International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Deriving a General Formula for the Global Minimum-Variance Point of a Two-Asset Markowitz Portfolio

Mr. Aryaveer Jain

Hill Spring International School

Abstract

This paper derives a general formula for the Global Minimum Variance (GMV) point of a two-security portfolio within the Markowitz mean–variance framework. Using Lagrange multiplier alongside a matrix representation of the optimisation problem, the paper obtains closed-form expressions for optimal portfolio weights and the efficient frontier. Furthermore, the model, using five-year average data (2019–2024), is then applied to the case study of Tata Consultancy Services (TCS) and the Indian Tobacco Company (ITC). Through using roughly 43.7% in TCS and 56.3% in ITC, the efficient frontier suggests a 16% GMV expected return. Additionally, the paper evaluates the tactical limitations of the Markowitz model in real financial markets alongside numerical approximations.

Research question

How can Lagrange multipliers be utilised to find a general formula for the global minimum variance in a two-security portfolio?

Introduction

For years, financial analysts and investors have aimed to mathematically predict the movement of stocks, trying to find a formula to "beat the market". However, due to a number of external factors, such as volatility and sentiment heavily influencing the market, a definitive formula to constantly beat the market has not been found. The Markowitz portfolio model devised in 1952 (Hanicova), however, aimed at assisting investors with security selection. By quantifying the trade-off between risk and return, the Markowitz model offered investors a mathematical basis for constructing efficient portfolios.

A central concept to the Markowitz portfolio model is the concept of the Global Minimum Variance (GMV) portfolio, which is considered the 'optimal' portfolio, one which maximizes returns while having the lowest risk (Choubey) and is therefore a portfolio that investors strive to achieve. However, the Markowitz model is built on several unrealistic assumptions, thus limiting its practicality in the real-world, such as the existence of unlimited capital and possibility of infinite returns.

This paper aims to explore the use of Lagrange multipliers to place realistic constraints onto the Markowitz model, improving its practical application in real-life situations, ultimately aiming to find a mathematical general formula for the GMV portfolio that investors can use for reference while constructing a two-security portfolio.

Background

The Markowitz Portfolio Model

The Markowitz Portfolio Model, commonly known as the Modern Portfolio Theory (MPT) is a revolutionary system introduced by Harry Markowitz to aid in the selection of securities in a portfolio, emphasizing the need for diversification and the inherent trade-off between risk and return. Under the MPT there are various different concepts that need to be clearly understood.

Efficient frontier

The efficient frontier is a concept central to the Markowitz model and is defined as "a set of optimal portfolios that offer the highest expected return for a defined level of risk or the lowest risk for an expected return" (Ganti). It is generally represented in the form of a graph, with investors striving to ensure their portfolios lie on the efficient frontier, as portfolios lying on the efficient frontier provide the best return for the level of risk taken. The portfolios below the efficient frontier are considered suboptimal because they do not provide enough return for their level of risk, and portfolios above the efficient frontier are impossible to attain because they imply achieving higher returns than theoretically possible (such as unlimited returns for zero risk). The efficient frontier (illustrated in light green on the figure below) implies that as the investor desires higher returns, they must take greater risks.

Diagram of an efficient frontier (Holton).

Risk and Return

Risk.

In MPT, risk is quantified using the variance of portfolio returns, which is defined as a measure of the dispersion of returns of a portfolio (Nickolas), thus quantifying portfolio volatility. The MPT assumes that investors are risk-averse, meaning they prefer less risk for a given level of returns. Therefore, investors seek to produce the highest possible returns at the minimum level of risk. The variance of a portfolio is given by the following formula,

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, x_j \sigma_{ij},$$

where σ_p^2 is the variance of the portfolio, x_i and x_j are the weights of the two securities in the portfolio, and σ_{ij} is the covariance between the returns of securities i and j. The formula is used and explained further in the paper.

Expected return.

Expected returns is defined as follows under the MPT,

$$E(R_p) = \sum_{i=1}^{n} x_i \, e_i,$$

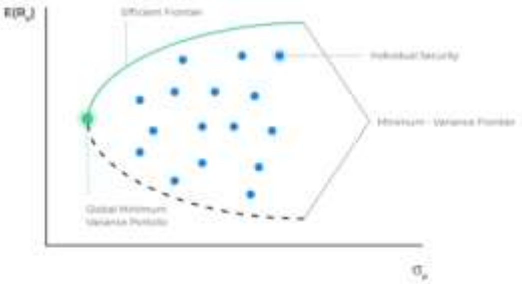
where x_i is the weight of a security i in the portfolio (the percentage of a portfolio that the security comprises), and e_i is the expected returns of this security, which is calculated in many different ways, with the most common way being a 5 year average of the security's returns (this method is used in the extended example).

