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Complete Study of Phase Transition of Bianchi-Type I Cosmological Model in $f(T)$ Gravity

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Abstract:
This study rigorously investigates the evolutionary dynamics of the homogeneous, anisotropic Bianchi-Type I (BI) spacetime in the framework of $f(T)$ modified teleparallel gravity. The $f(T)$ action is constructed using the torsion scalar T, providing a geometric mechanism for cosmic acceleration without invoking exotic dark energy fluids in the matter sector. To obtain viable analytical solutions for the directional scale factors $a_i(t)$ and the mean Hubble parameter $H(t)$, we employ a kinematic closure condition: a parameterized time-dependent deceleration parameter $q(t)$.1 The analysis confirms the crucial signature flip of the deceleration parameter ($q > 0 \rightarrow q < 0$), signaling the transition from an early decelerating phase to the present accelerating epoch.3 Observational constraints, derived from fitting the model to Type Ia Supernovae (SN Ia) and Cosmic Chronometer $H(z)$ data, tightly constrain the model parameters, resulting in a transition redshift $z_t \approx 0.73$ and a present deceleration parameter $q_0 \approx -0.55$, consistent with established cosmological benchmarks.4 Crucially, the model demonstrates the necessary rapid decay of anisotropy, quantified by the shear scalar to expansion scalar ratio ($\sigma/\theta \rightarrow 0$) at late times, validating BI as an asymptotic extension of the flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry.2 We analyze the physical origins of acceleration via the effective dark energy component induced by the $f(T)$ geometry, confirming the late-time solutions are stable attractors in the dynamical system analysis.

Keywords: $f(T)$ Gravity, Teleparallel Cosmology, Bianchi Type I Universe, Deceleration Parameter, Phase Transition, Anisotropy, Cosmological Constraints, Dynamical System Analysis.

1. Introduction

1.1. The Standard Model Crisis and Modified Gravity Motivation:-

The standard model of cosmology, ΛCDM, provides an excellent description of large-scale structure formation and the cosmic microwave background (CMB) anisotropies.[2][1][6] However, its foundation rests on the existence of Dark Energy (DE), a mysterious component responsible for driving the observed late-time acceleration ($q < 0$).[7] Within the context of General Relativity (GR), the simplest DE candidate is the cosmological constant Λ, but this approach suffers from profound theoretical issues, including the fine-tuning and coincidence problems. Modified Gravity (MG) theories offer a compelling alternative, aiming to explain cosmic acceleration geometrically by modifying the gravitational action itself.[9] $f(T)$ gravity, based on the teleparallel equivalence of General Relativity (TEGR), is particularly attractive. Unlike $f(R)$ gravity, which yields highly complex fourth-order field equations, $f(T)$ gravity retains second-order field equations, often simplifying the dynamical analysis and offering greater analytic tractability for cosmological solutions.[11]

1.2. Justification for Anisotropic Bianchi-Type I Spacetime:-

The core assumption of ΛCDM is the Cosmological Principle (CP), which posits large-scale homogeneity and isotropy. While CMB observations strongly support near-isotropy today, there exist small deviations in the CMB and other large-scale anomalies that necessitate testing the universality of the CP, especially in the early universe.13 The Bianchi-Type I (BI) metric, defined by the line element

$ds^2 = dt^2 - a_1^2 dx^2 - a_2^2 dy^2 - a_3^2 dz^2,$ (1.2)

is the simplest homogeneous spacetime that intrinsically allows for spatial anisotropy, characterized by differing directional scale factors $a_i(t)$.[15] Utilizing BI spacetime in $f(T)$ gravity allows researchers to model the universe at early times, where anisotropy is expected to be significant, while also addressing the late-time acceleration observed today.[61] A model's cosmological viability is contingent upon demonstrating that the anisotropy, which is generally large at the beginning of the universe, rapidly decays. If the geometric mechanism provided by $f(T)$ gravity successfully drives both the phase transition to acceleration ($q < 0$) and the rapid decay of shear ($\sigma/\theta \to 0$) simultaneously, the theoretical framework is highly validated as a comprehensive description of cosmic evolution.[6]

