

Linear Programming Approaches to Transportation Problems: An Application of Gomory's Cutting Plane Method

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Abstract:

This study investigates the optimization of a transportation problem involving three supply points and four demand centers by transforming it into a linear programming problem. The primary focus is to apply Gomory's Cutting Plane Method to obtain an optimal integer solution and compare it with the solution derived using the Vogel Approximation Method (VAM). The research begins by formulating the transportation problem in a linear programming framework, followed by the application of Gomory's method, which is particularly useful for integer programming problems where fractional solutions are not feasible. The Vogel Approximation Method, a widely used heuristic for generating good initial feasible solutions in transportation problems, serves as the comparative benchmark. Results indicate that while VAM provides quick and reasonably effective solutions, Gomory's Cutting Plane Method achieves an exact optimal integer solution. The comparative analysis highlights the strengths and limitations of each method in terms of solution quality and computational effort. The study provides valuable insights for decision-makers in logistics and supply chain management seeking efficient and accurate optimization techniques for distribution problems.

Keywords: Linear Programming Problem, Transportation Problem, Gomory's Cutting Plane Method, Vogel Approximation Method

INTRODUCTION:

Achieving balance is frequently essential for success in the transportation industry; however, the reality rarely conforms to simple calculations. Imagine a transportation puzzle with missing pieces, where the intersection of supply and demand is influenced by a myriad of complex factors. This situation exemplifies the unbalanced transportation problem, a mathematically sophisticated real-world challenge that requires establishing order within chaos. The transportation problem involves delivering goods from their origin to their destination at the lowest possible total cost while complying with supply and demand constraints [1–4]. Transportation expenses are a major concern for numerous companies, which allocate millions of dollars to the movement of goods from supply to demand [2]. As a result, effective planning in the transportation of goods and services is vital [5]. Investigating optimal costs for the transportation of items has emerged as a significant field of research [6]. The transportation problem is a subset of both linear programming and nonlinear programming issues [3,7], as well as mixed integer programming challenges [8]. Transportation problems are primarily divided into two categories: balanced transportation problems and unbalanced transportation problems. A transportation problem is classified as balanced when demand

matches supply, whereas it is deemed unbalanced in other cases [9]. Attaining the optimal solution for the transportation problem consists of two stages. The Initial Basic Feasible Solution (IBFS) serves as the initial step in pinpointing a starting solution for the transportation problem, representing a basic solution aimed at cost minimization [3,9,10]. The following stage concentrates on identifying the optimal solution using the IBFS, utilizing techniques such as the stepping stone or MODI method. The IBFS is essential for achieving a minimal or optimal total cost in transportation issues. Therefore, it is crucial to select a reliable method for determining the IBFS [2,11]. It should be as close as feasible to the optimal value in conjunction with the optimal solution [1–3,10,12].

Based on the examination of the historical context of Transportation Problems (TP), foundational figures such as Hitchcock [15], Koopmans [16], Dantzig [17], and Charnes et al. [18] have significantly contributed to the evolution of TP. This review is limited to prior research on Initial Basic Feasible Solution (IBFS) methods, as the primary aim of this study is to introduce an effective IBFS method for TP. A variety of researchers, including Kumar et al. [19], Hosseini [1], Ullah et al. [20], and Babu et al. [21], have proposed various IBFS techniques. Nonetheless, the identification of the IBFS is essential for attaining the optimal solution. Traditional approaches recommended for determining the IBFS encompass the Northwest Corner Method (NWCN) [22], the Least Cost Method (LCM) [23], and Vogel's Approximation Method (VAM) [24]. The NWCN utilizes an intuitive approach to distribute goods and services from sources to destinations. In the process of identifying the IBFS, the NWCN emphasizes the northwest corner (top-left) of the cost matrix, continuing this allocation until it reaches the final cell located at the bottom-right corner [9]. This method assigns resources to the current cell without considering the associated costs, which frequently results in a high value of IBFS. As an improvement to the NWCN, the LCM identifies the cell with the lowest cost in the cost matrix and allocates as many resources as feasible while adhering to supply and demand constraints [9]. However, this method faces challenges when there is a tie at the minimum cost and lacks a systematic approach to achieve the optimal solution. VAM is acknowledged as one of the most efficient methods for determining IBFS. This method is based on the calculation of penalties to determine the IBFS [9]. VAM follows a systematic framework and requires minimal computational time. However, it is crucial to recognize that VAM yields an approximate solution. Other methods suggested for identifying the IBFS of a TP include the Row Minimum Method (RMM) [25] and the Maximum Difference Method (MDM) [24], the Column Minimum Method (CMM) and the Extreme Difference Method (EDM) [26], and the Allocation Table Method (ATM) [23].

