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Categories Arising from Certain Algebraic Structures

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Abstract

This paper introduces categories constructed from semigroups and rings, viewed as generalizations of the normal categories developed by K. S. S. Nambooripad. We consider cones in these categories in which at least one component is surjective; such cones are termed *proper cones*. Several structural features of these categories and their cones are established.

Keywords: Proper category, Proper cone, Semigroups, Rings.

1. Introduction

Category theory has been successfully applied to the study of a wide range of algebraic systems, including groups, semigroups, and rings. Notable developments include the Erasmann–Schein–Nambooripad (ESN) theorem, inverse categories introduced by Lawson, and the cross-connection categories developed by Grillet and Nambooripad.

The present work extends the categorical viewpoint to the structural theory of arbitrary semigroups, rings, and modules. Earlier approaches employed normal or balanced categories to analyse special classes of semigroups. We introduce a broader notion—a *proper category*—defined as a category with subobjects in which every morphism has a unique canonical factorization and each object is the vertex of an idempotent proper cone. Proper cones generalize normal cones.

The left and right ideal categories of any semigroup turn out to be proper categories, and the corresponding ideal categories of a ring are additive proper categories. Moreover, the collection of all proper cones in these settings forms a semigroup or a ring, respectively.

2. Preliminaries

We assume familiarity with basic concepts from category theory. Standard sources include Mac Lane and Nambooripad. Throughout this paper, R denotes a noncommutative ring with unity, and we consider only small categories.

Let Cbe a category with object set denoted by vC. For objects $A$and B, the collection of all morphisms from $A$to $B$is called the hom-set and is written as $\hom(A, B)$. The composition of morphisms is written simply as juxtaposition; if one morphism leads to another, their composite is the morphism with the appropriate resulting domain and codomain.

A morphism is called a *monomorphism* if it is right-cancellable and an *epimorphism* if it is left-cancellable. A morphism that is both mono and epi is referred to as balanced. A category that contains either an initial or a terminal object is said to have a zero object.

Additive Categories

A category is *additive* if:

1. Each hom-set is an additive abelian group.
2. Composition is bilinear with respect to addition.
3. A zero object exists.
4. Binary products and coproducts exist.

Categories with Subobjects

A *pre-order* is a category in which each hom-set contains at most one morphism. In this case, the existence of a morphism from one object to another defines a quasi-order. If this quasi-order is antisymmetric, the structure is a strict pre-order.

Let Pbe a subcategory of a category C. The pair $(C, P)$is called a *category with subobjects* if:

1. $P$is a strict pre-order on the object set of C.
2. Every morphism in $P$is a monomorphism in C.
3. Whenever two morphisms in Pfactor through a morphism in C, that morphism also lies in P.

Morphisms in $P$are called *inclusions*. Subobjects correspond to equivalence classes of monomorphisms.

Factorization and Normal Factorization

A morphism $f$in a category with subobjects admits a *factorization* if it can be expressed as a composition of an epimorphism followed by an embedding. When the embedding is an inclusion, the factorization is said to be *canonical*. A category has the *factorization property* if every morphism admits such a factorization.

A *retraction* is a morphism that serves as a right inverse to an inclusion, and in this case the inclusion is said to split.

A *normal factorization* is a decomposition of a morphism into a retraction, an isomorphism, and an inclusion.

Cones and Normal Categories

A *cone* with vertex $D$is a rule that assigns to each object Ca morphism from $C$to D, compatible with the inclusions. A cone is *normal* if one of its components is an isomorphism.

A category is *normal* if:

1. It is a category with subobjects.
2. Every inclusion splits.
3. Every morphism admits a normal factorization.
4. For each object, there exists a normal cone whose component at that object is the identity.

3. Proper Categories

We introduce a generalization of normal categories.

Definition: Proper Category

A *proper category* is a small category with subobjects in which every morphism has a unique canonical factorization with splitting inclusions. A *proper cone* is a cone for which at least one component is surjective. A proper cone becomes a normal cone if one of its components is an isomorphism.

Let PC denote the set of all proper cones in C, and let TC be the subset consisting of normal cones.