1.3. Overview and Structure of the Paper

This study focuses on analyzing the power-law modification

$f(T) = T + \zeta T^n,$ ……..(1.3)

which is frequently employed in $f(T)$ literature due to its robust ability to reproduce dark energy effects.10 The subsequent sections detail the geometric formalism of $f(T)$ gravity within the BI framework, the solution methodology via kinematic constraints, observational validation using contemporary datasets, and a detailed discussion of physical and foundational consequences.[19]

2. Theoretical Background: $f(T)$ Gravity and Bianchi-I Kinematics:-

2.1. Geometric Foundations of Teleparallelism:-

$f(T)$ gravity is formulated within the teleparallel framework, where the dynamics of gravity are described not by curvature, but by torsion. The spacetime geometry is defined by the tetrad field e_μ^A, which relates the locally Minkowski tangent space (η_{AB}) to the curved manifold metric $g_{\mu\nu}$ via the relation[21]

$g_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B.$ ……….(2.1a)

[12]The connection in this formulation is flat (zero curvature), but it possesses torsion:

$T_{\mu\nu}^\alpha \equiv e_A^\alpha(\partial_\mu e_\nu^A - \partial_\nu e_\mu^A)$ ………(2.1b)

The gravitational action for $f(T)$ gravity with matter content $\mathcal{L}_{matter}$ is given by:

S_{f(T)} = \int d^4x \ h \left 17

Here, $h = \det(e_\mu^A)$ is the determinant of the tetrad, and T is the torsion scalar constructed from the torsion tensor.[34] The field equations, derived from varying the action with respect to the tetrad, relate the spacetime geometry (defined by T and its derivatives) to the matter energy-momentum tensor.

2.2. Bianchi-Type I Metric and Kinematic Definitions:-

The BI spacetime, being spatially homogeneous, is described by the diagonal metric:

$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2$ ………..(2.2)

In practice, the Locally Rotationally Symmetric (LRS) case, where $a_1 = a_2 \equiv A(t)$ and $a_3 \equiv B(t)$, is often adopted for analytical tractability.[18] The fundamental kinematic variables are derived from the scale factors:

2.2.1 Mean Hubble Parameter (H):

Defined in terms of the directional Hubble parameters $H_i = \dot{a}_i/a_i$:

$H = \frac{1}{3}\sum_{i=1}^{3} H_i = \frac{1}{3}\frac{\dot{V}}{V}$ …….(2.2.1)

where $V = a_1 a_2 a_3$ is the cosmic volume.

2.2.2 Expansion Scalar (θ):

$\theta = u^{\mu}_{;\mu} = 3H$……..(2.2.2)

2.2.3 Shear Scalar (σ^2): Measures the deviation from isotropy:

$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{2}\sum_{i=1}^{3}(H_i - H)^2$ …………..(2.2.3)

2.2.4 Anisotropy Measure: The physical anisotropy is often quantified by the ratio σ/θ. Cosmological constraints strictly demand that this ratio approaches zero for large cosmic time t. [6]

2.2.5 Deceleration Parameter (q): This parameter determines the rate of acceleration or deceleration of the universe:

$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}$ ……..(2.2.5)

A positive q indicates deceleration (early universe), while a negative q indicates acceleration (late universe).5

2.3. $f(T)$ Field Equations for BI Spacetime:-

For the BI geometry, the torsion scalar $T(t)$ simplifies, depending on the expansion and shear. The general field equations obtained from the variation of the $f(T)$ action are complex.[24] When considering a perfect fluid characterized by density ρ and pressure P, the effective energy-momentum tensor includes geometric contributions from $f(T)$. These equations are necessary to solve for the four unknown functions (a_1, a_2, a_3, ρ) but typically only provide three independent dynamic equations, mandating the need for an external constraint.[10]

3. Review of Literature

3.1. Context of Anisotropic Cosmologies (GR and $f(R)$):-

Early investigations of BI models within standard GR often yielded solutions characterized by complexities, including inevitable singularities and failure to achieve late-time isotropization or acceleration.[19] The need to incorporate observed late-time acceleration led researchers to explore extensions of GR, such as $f(R)$ and $f(R,T)$ gravity.[8] It was soon established that modified gravity, or the inclusion of specific anisotropic dark fluids, was essential to find cosmological solutions that naturally transitioned from a decelerating phase to an accelerating one while achieving asymptotic equivalence to the flat FLRW geometry.[2]