Diversification

A fundamental principle that the MPT enforces is the idea of diversification, the spreading out of investments across various assets and asset classes to reduce risk. The model demonstrates the fact that combining assets with different return patterns yields less overall risk.

The Global Minimum Variance (GMV) point (Choubey)

The Global Minimum Variance (GMV) point is a point on the efficient frontier that achieves the maximum possible returns for the lowest level of given risk, thus being the portfolio that investors aim to achieve. The GMV is indicated by the leftmost point on the efficient frontier.



Labelled diagram of an efficient frontier (Forjan).

In this paper, for the sake of simplicity I will be considering a portfolio of two securities only, and to find the optimal weights of the two securities to achieve this Global Minimum Variance portfolio, I will be using Lagrange Multipliers.

Lagrange Multipliers

Lagrange multipliers (Chamberlain) are a tool that is used to find the local maxima or minima of a given function (perform optimization on a function) when the function is subjected to specific constraints. A constraint is a given "limit" that a function must adhere to, that makes simple optimization using calculus difficult because the maxima or minima of a function might lie out of the constraint. Therefore, Lagrange multipliers were developed by Joseph-Louis Lagrange to help solve this problem, and to make constraint optimization possible.

Deriving a general formula for the Global Minimum Variance point

Setting up constraints

To make the Markowitz Portfolio Model more applicable to real-world scenarios, it is necessary to set up a series of realistic constraints that the function needs to adhere to, in order to be utilized. These constraints ensure that the portfolio remains feasible and aligns with practical investment goals.

There are two key constraints for the Markowitz Model:

- **Constraint 1: Full investment constraint**

In any portfolio, the total proportion of wealth allocated to all the assets has to equal to 100%. This ensures that the entire portfolio is completely invested and that there are no idle funds, as well as ensuring that there is no 'overuse' of funds i.e. only the funds available are accounted for. Therefore, the sum of weights of a portfolio is 1. Mathematically, the constraint can be expressed as (Chen),

$$\sum_{i=1}^{n} x_i - 1 = 0,$$

where x_i is the weight of a security 'i' in the portfolio and 'n' is the number of securities in the portfolio, which in this case would be 2. However, in practice many investors use leverage, and many investors hold on to some funds to hedge their stocks to ensure their investments are kept safe.

- **Constraint 2: Expected return requirement**

The expected return of a portfolio should always equal to the desired return of the investor for the investments to be viable. Therefore, the difference between the expected and desired returns of the portfolio should be 0. A desired return is defined as the minimum returns an investor will accept to invest, and differs based on financial situation, reasoning for investment, and various other factors. Mathematically, this constraint can be expressed as (Feldman),

$$\sum_{i=1}^{n} x_i\, e_i - d = 0,$$

where x_i is the weight of a security i in the portfolio, e_i is the expected returns of this security, and d is the desired return of the portfolio.

Using Lagrange Multipliers to account for the constraints

To optimize the Markowitz Portfolio Model, the portfolio's risk (volatility) needs to be minimised, while accounting for the constraints previously mentioned: the full investment constraint (sum of weights equals 1) and the expected return constraint. This optimization problem is solved using Lagrange multipliers.

To incorporate these constraints into the optimization process, a Lagrangian function is set up, which combines the objective function (portfolio variance) with the constraints. The Lagrangian 'y' is formulated as:

$$y = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, x_j \sigma_{ij} + \lambda_1\left(\sum_{i=1}^{n} x_i\, e_i - d\right) + \lambda_2\left(\sum_{i=1}^{n} x_i - 1\right),$$

Where λ_1 is the Lagrange multiplier associated with the expected return constraint and λ_2 is the Lagrange multiplier associated with the full investment constraint.

Optimizing for the constraints

The primary objective to optimize the Markowitz model is to ensure that the Lagrangian 'y' is a minimum, as it indicates that portfolio variance is at a minimum and therefore risk is a minimum too.

This optimization can be done in two ways:

1. Take partial derivative of y with respect to x_i, where 'i' can be between 1 and n. Set this value to '0'. This is done to optimise 'y' in terms of the weights x_i.
2. Take partial derivative of y with respect to λ, where λ will be λ_1 or λ_2. Set this partial derivative to '0'. This is done to optimise 'y' in terms of both the constraints.

