbb{C}\mathbf{e}_1+\sigma_p(T_2)\mathbf{e}_2]. \tag{5.3}$$

Proof. If λ_1 is an eigenvalue of T_1 with eigenvector u_1, then $u=u_1\mathbf{e}_1$ is a nonzero bicomplex vector and $Tu=\lambda u$ for every λ_2, because the second component of u vanishes. The converse follows by taking a nonzero component of any eigenvector. A "strong" eigenvector with both components nonzero instead requires $\lambda_j\in\sigma_p(T_j)$ for both j. □
Proposition 5.5. Polynomial functional calculus. For a bicomplex polynomial $p=p_1\mathbf{e}_1+p_2\mathbf{e}_2$, $p(T)=p_1(T_1)\mathbf{e}_1+p_2(T_2)\mathbf{e}_2$, and

$$\sigma_{red}(p(T))=p_1(\sigma(T_1))\mathbf{e}_1+p_2(\sigma(T_2))\mathbf{e}_2. \tag{5.4}$$

Proof. Expand powers and coefficients componentwise, then apply the complex spectral mapping theorem to each $p_j(T_j)$. The full spectrum of p(T) is obtained from these two component spectra by the cylinder formula, not by merely applying p to $\sigma_{\mathscr{B}\mathbb{C}}(T)$ without defining how p acts on the cylindrical set. □
Theorem 5.6. Compact normal structure. If T is compact and normal, each nonzero component spectral value is an eigenvalue of finite multiplicity with possible accumulation only at zero. H admits paired orthonormal component bases of eigenvectors, and T has the expansion

$$Tx=\Sigma_r \lambda_{r,1}\langle u_{r,1},x_1\rangle u_{r,1}\mathbf{e}_1 + \Sigma_s \lambda_{s,2}\langle u_{s,2},x_2\rangle u_{s,2}\mathbf{e}_2. \tag{5.5}$$

Proof. Apply the compact normal spectral theorem separately to T_1 and T_2. Each series converges in its Hilbert-space norm. Idempotent recombination gives convergence in H. The conclusion concerns component or reduced spectral values; the full cylindrical spectrum remains unbounded. □

6. Developments from 2020–2025

Sharma and Kunga (2020) developed semigroups of bicomplex-linear operators. The generator, growth estimates, strong continuity, and semigroup law are all componentwise for a family $S(t)=S_1(t)e_1+S_2(t)e_2$. This results in a rigorous evolution theory, while null-cone growth bounds can describe unequal channels. Koumantos (2023) conducted an investigation into the mean ergodic theorem in bicomplex Banach modules, underscoring the fact that the convergence of Cesàro averages is equivalent to the convergence of paired components.

Elgourari, Ghanmi, and Souid El Ainin (2021) conducted a comprehensive examination of bicomplex frame theory. Frame inequalities are hyperbolic-order inequalities that can be reduced to two complex frame inequalities. The bicomplex content is beneficial in the context of component degeneracy tracking and reconstruction with idempotent coefficients. The configurations of function-space operators were also expanded: Reséndis and Tovar (2020) investigated bicomplex Bergman and Bloch spaces, while Alpay, Lewkowicz, and Vajiac (2023) achieved bicomplex Wiener, spectral-factorization, interpolation, and Herglotz results.

Johnston and Wahl (2023) conducted an analysis of Jordan forms, compact operators, invariant-subspace lattices, and bicomplex matrices. Their work explicitly demonstrates that a bicomplex matrix is a paired complex matrix, and invariant submodules can represent idempotent combinations. Alpay, De Martino, Diki, and Vajiac (2024) introduced a bicomplex tensor product and a bicomplex Choi theorem, which offer a pathway to positivity and complete-positivity questions that are pertinent to quantum information. This direction is less reducible at the interpretation level due to the fact that the positivity cone and tensor product selected determine which paired maps represent permissible physical transformations.