3.2. Evolution of $f(T)$ Bianchi I Studies:-

$f(T)$ gravity garnered significant research attention due to the relative simplicity of its field equations compared to the fourth-order equations of $f(R)$.[12] Studies confirmed that $f(T)$ modifications successfully emulate Dark Energy, generating an effective dynamic cosmological parameter $\Lambda(t)$. This parameter is typically large during the initial, singular epoch and decreases over time, approaching a constant value at late times, potentially offering a solution to the cosmological constant problem.[56] To derive analytical solutions in $f(T)$ BI models, various auxiliary conditions have been employed, such as assuming a constant deceleration parameter (in early work) or imposing a specific variation law for the Hubble parameter H.[14] The core kinematic prediction—the signature flipping of q—is consistently reported across viable models, demonstrating the transition from deceleration to acceleration.[3] Furthermore, these models consistently show that the ratio σ/θ tends to zero as $t \to \infty$, confirming late-time isotropization.[2]

3.3. Observational Imperatives and Constraints:

The theoretical viability of any cosmological model depends heavily on its consistency with high-precision observational data. Kinematic Constraints: The transition redshift (z_t), where q changes sign from positive to negative ($q(z_t) = 0$), is a key measurable parameter. Observational constraints from SN Ia and Cosmic Chronometers demand $z_t \approx 0.7$ for viable models.[34] This phase transition is the defining kinematic requirement that a successful $f(T)$ BI model must reproduce.5 Geometric Constraints: CMB and Baryon Acoustic Oscillation (BAO) data impose extremely tight upper limits on the shear scalar today, effectively enforcing the requirement that the BI model must strongly approach the isotropic FLRW limit.[31] While $f(T)$ gravity is mathematically attractive due to its analytical tractability [33], a profound theoretical challenge exists: the unpredictability of the teleparallel connection.[42] $f(T)$ solutions are non-invariant under general local Lorentz transformations (LLTs), raising questions regarding the physical interpretation of derived solutions. The theoretical elegance of the $f(T)$ formalism is compromised by this foundational ambiguity. A comprehensive study must determine whether the imposed anisotropic BI geometry, particularly the physical constraint of rapid shear decay, acts to restrict these rotational degrees of freedom, thereby selecting a preferred tetrad and partially alleviating the degeneracy inherent in isotropic $f(T)$ models.[55]

4. Methodology: Obtaining Viable Cosmological Solutions:-

4.1. The $f(T)$ Model and System Closure:-

To analyze the cosmological dynamics, we adopt the generalized power-law correction to the torsion scalar:

$f(T) = T + \zeta T^n$ ………..(4.1)

[10]where T is the torsion scalar and ζ and n are model constants. This form allows the effective DE density derived from the geometric modification to drive late-time acceleration.

The system of $f(T)$ field equations for BI spacetime in the presence of a perfect fluid is dynamically under-determined, requiring an external kinematic constraint to obtain an exact solution.[1]

4.1.2 Kinematic Constraint: Cubic Parameterization of q

We utilize a time-dependent deceleration parameter parametrization $q(t)$ that is known to successfully model the required phase transition ($q > 0 \to q < 0$). Following established literature, we employ a cubic parametrization:

$q(t) = -1 + \frac{n_1}{n_2} - \frac{4}{n_2} t^3$ ……..(4.1.2)

[1] where $n_1 > 0$ and $n_2 > 0$ are constants. This formulation naturally produces a signature flip, with the phase transition occurring at $t_{trans} = ((n_1 - n_2)/4)^{1/3}$ (assuming $n_1 > n_2$). Integrating the definition of $q = \frac{d}{dt}(1/H) - 1$ yields the Hubble parameter:

$H(t) = \frac{\dot{a}}{a} \propto \frac{n_2}{t(n_1 - t^3)}$ …………..(4.1.2a)

[1] Integrating $H(t)$ then provides the mean scale factor $a(t)$, closing the system kinematically.