The continuous allocation table technique (IAM) [6], the maximum difference extreme difference technique (MDEDM) [27], the total differences technique (TDM) [1], the demand-based allocation technique (DBAM) [28], among others, are examined. Most of the previously mentioned techniques undergo thorough derivations before the calculation of the Initial Basic Feasible Solution (IBFS). In some cases, certain methods produce an IBFS that significantly diverges from the optimal solution. This paper presents a robust strategy referred to as the maximum range method (MRM), which generates an IBFS with minimal accuracy error. The proposed strategy was assessed against various existing methods to evaluate its dependability. The MRM seems to be closely related to the extreme difference method (EDM) [26].

Preliminaries

In this section some basic definitions are reviewed

Formulation of Transportation problem in Linear Programming Problem

Given m origins and n destinations, the transportation problem can be formulated as the following linear programming problem model:

Minimize: $\sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$

Subject to constraint:

$$\sum_{j=1}^n x_{ij} \leq a_i \quad i=1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} \geq b_j \quad j=1, 2, \dots, n$$

$$x_{ij} \geq 0 \quad \text{for all } i \text{ and } j$$

Where x_{ij} is the amount of units of shipped from origin I to destination j and c_{ij} is the cost of shipping one unit from origin i to destination j. The amount of supply at origin is a_i and the amount of destination j is b_j . The objective is to determine the unknown x_{ij} that will the total transportation cost while satisfying all the supply and demand constraints.

Balanced Transportation Problem

The model above is said to be a balanced transportation problem whenever

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

Otherwise, it is an unbalanced transportation problem if

$$\sum_{i=1}^m a_i \neq \sum_{j=1}^n b_j$$

Note that for the existence of a feasible solution to the transportation problem, it is a necessary condition that the total supply should be equal to the total demand

Gomory’s Plane Method:

Step 1 (a): Change Transportation in Linear Programming Problem.

(b): If any constraint contains non-integer coefficient, then convert it into integer.

(c): Solve the given problem using Simplex method, ignore the integer.

Step 2 (a): Examine the optimal solution. if all the basic variables have integer values, then terminate the process.

(b): Otherwise, construct a Gomory's fractional cut from the row, which has the largest fractional part, and add it to the original set of constraints.

Gomory’s constraints $f_r = s_g - \sum f_r x$

Step 3 (a): Now add this constraint to optimal simplex table.

(b): Find a new optimal solution using dual simplex method. and then go to step-2.

Numerical Problems: A pharmaceutical manufacturing company located in the mandideep industrial area needs to distribute its products from three production plants to four regional distribution centers. The goal is to minimize transportation costs while meeting the demand at each destination.

Table 1: Supply Capacities of Plant

Production Plant	Capacity
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Plant A	5200 units
Plant B	6900 units
Plant C	6300 units

Table 2: Requirement of Distribution Center

Distribution Center	Requirement
Indore	4500 units
Bhopal	3700 units
Jabalpur	5700 units
Gwalior	4500 units

Table3: Transportation cost of product (per unit)

	Indore	Bhopal	Jabalpur	Gwalior
Plant A	4	6	9	5
Plant B	5	4	7	6
Plant C	6	3	4	5

This analyze the transportation problem for a pharmaceutical industry which is produce medicines. Here, obtained the following transportation model to determine an optimal solution to minimize the transportation cost

Table 4: Transportation Model

	Indore	Bhopal	Jabalpur	Gwalior	Supply
Plant A	4	6	9	5	5200
Plant B	5	4	7	6	6900
Plant C	6	3	4	5	6300
Demand	4500	3700	5700	4500	

Here, we are going to apply Vogel Approximation and MODI Method in transportation model to obtained feasible and optimal solution respectively, then we get following decision variable

$$x_{11} = 4500, x_{14} = 700, x_{22} = 3100, x_{24} = 3800, x_{32} = 600, x_{33} = 5700$$

With the help of above decision value we obtained the optimal solution is as follows

$$Min Z = 4 * 4500 + 5 * 700 + 4 * 3100 + 6 * 3800 + 3 * 600 + 4 * 5700 = Rs 81300$$

Linear Programming Formulation of Transportation Problem

Now, According to Step 1(a) we are going to change our transportation problem in linear programming formulation with the help of Table 4, then we obtained

Objective

$$Min Z = 4x_{11} + 6x_{12} + 9x_{13} + 5x_{14} + 5x_{21} + 4x_{22} + 7x_{23} + 6x_{24} + 6x_{31} + 3x_{32} + 4x_{33} + 5x_{34}$$

Constraints

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 5200$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 6900$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 6300$$

$$x_{11} + x_{21} + x_{31} \geq 4500$$

$$x_{12} + x_{22} + x_{23} \geq 3700$$

$$x_{13} + x_{23} + x_{33} \geq 5700$$

$$x_{14} + x_{24} + x_{34} \geq 4500$$

$$x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34} \geq 0$$

Now according Step 1(c), we are going to introduce slack, surplus and artificial variable to solve our problem, so we have