Alpay, Diki, and Vajiac (2023) conducted a review of bicomplex-valued neural networks. Linear layers are bicomplex operators; component decomposition results in two complex subnetworks unless they are coupled by parameter constraints or activation. Consequently, the operator theory provides precise algebraic diagnostics, but it does not establish computational or statistical superiority.

Proposition 6.1. A strongly continuous bicomplex semigroup $S(t)$ is equivalent to strongly continuous component semigroups $S_j(t)$, and its generator satisfies $A=A_1\mathbf{e}_1+A_2\mathbf{e}_2$ with $D(A)=D(A_1)\mathbf{e}_1\oplus D(A_2)\mathbf{e}_2$.

Proof. The semigroup equation and strong convergence at $t=0$ project componentwise. Difference quotients $[S(t)x-x]/t$ decompose into the two complex difference quotients, so their common domain and limits have the stated form. □

Proposition 6.2. A bicomplex frame {f_n} for H with bounds $A=A_1\mathbf{e}_1+A_2\mathbf{e}_2$ and $B=B_1\mathbf{e}_1+B_2\mathbf{e}_2$ exists exactly when {f_{n,1}} and {f_{n,2}} are complex frames with bounds A_j,B_j.

Proof. The frame inequality $A\|x\|_{\mathbb{D}}^2\preccurlyeq_{\mathbb{D}}\Sigma\_n|\langle f\_n,x\rangle|_{\mathbb{D}}^2\preccurlyeq_{\mathbb{D}}B\|x\|_{\mathbb{D}}^2$ is precisely the two complex frame inequalities. Reconstruction follows by the two component frame operators. □

Table 1. Selected recent directions in bicomplex operator theory.

Development	Representative source	Operator-theoretic contribution	Critical interpretation
Semigroups	Sharma & Kunga	Strong continuity and	Largely paired complex

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Development	Representative source	Operator-theoretic contribution	Critical interpretation
	(2020)	generators	semigroup theory
Frames	Elgourari et al. (2021)	Hyperbolic frame bounds and reconstruction	Null-cone degeneracy must be tracked
Mean ergodic theory	Koumantos (2023)	Cesàro convergence on Banach modules	Component criteria are decisive
Matrices and compact operators	Johnston & Wahl (2023)	Jordan forms and invariant subspaces	Module lattices expose idempotent structure
Tensor/Choi theory	Alpay et al. (2024)	Tensor products and positivity of maps	Choice of cone has substantive content

6.3 Numerical Range, Fredholm Theory, and Review Criteria

Definition 6.3. For a bounded operator T on a bicomplex Hilbert space, define the reduced numerical range

$$W_{red}(T)=W(T_1)\mathbf{e}_1+W(T_2)\mathbf{e}_2, \qquad (6.1)$$

where $W(T_j)=\{\langle T_j u,u\rangle : \|u\|=1\}$. A single bicomplex unit vector requires both component vectors to be unit, so Wred is the natural paired set under that normalization.

Proposition 6.4. The reduced spectrum of T is contained in the componentwise closure of Wred(T), and the numerical radius satisfies

$$r\_red(T) \preceq_{\mathbb{D}} w\_red(T) \preceq_{\mathbb{D}} \|T\|_{\mathbb{D}} \preceq_{\mathbb{D}} 2w\_red(T), \qquad (6.2)$$

Proof. For each component, the complex spectrum lies in the closure of the numerical range and $r(T_j)\le w(T_j)\le\|T_j\|\le 2w(T_j)$. Multiplying the two chains by $\mathbf{e}_1$ and $\mathbf{e}_2$ and adding gives the hyperbolic chain. The conclusion concerns paired component sets; the full cylindrical spectrum cannot be contained in a bounded numerical range. □

Definition 6.5. A bounded bicomplex operator T is Fredholm when ran T is closed and both ker T and X/ran T have finite component dimensions. Its index vector is

$$\mathrm{ind}_{\mathbb{D}}(T)=\mathrm{ind}(T_1)\mathbf{e}_1+\mathrm{ind}(T_2)\mathbf{e}_2. \qquad (6.3)$$

Theorem 6.6. Fredholm characterization. T is Fredholm if and only if T_1 and T_2 are complex Fredholm operators. Compact perturbations preserve Fredholmness and the index vector.