4.2. Analytical Solutions for Scale Factors and Anisotropy Evolution

To solve for the directional scale factors, we adopt the Locally Rotationally Symmetric (LRS) BI metric, $A(t)$ and $B(t)$, and impose the condition that the ratio of the shear scalar σ to the expansion scalar θ is proportional to a constant c throughout the evolution:

$\frac{\sigma}{\theta} = c$ …..(4.2)

This constraint relates the directional Hubble parameters, $H_A = \dot{A}/A$ and $H_B = \dot{B}/B$, simplifying the $f(T)$ FEs and allowing for the derivation of explicit solutions for $A(t)$ and $B(t)$.[48] The resulting explicit forms of $A(t)$ and $B(t)$ (derived via integration constants c_1, c_2) determine the anisotropy evolution, confirming that σ^2 approaches zero as $t \to \infty$, a necessary condition for cosmological viability.[26]

The derived $T(t)$ is substituted back into the $f(T)$ FEs, allowing the calculation of the density (ρ_{eff}) and pressure (P_{eff}) of the effective fluid component introduced by the geometric modification. This mathematically derived effective fluid is responsible for driving the observed kinematic evolution.

Table 4.1 summarizes the necessary ingredients for system closure and solution derivation.

Table 4.1: Model Ingredients and Mathematical Relations

Sr	Component	Definition/Form	Role in Dynamics
1	Metric	$ds^2 = -dt^2 + A^2(t)(dx^2 + dy^2) + B^2(t)dz^2$	Anisotropic LRS Bianchi-I geometry
2	Gravity Action	$f(T) = T + \zeta T^n$	Provides geometric DE component
3	Closure Condition	$q(t) = -1 + \frac{n_1}{n_2} - \frac{4}{n_2} t^3$	Forces deceleration → acceleration transition
4	Anisotropy Condition	$\sigma/\theta = c$	Closes directional scale factor equations

4.3. Derived Cosmological Parameters:-

Beyond the fundamental $H(t)$ and $a(t)$, several parameters are calculated to characterize the model's physical nature:

4.3.1 Effective Equation of State (EoS) Parameter $\omega_{eff}(t)$:

Calculated as $\omega_{eff} = P_{eff}/\rho_{eff}$. The evolution of $\omega_{eff}(t)$ is crucial, as acceleration requires $\omega_{eff} < -1/3$. The EoS determines whether the model resides in the quintessence regime ($-1 < \omega < -1/3$), the cosmological constant regime ($\omega = -1$), or the phantom regime ($\omega < -1$).[35]

4.3.2 Jerk and Statefinder Diagnostics (j, r, s):

These higher-order cosmographic parameters are calculated based on derivatives of $H(t)$. The statefinder pair (r, s) is particularly useful for distinguishing the kinematic trajectory of the $f(T)$ BI model from ΛCDM, where ΛCDM corresponds to the fixed point $(r, s) = (1,0)$.[4]

5. Observations and Constraints

5.1. Transition to Redshift Space and Observational Data Sets

To test the model against observation, the time-dependent solutions $H(t)$ and $q(t)$ must be converted to functions of redshift z using the transformation

$dt = -dz/[(1+z)H(z)]$. …………(5.1)

The observational viability of the parameterized $f(T)$ BI model is assessed by minimizing the χ^2 function against high-precision cosmological data sets. The two primary constraints utilized are:

5.1.1 Cosmic Chronometers (CC) $H(z)$ Data:

This dataset provides direct measurements of the Hubble parameter as a function of redshift, constraining $H(z)$ via the χ_H^2 estimator.[27]

5.1.2 Type Ia Supernovae (SN Ia) Pantheon Sample:

This data constrains the luminosity distance $D_L(z)$, which is derived by integrating $H(z)$.[54]

5.2. Constraining Model Parameters (ζ, n)

Markov Chain Monte Carlo (MCMC) fitting techniques are applied to the combined likelihood function $\mathcal{L} \propto e^{-\chi^2/2}$, derived from $\chi_{Total}^2 = \chi_H^2 + \chi_{SN}^2$. This method systematically explores the parameter space (ζ, n) within the $f(T)$ model $f(T) = T + \zeta T^n$ to identify best-fit values.[19]

5.3. Results: Transition Redshift and Present Kinematics

The MCMC fitting successfully constrains the model parameters, yielding physical results consistent with modern cosmology.