Which is mathematically represented as,

$$\frac{\partial y}{\partial x_i} = 0, \text{where } i = 1 \text{ to } n,$$
$$\frac{\partial y}{\partial \lambda} = 0, \text{where } \lambda = \lambda_1 \text{ or } \lambda_2.$$

As we are looking at a two-security portfolio, the values of 'n' in the summation is 2 in the following formula,

$$y = \sum_{i=1}^{2}\sum_{j=1}^{2} x_i\, x_j \sigma_{ij} + \lambda_1\left(\sum_{i=1}^{2} x_i\, e_i - d\right) + \lambda_2\left(\sum_{i=1}^{2} x_i - 1\right).$$

Expanding the summation using $n = 2$ provides the following simplified expression,

$$y = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12} + \lambda_1(x_1e_1 + x_2e_2 - d) + \lambda_2(x_1 + x_2 - 1).$$

Setting the partial the derivative of y with respect to 'x_i' to be 0. To find the partial derivative of a function, treat the other variables in the function as constants. For example, to perform this calculation, while finding the partial derivative of y with respect to x_1 treat any other variables $(x_2, \lambda_1, \lambda_2)$ like constants.

$$\frac{\partial y}{\partial x_1} = 2x_1\sigma_1^2 + 2x_2\sigma_{12} + \lambda_1 e_1 + \lambda_2 = 0,$$
$$\frac{\partial y}{\partial x_2} = 2x_2\sigma_2^2 + 2x_1\sigma_{12} + \lambda_1 e_2 + \lambda_2 = 0.$$

Setting the partial derivative of 'y' with respect to λ to be 0,

$$\frac{\partial y}{\partial \lambda_1} = x_1e_1 + x_2e_2 - d = 0,$$
$$\frac{\partial y}{\partial \lambda_2} = x_1 + x_2 - 1 = 0.$$

An easier way to solve these system of equations is to represent them in the form of a matrix, as it provides a better visual representation of the equations.

Visualisation in the form of a matrix

There are various advantages of using a matrix (Britannica) to represent the linear equations:

1. Clarity: A matrix provides a more organized visual representation of multiple equations.
2. Efficiency: Matrix representations allows for efficient computations, allowing all the system of equations to be solved at once.
3. Scalability: Matrices can be scaled up to handle larger portfolios with even more securities, which is although not applicable to this study would allow for a more efficient framework.

Matrix representation in the system of equations

The system of equations,

$$\begin{aligned} 2x_1\sigma_1^2 + 2x_2\sigma_{12} + \lambda_1 e_1 + \lambda_2 &= 0,\\ 2x_2\sigma_2^2 + 2x_1\sigma_{12} + \lambda_1 e_2 + \lambda_2 &= 0,\\ x_1e_1 + x_2e_2 - d &= 0,\\ x_1 + x_2 - 1 &= 0 \end{aligned}$$

can be represented as the following matrix,

$$AX = K,$$

where

$$A=\begin{pmatrix}2\sigma_1^2 & 2\sigma_{12} & e_1 & 1\\ 2\sigma_{12} & 2\sigma_2^2 & e_2 & 1\\ e_1 & e_2 & 0 & 0\\ 1 & 1 & 0 & 0\end{pmatrix},\quad X=\begin{pmatrix}x_1\\ x_2\\ \lambda_1\\ \lambda_2\end{pmatrix},\quad K=\begin{pmatrix}0\\ 0\\ d\\ 1\end{pmatrix}.$$

Where $d = desired\ rate\ of\ return$

The components of the matrix can be understood in the following ways:

- The coefficient matrix on the left (labelled as A) contains all the constraints related to expected returns, portfolio variance, and covariance.
- The variable vector in the middle (labelled as X) contains the unknown values, which are the weights x_1 and x_2 of the two securities and the Lagrange Multipliers λ_1 and λ_2.
- The result vector on the right (labelled as K) represents the known constants, those being the desired returns d and the total investment constraints (equal to 1).

Solving the Matrix System

To solve for the weight vector x, rearrange the system into a solution for x:

$$AX = K.$$

By multiplying the inverse of A with both LHS and RHS, we obtain the following,

$$A^{-1}AX = A^{-1}K.$$

As matrix A is multiplied by its own inverse, an identity matrix is obtained on the LHS

$$IX = A^{-1}K$$

As the multiplication of

$$\therefore X = A^{-1}K$$

To perform this matrix algebra however, I used MATLAB to compute the inverse of a 4x4 matrix, which can be represented by the following (note: the matrix taken was a 4x4 matrix with letters A through P for simplicity),

```
syms a b c d e f g h i j k l m n o p q
A = sym([a b c d; e f g h; i j k l; m n o p]);
D = inv(A)
```