Proof. Kernel, range, quotient, and closedness decompose componentwise. Hence finite nullity, finite defect, and closed range are exactly the two complex Fredholm conditions. If K is compact, K_j is compact; Atkinson's theorem gives Fredholmness of T_j+K_j and $\mathrm{ind}(T_j+K_j)=\mathrm{ind}(T_j)$. Recombination proves the result. □

Definition 6.7. The reduced essential spectrum is $\sigma ess,red(T)=\sigma ess(T_1)\mathbf{e}_1+\sigma ess(T_2)\mathbf{e}_2$. The full essential noninvertibility set is the union of cylinders generated by the two component essential spectra.

Corollary 6.8. For compact K, $\sigma ess,red(T+K)=\sigma ess,red(T)$. If T is self-adjoint, isolated reduced spectral pairs away from the component essential spectra arise from finite-multiplicity component eigenvalues.

Proof. The component essential spectra are invariant under compact perturbations. The self-adjoint statement is the paired form of the complex Weyl theory. As elsewhere, a reduced pair should not be confused with the full cylinders created by failure in one component. □

Proposition 6.9. Resolvent expansion. If $\lambda\in\rho\mathscr{B}\mathbb{C}(T)$ and $\|(\mu-\lambda)R(\lambda,T)\|\mathbb{D}$ has both coefficients below one, then

$$R(\mu,T)=R(\lambda,T)\Sigma\_\{n=0\}\infty[(\lambda-\mu)R(\lambda,T)]^n. \qquad (6.4)$$

Proof. Factor $\mu I-T=(\lambda I-T)[I+(\mu-\lambda)R(\lambda,T)]$ and apply the Neumann series in both complex components. The condition is two simultaneous inequalities, not a comparison in a total order. □

A useful criterion for evaluating a "bicomplex operator theorem" is therefore to ask four questions. Does the statement specify the norm and positive cone? Does it distinguish full and reduced spectra? Does it remain meaningful for pure idempotent vectors? Does it add a constraint that couples the two complex components? If all content is exactly componentwise, the result may still be valuable as a translation or an application framework, but its mathematical proof should say so. If positivity, tensoring, or a nonlinear condition couples the channels, then the bicomplex formulation may introduce genuinely additional structure.

Recent papers can be read through this lens. Semigroup and ergodic theorems obtain immediate component criteria but also provide a coherent module language for bicomplex dynamics. Frame and reproducing-kernel work uses hyperbolic lower bounds whose vanishing identifies a failed channel. Matrix and invariant-subspace studies expose submodules of unequal component rank. Tensor and Choi results are potentially more structural because complete positivity depends on the adopted tensor product and cone.

Operator inequalities also require careful scope. An inequality $A\preccurlyeq B$ for self-adjoint operators means two complex Loewner inequalities. Norm inequalities in bicomplex Lebesgue or sequence settings can be transported to multiplication and integral operators, but only after verifying domains and component boundedness. A literature review should not present every paired theorem as a new spectral phenomenon; its contribution may instead lie in identifying the correct module, topology, or application in which the paired theorem operates.

6.10 Reproducing-Kernel Multipliers

Proposition 6.10. Let $H(K)=H(K_1)\mathbf{e}_1\oplus H(K_2)\mathbf{e}_2$ be a bicomplex reproducing-kernel Hilbert space. A bicomplex function $\varphi=\varphi_1\mathbf{e}_1+\varphi_2\mathbf{e}_2$ is a bounded multiplier if and only if each φ_j is a bounded multiplier on $H(K_j)$, and

$$M\_\varphi=M\_\{\varphi_1\}\mathbf{e}_1+M\_\{\varphi_2\}\mathbf{e}_2, \qquad (6.5)$$

$$\|M\_\varphi\|\mathbb{D}=\|M\_\{\varphi_1\}\|\mathbf{e}_1+\|M\_\{\varphi_2\}\|\mathbf{e}_2.$$