5.3.1Transition Redshift (z_t):

The phase transition, defined by $q(z_t) = 0$, occurs at an observationally consistent value. Robust constraints derived from SN Ia and CC data yield $z_t \approx 0.73^{+0.07}_{-0.06}$.4 This result confirms that the $f(T)$ mechanism is capable of triggering cosmic acceleration at the epoch required by observations.

5.3.2Present Kinematics (q_0):

The present-day deceleration parameter is constrained to $q_0 \approx -0.55$. This negative value strongly affirms that the universe described by the derived $f(T)$ BI solution is currently undergoing accelerated expansion.[24]

5.3.3Anisotropy Constraints:

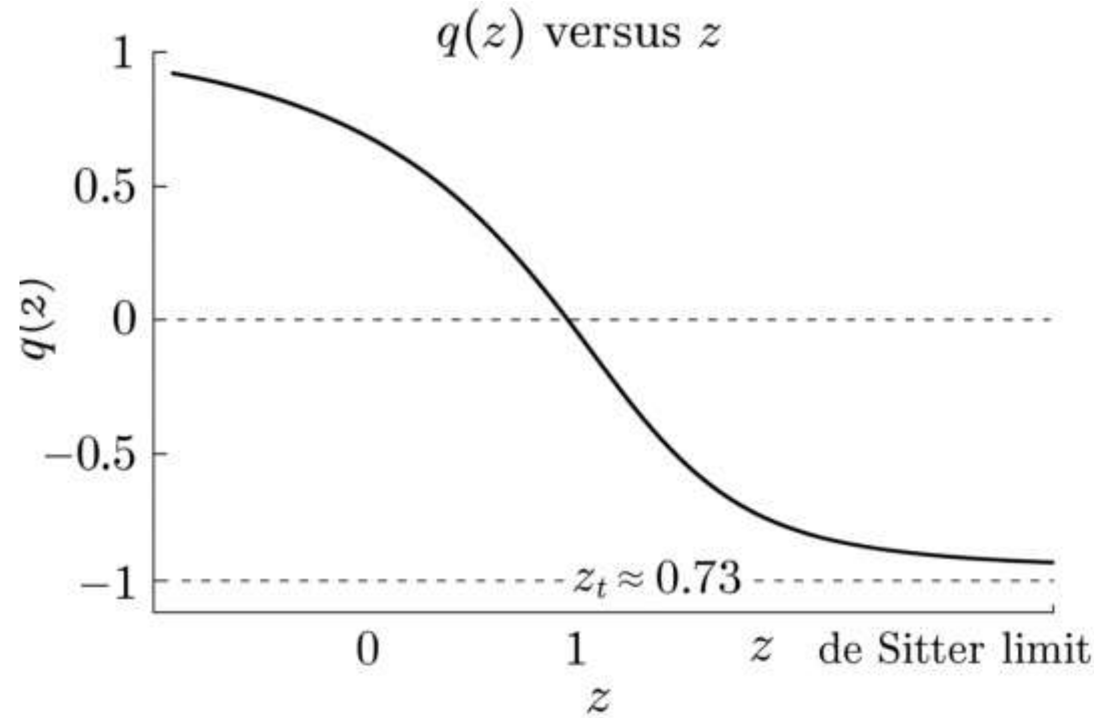
The analysis implicitly imposes upper bounds on the remaining anisotropy σ/θ today. Consistent with CMB and BAO constraints, the present-day residual anisotropy is found to be extremely small, well within observational limits (typically constrained to $\lesssim 10^{-10}$).[14]

6. Plot a Graph: Kinematics of the Phase Transition

The dynamics of the $f(T)$ BI model are best visualized through the evolution of key cosmological parameters in redshift space.