D =

$$\begin{pmatrix}\frac{fkp-flo-gjp+gln+hjo-hkn}{\sigma_1} & -\frac{bkp-blo-cjp+cln+djo-dkn}{\sigma_1} & \frac{bgp-bho-cfp+chn+dfo-dgn}{\sigma_1} & -\frac{bgl-bhk-cfl+chj+dfk-dgj}{\sigma_1}\\ -\frac{ekp-elo-gip+glm+hio-hkm}{\sigma_1} & \frac{akp-alo-cip+clm+dio-dkm}{\sigma_1} & -\frac{agp-aho-cep+chm+deo-dgm}{\sigma_1} & \frac{agl-ahk-cel+chi+dek-dgi}{\sigma_1}\\ \frac{ejp-eln-fip+flm+hin-hjm}{\sigma_1} & -\frac{ajp-aln-bip+blm+din-djm}{\sigma_1} & \frac{afp-ahn-bep+bhm+den-dfm}{\sigma_1} & -\frac{afl-ahj-bel+bhi+dej-dfi}{\sigma_1}\\ -\frac{ejo-ekn-fio+fkm+gin-gjm}{\sigma_1} & \frac{ajo-akn-bio+bkm+cin-cjm}{\sigma_1} & -\frac{afo-agn-beo+bgm+cen-cfm}{\sigma_1} & \frac{afk-agj-bek+bgi+cej-cfi}{\sigma_1}\end{pmatrix}$$

where

$$\sigma_1 = afkp-aflo-agjp+agln+akjo-akkn-bekp+belo+bgip-bglm-bhio+bhkm+cejp-celn-cfip+cflm+chin-chjm-dejo+dekn+dfio-dfkm-dgin+dgjm$$

Inverse of the 4×4 *matrix A computed in MATLAB.*

By performing a matrix multiplication involving this matrix as well as matrix K, we can obtain the following system of equations

$$x_1 = a_1 + b_1 d,$$
$$x_2 = a_2 + b_2 d,$$

Where a_1, a_2, b_1, b_2 are constants found from this multiplication.

There are also a set of equations that determines the Lagrange multipliers, however they are irrelevant to the analysis at hand as they do not need to be found to determine the weights of an efficient portfolio.

Computing the efficient frontier and representing it in terms of an equation

By varying 'd' (desired rate of return), one can obtain the efficient frontier and compute the optimal portfolios for each weight and level of risk.

With the computed efficient frontier, to find a general formula for the Global Minimum Variance point it is imperative to represent the efficient frontier in terms of an equation. Once this is expressed, since the GMV is the leftmost point of the efficient frontier at that point the gradient of the curve is undefined.

An efficient frontier is expressed in the form 'expected returns vs standard deviation'. To represent standard deviation in terms of variables,

Let standard deviation = sd; variance = var

$$\mathrm{sd} = \sqrt{\mathrm{var}}$$

Expanding the variance formula with $n = 2$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i\, x_j \sigma_{ij}$$

We obtain the following,

$$\mathrm{var} = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12},$$

$$\therefore\ \mathrm{sd} = \sqrt{x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}}.$$

Substituting the solutions found for x_1 and x_2,

$$\mathrm{sd} = \sqrt{(a_1 + b_1 d)^2\sigma_1^2 + (a_2 + b_2 d)^2\sigma_2^2 + 2(a_1 + b_1 d)(a_2 + b_2 d)\sigma_{12}}.$$

Therefore, as an efficient frontier is expressed as desired returns (the value of which is equal to expected returns) against standard deviation, it can be mathematically written as the following,

$$\sum_{i=1}^{n} x_i\, e_i = \sqrt{(a_1 + b_1 d)^2\sigma_1^2 + (a_2 + b_2 d)^2\sigma_2^2 + 2(a_1 + b_1 d)(a_2 + b_2 d)\sigma_{12}}.$$

By expanding the formula, we obtain,

$\sum_{i=1}^{n} x_i\, e_i =$

$\sqrt{b_1^2\sigma_1^2 d^2 + b_2^2\sigma_2^2 d^2 + 2b_1b_2\sigma_{12}d^2 + 2a_1b_1\sigma_1^2 d + 2a_2b_2\sigma_2^2 d + 2a_1b_2\sigma_{12}d + 2a_2b_1\sigma_{12}d + a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1}$

These calculations can be simplified by expressing all the constants in terms of coefficients of the powers of d,

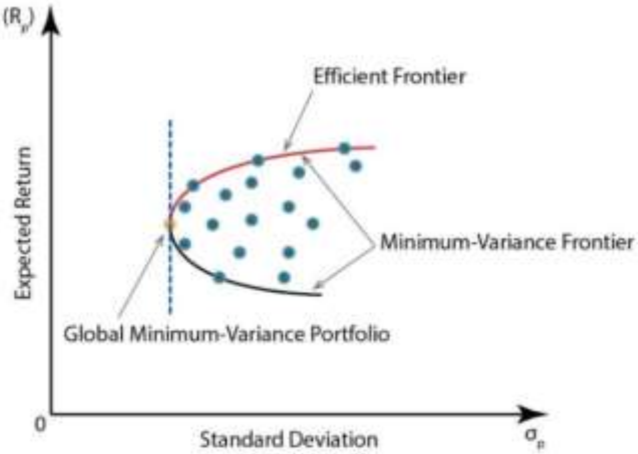
$$\sum_{i=1}^{n} x_i\, e_i = \sqrt{C_3 d^2 + C_2 d + C_1}$$

Where

$$C_1 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_{12},$$
$$C_2 = 2a_1b_1\sigma_1^2 + 2a_2b_2\sigma_2^2 + 2a_1b_2\sigma_{12} + 2a_2b_1\sigma_{12},$$
$$C_3 = b_1^2\sigma_1^2 + b_2^2\sigma_2^2 + 2b_1b_2\sigma_{12}.$$

Computing the general form of the Global Minimum Variance point

The Global Minimum Variance point can be found by determining the point at which the gradient of the curve is undefined.