Objective

$$\begin{aligned} \text{Min } Z = & 4x_{11} + 6x_{12} + 9x_{13} + 5x_{14} + 5x_{21} + 4x_{22} + 7x_{23} + 6x_{24} + 6x_{31} + 3x_{32} + 4x_{33} + 5x_{34} \\ & + 0S_1 + 0S_2 + 0S_3 + 0S_4 + 0S_5 + 0S_6 + 0S_7 + MA_1 + MA_2 + MA_3 + MA_4 \end{aligned}$$

Constraints

$$x_{11} + x_{12} + x_{13} + x_{14} + S_1 \leq 5200$$

$$x_{21} + x_{22} + x_{23} + x_{24} + S_2 \leq 6900$$

$$x_{31} + x_{32} + x_{33} + x_{34} + S_3 \leq 6300$$

$$x_{11} + x_{21} + x_{31} - S_4 \geq 4500$$

$$x_{12} + x_{22} + x_{23} - S_5 \geq 3700$$

$$x_{13} + x_{23} + x_{33} - S_6 \geq 5700$$

$$x_{14} + x_{24} + x_{34} - S_7 \geq 4500$$

$$x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}, S_1, S_2, S_3, S_4, S_5, S_6, S_7, A_1, A_2, A_3, A_4, \geq 0$$

Now after apply all the steps of Gomory's Cutting Plan Method we obtained the following table to obtained the optimal solution after the 6 iterations

		C_j	4	6	9	5	5	4	7	6	3	4	5	0	0	0	0	0	0	0	M	M	M	M		
B	C_B	X_B	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	A_4	
x_{14}	5	4500	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	-1	0	0	0	1
S_2	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	-1	-1	-1	-1
x_{33}	4	5700	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	-1	0	0	0	1	0
x_{11}	4	700	1	1	1	0	0	0	0	-1	0	0	0	-1	1	0	0	0	0	0	0	1	0	0	0	-1
x_{32}	3	600	0	0	-1	0	0	0	-1	0	1	1	0	1	0	0	1	0	0	1	0	0	0	0	-1	0
x_{22}	4	3100	0	1	1	0	0	1	1	0	-1	0	0	-1	0	0	-1	0	-1	-1	0	0	1	1	0	0
x_{21}	5	3800	0	-1	-1	0	1	0	0	1	1	0	0	1	-1	0	0	-1	0	0	-1	1	0	0	0	1
		Z_j	4	3	4	5	5	4	5	6	4	3	4	5	-1	0	-1	-5	-4	-5	-6	5	4	5	6	
		$Z_j - C_j$	0	-3	-5	0	0	0	-2	0	-2	0	0	0	-1	0	-1	-5	-4	-5	-6	-M+5	-M+4	-M+5	-M+6	

Table 5: Final Iteration Table of Gomory's Cutting Plan Method

Since all $Z_j - C_j \leq 0$

Hence optimal solution is arrived with value of variable as

$$x_{11} = 700, x_{14} = 4500, x_{21} = 3800, x_{22} = 3100, x_{32} = 600, x_{33} = 5700$$

Optimal Solution

$$\text{Min } Z = 4 * 700 + 5 * 4500 + 5 * 3800 + 4 * 3100 + 3 * 600 + 4 * 5700 = \text{Rs } 81300$$

i.e. after the implementation of both methods, we get the following comparative method

Table 2: Requirement of Distribution Center

Method	Decision Variable	Optimal Solution
Vogel Method Approximation	$x_{11} = 4500, x_{14} = 700, x_{22} = 3100,$ $x_{24} = 3800, x_{32} = 600, x_{33} = 5700$	Rs 81300
Gomory's Method Cutting Plan	$x_{11} = 700, x_{14} = 4500, x_{21} = 3800,$ $x_{22} = 3100, x_{32} = 600, x_{33} = 5700$	Rs 81300

Conclusion:

The research successfully formulated a transportation problem involving three supply points and four demand centers into a linear programming model. The application of Gomory’s Cutting Plane Method yielded an optimal integer solution, demonstrating its capability to handle integer constraints effectively in transportation problems. In comparison, the Vogel Approximation Method produced a feasible and practical solution more rapidly but without guaranteeing optimality. The comparative evaluation underscores that while VAM is suitable for obtaining quick and near-optimal solutions with less computational complexity, Gomory’s method provides precise optimization, essential for applications where exact solutions are critical. The study emphasizes the importance of method selection based on problem context, computational resources, and the need for solution accuracy. These findings are significant for enhancing decision-making in logistics and distribution systems, suggesting that Gomory’s Cutting Plane Method is preferable when exact optimization is required despite its higher computational effort.

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