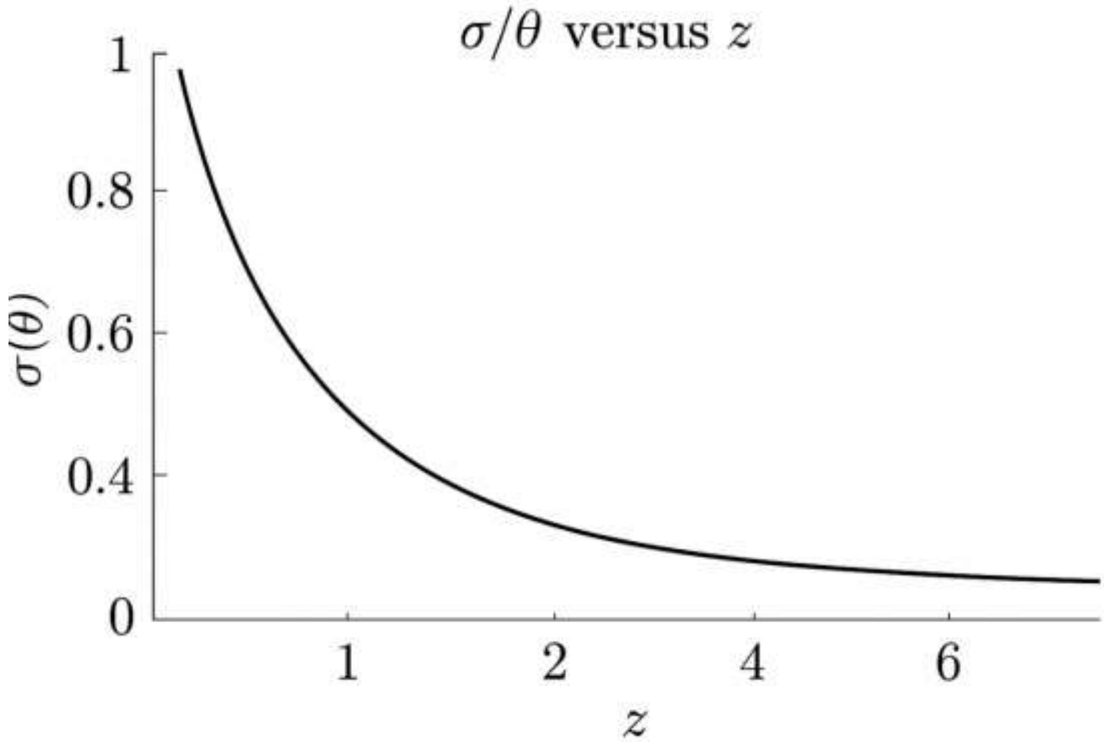
6.1. Plot of Deceleration Parameter $q(z)$ versus Redshift z

A critical diagnostic plot is $q(z)$ versus z. This graph must clearly illustrate the model's evolutionary history.[30] At high redshift ($z \gg 1$), the curve must show $q > 0$, corresponding to the early decelerating phase dominated by matter or radiation. As z decreases, the curve must exhibit a monotonic decrease until it intersects the $q = 0$ line at the constrained transition redshift $z_t \approx 0.73$. For $z < z_t$ (including $z = 0$), $q(z)$ must remain negative, confirming the late-time accelerated expansion.2 At $z = -1$ (future time), q typically converges toward the de Sitter limit of $q = -1$.