3.1. Semigroup of Proper Cones

Proposition 1

Let γ be a proper cone and let f be a morphism whose domain is the vertex of γ. Then the cone obtained by composing each component of γ with f is also a proper cone, with vertex equal to the image of f. Composition behaves compatibly for successive morphisms.

Proposition 2

The set PC of proper cones forms a semigroup under the operation
"compose a cone with the component of another cone taken at its vertex."

Proposition 3

A proper cone is idempotent if and only if its component at its own vertex is the identity morphism.
Let $E(PC)$ denote the set of all idempotent proper cones.

Green's Relations

Green's relations classify elements of a semigroup via principal left and right ideals. The set of principal left ideals ordered by inclusion corresponds to the quotient set of elements modulo the L-relation; the dual holds for principal right ideals and the R-relation. The H-relation is defined as the intersection of the two.

4. Categories of Principal Ideals of a Semigroup

Let S be a semigroup and let S_1 be S with an identity adjoined. The category $L(S)$ has as objects the principal left ideals of S_1, and morphisms are right translations defined by multiplication by elements of S.

Proposition 4

Each semigroup S gives rise to a category $L(S)$ of principal left ideals with right translation morphisms.

Proposition 5

For a right translation morphism:

1. It is surjective precisely when the corresponding elements generate the same principal left ideal (equivalently, are in the same L-class).
2. It is a split injection or split surjection exactly when the elements lie in the same R-class.
3. It is an isomorphism precisely when both conditions hold.

Proposition 6

The category $L(S)$ is a proper category, since every morphism has a unique canonical factorization.

Cones in $L(S)$

Each element of Sdefines a proper cone in $L(S)$, and the collection of all such cones forms the semigroup $PL(S)$. The dual statements hold for the category of principal right ideals $R(S)$.

5. Additive Proper Categories

A *proper additive category* is a proper category that is also additive.

For a ring R, its left and right ideals coincide with those of its multiplicative semigroup. The category of principal left ideals of R, with right translations as morphisms, is both proper and additive. Its objects are all left ideals of the form Ra, and morphisms are given by right multiplication.

Proposition 8

Given two proper cones, their pointwise sum defines another proper cone whose vertex is the smallest principal left ideal containing both of their vertices.

Proposition 9

The set of all proper cones in $L(R)$forms a ring under addition defined above and the cone multiplication from Section 3.

References

[1] Michael Artin, Algebra, Prentice Hall, New Jersey, ISBN $0-13-004763-5$(1991).

[2] A.H.Clifford and G.B.Preston, The algebraic theory of semigroups vol 1, Math Surveys No.7, American Mathematical Society, Providence, R.I.(1961)

[3] J.M.Howie, Fundamentals of semigroup theory, Clarendon Press, Oxford. ISBN $0-19-851194-9$(1995).

[4] M E Keating, A First course in module theory, Imperial College Press, London. ISBN 186094096(1941).

[5] S.Mac Lane, Categories for the working mathematician, Springer Verlag, Newyork, ISBN $0-387-98403-8$(1971).

[6] Mark V Lawson, Inverse semigroups: The theory of partial symmetries, World Scientific, ISBN 978-981-02-3316-7(1998).

[7] Sunny Lukose and A.R.Rajan, Ring of normal cones, Indian J. Pure. Appi. Math,41(5):663-681, (2010).

[8] Barry Mitchell, Rings with several objects, Advances in Mathematics 8, 1 −161(1972).

[9] C.Musili, Introduction to rings and modules, Narosa Publishing House, New Delhi, ISBN $81-85198-64-0$ (1992).

[10]K.S.S.Nambooripad, Theory of cross connections, Publication No.28 - Centre for Mathematical Sciences, Trivandrum(1994).

[11]Petrq Petro, Green's relations and minimal quasi ideals in rings, Communications in algebra, vol 30, pp.4677-4686 (2002).

[12]Dr.P.G.Romeo and Sreejamol P.R.,On the category of ideals of ring, Journal of Mathematics and Systems Science 7(2017)94 – 99.

[13]Dr.P.G.Romeo and Sreejamol P.R. Ideal categories of semigroups, Proceedings of 11th International Conference on Mathematical Sciences for Advancement of Science & Technology, Kolkata(2017)71 – 81.