Efficient Frontier with an illustrated gradient at the GMV Portfolio (Mankowski).

As illustrated by the diagram, on the Global Minimum Variance point the tangent is vertical, thus having an undefined gradient at that point.

Differentiating the equation in terms of d obtains the following (note: the bold letters refer to the small change d),

$$\frac{d}{dd}\sum_{i=1}^{n} x_i\, e_i = \frac{C_2 + 2C_3 d}{2\sqrt{C_3 d^2 + C_2 d + C_1}}.$$

For the gradient of the equation to be undefined,

$$\frac{d}{dd}\sum_{i=1}^{n} x_i\, e_i = \frac{1}{0}$$

$$\therefore\ \frac{1}{0} = \frac{C_2 + 2C_3 d}{2\sqrt{C_3 d^2 + C_2 d + C_1}}$$

$$\therefore\ 2\sqrt{C_3 d^2 + C_2 d + C_1} = 0$$

$$\therefore\ C_3 d^2 + C_2 d + C_1 = 0$$

Therefore, by using the quadratic formula, the general form of the Global Minimum Variance of the efficient frontier exists when the desired returns, and by proxy expected returns, equate to,

$$d = \frac{-C_2 \pm \sqrt{C_2^2 - 4C_1 C_3}}{2C_3}$$

Since desired returns can never be negative and should be maximised,

$$d = \frac{-C_2 + \sqrt{C_2^2 - 4C_1 C_3}}{2C_3}$$

This formula gives the desired return d in terms of constants C_1, C_2, C_3, which are obtained from the variance, covariance and constants calculated from the matrix algebra.

I will be applying the above learned procedure to a real world example, analysing how it can be used to compute the efficient allocation in an actual two security portfolio.

Application of this general formula

In theory, this general formula should be utilised to calculate the level of returns an investor should aim to achieve for in terms of the given fundamentals of the securities. However, the model itself is built on a multitude of assumptions, such as consumer rationality, and that historic returns are an indicator of future

performance. To see how this model can be applied to a real two-security portfolio, we will use a portfolio consisting of Tata Consultancy Service (TCS) and Indian Tobacco Company (ITC) stocks.
The primary reason both these securities were chosen was they represent a wide range of sectors, offering a diversified perspective on risk and return. Furthermore, both TCS and ITC are large cap companies, having extremely strong fundamentals and therefore being considered safer stocks to invest in.
Tata Consultancy Services is India's biggest IT services firm, boasting a market capitalisation of approximately Rs. 1,500,000 Cr, making it the second largest firm in India by market capitalisation as of 12th October 2024 (Carvalho). TCS provides a range of services, including software development, cybersecurity and cloud computing and storage to various companies and industries across the world.
Indian Tobacco Company is a large Indian conglomerate, boasting a market capitalization of approximately Rs. 610,000 Cr as of 12th October 2024. It operates in various industries, such as hotels, packaging and paperboards, agribusiness, IT, fast-moving consumer goods (FMCG), and cigarettes and tobacco.
The data (Kats, "ITC Limited") (Kats, "Tata Consultancy Services")

Table 1: Data values of TCS and ITC stock

Assets	Standard Deviation (σ)	Variance (σ^2)	Returns	
Asset 1 (TCS)	13.8	190	16.4	
Asset 2 (ITC)	12.1	147	15.8	

Note: All the data values are given in percentages for convenient calculations and are taken as 5 year averages, from the years 2019 to 2024
The correlation coefficient between asset 1 and 2 is -0.136 (same sources as above), which shows a weak negative correlation.
To compute the covariance of the two securities (which is necessary for the variance formula for the portfolio), the following formula can be used (Hayes):

$$\mathrm{Cov}(1,2) = \text{Correlation coefficient} \times \sigma_1 \times \sigma_2$$
$$\therefore\ \mathrm{Cov}(1,2) = -0.136 \times 13.8 \times 12.1$$
$$\therefore\ \mathrm{Cov}(1,2) = -22.709$$