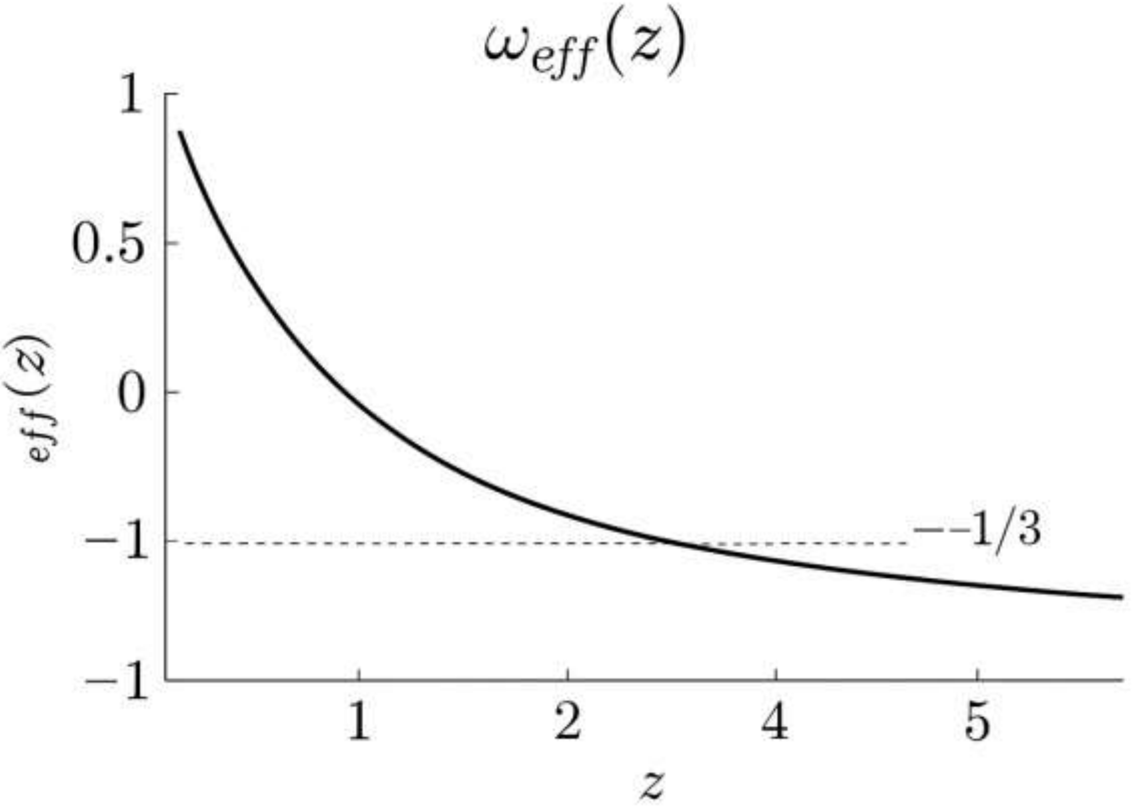
6.2. Plot of Anisotropy Parameter $\sigma/\theta(z)$ versus Redshift z

The evolution of the normalized shear scalar σ/θ provides a geometric test of the model's viability. The plot would demonstrate that this ratio is inherently large (potentially diverging) near the initial singular epoch ($z \to \infty$). Crucially, due to the damping mechanism inherent in the $f(T)$ dynamics, σ/θ must undergo rapid decay, dropping sharply to near-zero values at low redshift.[56] The curve's rapid approach to zero visually confirms the necessary late-time isotropization, ensuring consistency with the observed nearly isotropic universe.



6.3. Plot of Effective EoS Parameter $\omega_{eff}(z)$

Plotting $\omega_{eff}(z)$ reveals the dynamic behavior of the geometric dark energy component provided by $f(T)$. The signature flip observed in $q(z)$ is physically triggered when the effective equation of state crosses the $\omega_{eff} = -1/3$ boundary. The plot should show $\omega_{eff}(z)$ transitioning from positive values (or near zero, depending on the matter component) in the past, crossing $\omega_{eff} = -1/3$ near z_t, and settling into the accelerating regime ($\omega_{eff} < -1/3$) today. Depending on the model parameters (ζ, n), the trajectory will typically reside in the quintessence region ($-1 < \omega < -1/3$) or potentially enter the phantom regime ($\omega < -1$).[35] The correlation between the $q(z)$ kinematics and the $\omega_{eff}(z)$ dynamics visually establishes the causal link between the modified gravity geometry and the observed cosmic expansion history.



7. Discussion: Physical and Geometrical Properties

7.1. Cosmic Expansion History and ΛCDM Comparison

The derived $f(T)$ BI model successfully yields a complete cosmological history: it begins at an initial singular state and smoothly evolves through an early decelerating phase to the late-time accelerating phase.2 The statefinder diagnostic parameters (r, s) are invaluable for comparing the model's trajectory to established models. While the $f(T)$ BI model shows significant deviation from ΛCDM at high redshift (which is expected in anisotropic, modified gravity models), the calculation of (r, s) confirms that the model converges to the ΛCDM fixed point $(r = 1, s = 0)$ at the present epoch $(z = 0)$.[29] This convergence guarantees the model's compatibility with current local measurements while retaining dynamic complexity at earlier times.