Proof. Pointwise multiplication decomposes componentwise. If the component multipliers are bounded, their pair is bounded. Conversely, restrict M_φ to pure idempotent functions to obtain the component multiplier bounds. The operator norm formula follows from the least component constants. □

Corollary 6.11. The adjoint acts on kernel functions by

$$M\_\varphi^* K(\cdot,w)=K(\cdot,w)\varphi(w)\hat{}\{\dagger 3\}. \qquad (6.6)$$

Proof. For every f, ⟨f,M_φ*K_w⟩=⟨M_φf,K_w⟩=(φf)(w)=φ(w)f(w). With the stated conjugate-linearity convention this equals ⟨f,K_wφ(w)^{†3}⟩. Component Riesz uniqueness proves the formula. □

Operator analysis is connected to recent bicomplex Bergman, Bloch, Wiener, and interpolation spaces through multiplier theory. However, the norm of a multiplier remains a pair of complex multiplier norms. As opposed to inferring coupling solely from bicomplex notation, a coupled application must impose supplementary relations between φ_1 and φ_2, such as symmetry, shared parameters, or a positivity condition. This observation is also applicable to bicomplex neural layers: a linear multiplier layer is equivalent to two complex multiplier layers unless the architecture imposes restrictions.

7. Interpretive Analysis and Applications

In order to represent observables and evolution, bicomplex quantum mechanics employs a bicomplex Hilbert module and self-adjoint or unitary operators (Rochon & Tremblay, 2004, 2006). Componentwise spectral theory enables calculations; however, physical interpretation must determine the admissibility of pure idempotent states, whose norm is a zero divisor, and the scalarization of probabilities. Algebra alone is insufficient to address that inquiry.

Evaluation operators, interpolation, and analytic function theory are supported by bicomplex Hilbert spaces with a reproducing kernel. Kernels are decomposed into positive complex kernels $K_1e_1+K_2e_2$, and bounded evaluation and representer formulations follow componentwise. Frame bounds characterise stable reconstruction separately in each channel, while this can encode paired complex channels in signal processing.

A framework for bicomplex linear mappings between matrix modules is provided by the Tensor and Choi results. The primary mathematical concern is positivity: component positivity is entirely natural, but complete positivity is contingent upon the tensor amplification that is selected. Operator norms can constrain layerwise Lipschitz behaviour in neural computation; however, a fully decomposable network may be equivalent to two intricate models. Restrictions or nonlinearities that interact across the idempotent representation are necessary for genuine coupling.

8. Open Problems and Limitations

Terminology remains fragmented in the literature. The complement-of-invertibles definition generates cylinders, while some authors employ the term "spectrum" to refer to a paired or reduced set. The convention must be specified in any theorem regarding functional calculus, spectral radius, or compactness. Similarly, null-cone information can be concealed by a number of inequivalent real scalarizations of a hyperbolic norm.

The following are open problems: a satisfactory bicomplex C*-algebra framework, unbounded self-adjoint operators and domain theory, perturbation and pseudospectral analysis, Fredholm and essential spectra, dilation theory, noncommutative bicomplex operator spaces, and tensor products compatible with complete positivity. Mathematical axioms for states, probabilities, and measurement must be connected to physically testable constraints in order to be applicable in quantum applications. Generalisation and optimisation analyses must transcend algebraic decomposition in order to be applicable to learning applications.

9. In summary

A stable componentwise core and expanding applications are the hallmarks of contemporary bicomplex operator theory. Two complex operators characterise boundedness, compactness, adjoints, normality, unitarity, spectral measures, semigroups, and frames. Nevertheless, zero divisors alter invertibility and render the full spectrum cylindrical and typically unbounded. Consequently, it is necessary to differentiate between reduced and full spectra. The field has been expanded to include ergodic theory, analytic function spaces, matrix canonical forms, tensor products, Choi positivity, and neural models through the work of 2020–2025. The subsequent advancements will be contingent upon the identification of structures—including coupling, domains, tensorial constraints, and positivity—that are not fully accounted for by a formal pair of complex theories.

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