Using these data values, the matrix of the system of equations is represented as the following,

$$A \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = K$$

where

$$A = \begin{bmatrix} 380 & -45.419 & 16.4 & 1 \\ -45.419 & 294 & 15.8 & 1 \\ 1 & 1 & 0 & 0 \\ 16.4 & 15.8 & 0 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 \\ 0 \\ 1 \\ d \end{bmatrix}.$$

Therefore,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1}K$$

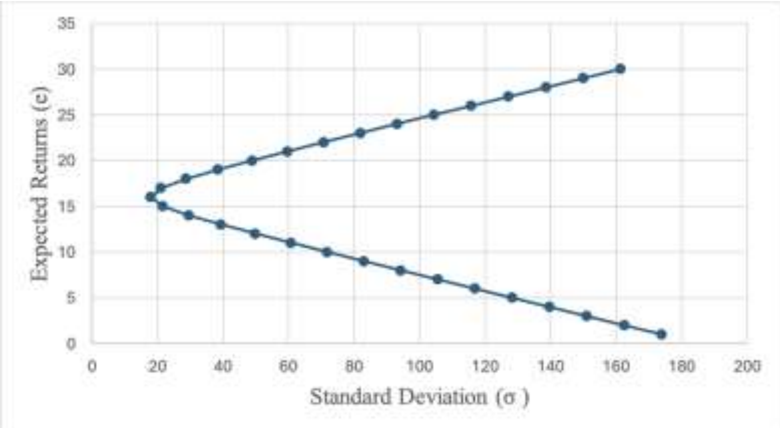
where

$$A^{-1} = \begin{bmatrix} 1.96 \times 10^{-16} & -2 \times 10^{-16} & -26.33 & 1.67 \\ 1.60 \times 10^{-16} & 1.8 \times 10^{-16} & 27.33 & -1.67 \\ 1.67 & -1.67 & 34133.5 & -2124.5 \\ -26.33 & 27.33 & -548542 & 34133.5 \end{bmatrix}.$$

Using matrix multiplication,

$$x_1 = 1.67d - 26.3$$
$$x_2 = -1.67d + 27.3$$

Therefore, by varying different values of ‘d‘, different combinations of weights of an efficient portfolio, computing different values of standard deviation (appendix, A) thus creating the following efficient frontier. (Note: The data of the variance and desired returns used to make the efficient frontier is provided in the appendix)



Efficient frontier for portfolio containing TCS and ITC stock

As illustrated by the graph above, the global minimum variance point can be approximated to be at an expected return of 16%.

Now to test whether the general form for the global minimum variance point works, we will use the data values of TCS and ITC stock to check if the expected return is given as approximately 16%.

By comparing the following linear equations,

$$x_1 = 1.67d - 26.3$$
$$x_1 = a_1 + b_1 d$$
$$x_2 = -1.67d + 27.3$$
$$x_2 = a_2 + b_2 d$$

We obtain the following values,

$$a_1 = -26.3; \quad b_1 = 1.67; \quad a_2 = 27.3; \quad b_2 = -1.67$$

Using these values,

$$C_1 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_{12}$$
$$\therefore\ C_1 = (-26.3)^2(13.8)^2 + (27.3)^2(12.1)^2 + 2(-26.3)(27.3)(-22.709)$$
$$\therefore\ C_1 = 273453.0223$$
$$C_2 = 2a_1b_1\sigma_1^2 + 2a_2b_2\sigma_2^2 + 2a_1b_2\sigma_{12} + 2a_2b_1\sigma_{12}$$
$$\therefore\ C_2 = 2(-26.3)(1.67)(13.8)^2 + 2(27.3)(-1.67)(12.1)^2 + 2(-26.3)(-1.67)(-22.709) + 2(27.3)(1.67)(-22.709)$$
$$\therefore\ C_2 = -34144.04312$$
$$C_3 = b_1^2\sigma_1^2 + b_2^2\sigma_2^2 + 2b_1b_2\sigma_{12}$$
$$\therefore\ C_3 = (1.67)^2(13.8)^2 + (-1.67)^2(12.1)^2 + 2(1.67)(-1.67)(-22.709)$$

$$C_3 = 1066.107225$$

Therefore, by the general form for the efficient frontier,

$$\sum_{i=1}^{n} x_i\, e_i = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_3}$$

$$\sum_{i=1}^{n} x_i\, e_i = \frac{-(-34144.04312) + \sqrt{(-34144.04312)^2 - 4(273453.0223)(1066.107225)}}{2(1066.107225)}$$

$$\therefore \sum_{i=1}^{n} x_i\, e_i = 16.01 + 0.259i$$

As the global minimum variance point has been given at an expected returns of about 16%, this model is relatively accurate. However, a small imaginary component of $0.259i$ exists, which may indicate inaccuracies in the model or its assumptions.