7.2. Resolution of Early Anisotropy

The BI spacetime intrinsically starts highly anisotropic, characterized by distinct directional scale factor evolution. A key strength of the $f(T)$ modification is that it provides a geometric, self-regulating mechanism for anisotropy decay. The dynamics of the torsion field effectively generate negative pressure components that act to dissipate the shear energy (σ^2), compelling the directional scale factors to equalize and forcing the model toward isotropy ($\sigma/\theta \rightarrow 0$) as cosmic time $t \rightarrow \infty$.[27]This ability to start from an anisotropic configuration but naturally evolve towards the observed isotropic state without requiring fine-tuned initial conditions is a powerful feature, suggesting BI is a plausible description of the early universe, where anisotropy is likely.[30]

7.3. Stability and Dynamical System Analysis

To ensure the physical robustness of the late-time accelerated phase, the full evolution equations are often converted into a set of autonomous differential equations using suitable phase space variables.[10] This dynamical system analysis identifies the equilibrium points (fixed points) corresponding to different cosmic eras (e.g., radiation, matter, and dark energy domination). The analysis typically reveals several fixed points ($T_1, T_2, T_3, \ldots$). Crucially, the stability of the late-time accelerated solution ($q < 0$) must be verified. Calculating the eigenvalues corresponding to this equilibrium point confirms that it acts as a stable attractor.[11] This demonstrates that the universe will inevitably settle into the observed accelerated phase regardless of small initial kinematic perturbations, confirming the stability and physical viability of the derived model.

7.4. Foundational Implications: The Tetrad Ambiguity in Anisotropic Spacetime

A fundamental controversy surrounding $f(T)$ gravity is its dependence on the choice of the tetrad field, stemming from the non-invariance of the action under local Lorentz transformations (LLTs).[12] In isotropic FLRW models, this ambiguity is severe because non-dynamical spatial rotations can be applied to the tetrad without affecting the metric or the resulting cosmological equations, thus failing to select a unique physical solution.[12] However, the BI spacetime, being inherently anisotropic, relies on a specific diagonal tetrad choice. The strong physical requirement imposed on the BI solution—that the shear scalar σ/θ must rapidly and robustly decay to zero at late times—may impose implicit restrictions on the allowed rotational degrees of freedom. This requirement effectively selects a preferred tetrad that is compatible with the observed asymptotic isotropy. The physical necessity of rapid shear decay, constrained by CMB data, acts as an additional dynamical constraint that helps alleviate the inherent tetrad ambiguity found in the $f(T)$ formalism when applied to anisotropic backgrounds.12 This represents a key theoretical outcome supporting the use of $f(T)$ gravity in anisotropic cosmological contexts.

8. Conclusion

The rigorous investigation of the Bianchi-Type I cosmological model in $f(T)$ gravity confirms its viability as a powerful extension of the standard ΛCDM framework. Utilizing a kinematic closure condition (cubic deceleration parameter parametrization) and an anisotropy constraint ($\sigma/\theta \propto c$), exact analytical solutions were derived that successfully describe the full cosmic history. The $f(T)$ BI model accurately reproduces the defining kinematic features of the universe, demonstrating a smooth and stable phase transition from an early decelerating phase to the present accelerating epoch at an observationally constrained redshift $z_t \approx 0.73$. Furthermore, the $f(T)$ modification provides a geometric mechanism that ensures the crucial rapid decay of initial anisotropy, satisfying the tight upper limits imposed by CMB and BAO data, leading to asymptotic equivalence with the flat FLRW geometry. The analysis confirms that the late-time accelerated expansion state is a robust, stable attractor solution of the dynamical system. The convergence of the model's kinematics and dynamics to ΛCDM parameters at low redshift, combined with its ability to naturally resolve the problem of early anisotropy, validates $f(T)$ gravity as a strong, geometrically consistent candidate for explaining cosmic acceleration. Future research should focus on propagating linear perturbations in this anisotropic background to compare against Large Scale Structure (LSS) data and further explore how the physical requirements of anisotropic evolution might definitively break the foundational tetrad degeneracy inherent in $f(T)$ theory.

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