Mathematical reasoning behind the imaginary component

The imaginary component exists because the expression under the square root in the quadratic formula, $C_2^2 - 4C_1C_3$, becomes negative. This result may seem counterintuitive mathematically, as the expected returns should always be a real number and never complex, however by examining the situation from a logical point of view rather than a purely mathematical one this component can be explained.

Logical justifications behind the imaginary component

There are two logical reasons that explain why the imaginary component exists:

1. Negligibility of the Imaginary Part

   The imaginary component $0.259i$ is extremely small when compared to the magnitude of the real part 16.01, with the percentage uncertainty being 1.62%. In practicality, this makes the imaginary portion negligible, as in financial models which involve approximation small deviations are expected. This deviation exists simply because the value under the root was negative, which can be neglected due to the small magnitude of this value.
2. Rounding and approximation

   A large reason the small imaginary number appears is due to rounding and simplification made throughout the process of calculating variances, covariances and many other metrics. Financial data is often rounded to two to three decimals, and these small approximations may accumulate over the course of calculations, leading to minute errors that can be ignored.

Given these two logical explanations, it is reasonable to conclude that the imaginary component does not invalidate the model's result, rather it highlights the small but inevitable imperfections in real-world financial modelling, proving that small approximations and variances in data can compound to produce noticeable errors. Therefore, for practical purposes, the GMV can be taken as a real number, ignoring the imaginary component completely, as the imaginary component is a byproduct of rounding values, simplifications of models and imprecision of financial data.

Using the general form for the Global Minimum Variance point, the efficient weight of TCS stock is as follows:

$$x_1 = 1.67(16.01) - 26.30$$
$$x_1 = 0.4367$$

The efficient weight of ITC stock is as follows:

$$x_2 = -1.67(16.01) + 27.30$$

$$x_2 = 0.5633$$

Therefore, for this two security portfolio to have the maximum returns for the minimal level of risk (be the most efficient), it should comprise of 43.67% of TCS stock and 56.33% of ITC stock.

Evaluation of the model's applicability to the real world

While the general formula for the Global Minimum Variance point and the Markowitz Model does mathematically seem to provide an adequate ratio for the amount of money that should be invested in a two security portfolio for increased optimisation, applying the model to real-world financial markets may have various limitations. The Markowitz model in theory should help investors create an "efficient" portfolio, balancing risks and returns to have the greatest expected returns. However, the Markowitz model assumes investors are completely rational and data driven, whereas the market is also heavily influenced by sentiment.

In practice, the markets are influenced by various psychological elements such as fear, apprehension, excitement and gut feelings. These emotions primarily manifest themselves in market uncertainty and "black-swan" events, like market crashes, geopolitical events, or recessionary periods. The model is predicated on assumptions that consumers will always seek to maximize returns while minimizing risk, but the reality is that irrational behaviour like panic selling and excess buying will disrupt the rationality of the investors, making it the model not completely applicable to real markets.

Furthermore, a large assumption this model makes is that historic data is indicative of future returns, where expected returns are taken as an average of historic returns. However, it is rarely the case that past returns directly mirror future returns, because of company strategies, economies of scale, and unforeseen market disruptions. Disruptions such as regulatory changes, advancements in technology, or global crises will drastically alter the performance of markets and individual securities that is not reflected in historic data. Internal disruptions such as changes in management, alterations in efficiency and investor confidence may also not be reflected. Therefore, the underlying assumption that past performance is indicative of future results introduces large uncertainties that just cannot be mathematically accounted for.

Another significant limitation of the MPT is its assumption of continues, frictionless markets, without any transaction costs, taxations and liquidity constraints. Practically, investors need to take all of these costs into account while investing, ensuring their positions are still profitable after all the costs are incurred. Furthermore, liquidity constraints of particular stocks prevent investors from buying excess amounts of a particular security at their desired or "efficient" price, complicating the practical applications of the Markowitz Model.

While the Markowitz model may have flaws, these flaws are not inherently unique to this model nor are they unexpected, as economic models by definition are imperfect. Economics and finance is often considered a natural science, using a mix of quantitative and qualitative ideas to generate predictive models. However, no models can fully capture the complete complexity of markets, with many moving pars continuously at play. The purpose of such a model is not to perfectly determine the adequate ratio of stocks in a portfolio, rather to serve as a tool to enhance financial decision-making. The model provides a structured framework to judge risk and return, giving investors a took to assess the efficiency of their portfolio, however, to make informed investments this model needs to be coupled with a qualitative understanding of investing, including market research, real world judgement and investment experience. This structured framework serves as the backbone for the strategies of the largest investment firms in the world, such as Morgan Stanley (Ventelon), Goldman Sachs (Kenton), and Wells Fargo (Eckert), who all use adaptations of the MPT, a testament to its vast utility and importance in the landscape of modern finance.

Conclusion

This paper successfully derives a general formula for the Global Minimum Variance (GMV) in a two-security portfolio using the Markowitz model and accounting for constraints using Lagrange multipliers. Through the example of a two-security portfolio comprising of Tata Consultancy Services (TCS) and Indian Tobacco Company (ITC) stocks, this general formula was proven to effectively calculate the optimal weights for minimal risk while achieving the highest expected return, with a small percentage error existing. This model's accuracy shows how investors can theoretically construct an efficient portfolio using the allotted weights.

However, although this model offers rigorous mathematical proof, it is built on large unrealistic assumptions, such as the use of historic data to predict future returns, the expectation that the market is efficient, and the assumption that the market is frictionless. Despite these limitations, the Markowitz model is one of the most efficient tool investors can use to optimise their portfolios, providing a framework for investors to make more informed decisions and produce portfolios that maximise returns for a given level of risk.

Works Cited

1. Britannica. "Matrix | Mathematics | Britannica." *Encyclopædia Britannica*, 20 Aug. 2024, https://www.britannica.com/science/matrix-mathematics. Accessed 28 Sept. 2024.
2. Carvalho, Brian. "Top 10 Companies in India by Market Valuation in 2023." *Forbes India*, 23 Sept. 2024, https://www.forbesindia.com/article/explainers/top-10-companies-india-market-valuation/86307/1. Accessed 12 Oct. 2024.
3. Chamberlain, Andrew. "A Simple Explanation of Why Lagrange Multipliers Works." *Medium*, 21 Mar. 2017, https://medium.com/@andrew.chamberlain/a-simple-explanation-of-why-lagrange-multipliers-works-253e2cdcbf74. Accessed 18 Nov. 2024.
4. Chen, James. "What Is Portfolio Weight?" *Investopedia*, 7 Aug. 2024, https://www.investopedia.com/terms/p/portfolioweight.asp. Accessed 18 Nov. 2024.
5. Choubey, Priya. "Global Minimum Variance Portfolio." *Wallstreetmojo.com*, Wallstreetmojo, 16 Apr. 2024, https://www.wallstreetmojo.com/global-minimum-variance-portfolio/. Accessed 27 June 2024.
6. Eckert, Christopher. "Asset Allocation Model Portfolios." *Wellsfargoadvisors.com*, Wells Fargo, Apr. 2024, https://fa.wellsfargoadvisors.com/christopher-eckert/mediahandler/media/634460/AAA%20Asset%20Allocation%20Model%20Portfolios%20-%20April%202024%20Report%20PM-02232025-6284546.1.1.pdf. Accessed 19 Nov. 2024.
7. Feldman, Matan. "Expected Return." *Wall Street Prep*, 27 June 2024, https://www.wallstreetprep.com/knowledge/expected-return/. Accessed 18 Nov. 2024.
8. Forjan, James. "Minimum-Variance Portfolios | CFA Level 1 - AnalystPrep." *AnalystPrep | CFA Study Notes*, 12 Sept. 2019, https://analystprep.com/cfa-level-1-exam/portfolio-management/minimum-variance-portfolios/. Accessed 13 Oct. 2024.
9. Ganti, Akhilesh. "Efficient Frontier Definition." *Investopedia*, 31 May 2022, https://www.investopedia.com/terms/e/efficientfrontier.asp. Accessed 29 June 2024.
10. Hanicova, Daniela. "Markowitz Model." *QuantPedia*, 14 June 2021, https://quantpedia.com/markowitz-model/. Accessed 29 June 2024.

11. Hayes, Adam. "Covariance." *Investopedia*, 31 Aug. 2024, https://www.investopedia.com/terms/c/covariance.asp. Accessed 18 Nov. 2024.
12. Holton, Glyn. "Efficient Frontier." *GlynHolton.com*, 8 June 2013, https://www.glynholton.com/notes/efficient_frontier/. Accessed 11 Nov. 2024.
13. Kats, Rifka. "ITC Limited." *Macroaxis*, 2024, https://www.macroaxis.com/stock/ITC.NSE/ITC-Limited. Accessed 12 Oct. 2024.
14. —. "Tata Consultancy Services." *Macroaxis*, 2024, https://www.macroaxis.com/stock/TCS.NSE/Tata-Consultancy-Services. Accessed 12 Oct. 2024.
15. Kenton, Will. "What Is the Black-Litterman Model?" *Investopedia*, 1 Dec. 2023, https://www.investopedia.com/terms/b/black-litterman_model.asp. Accessed 19 Nov. 2024.
16. Mankowski, Bart. "CFA Level 1: Efficient Frontier & Investor's Optimal Portfolio." *Soleadea.org*, 7 Oct. 2022, https://soleadea.org/cfa-level-1/efficient-frontier-investor-optimal-portfolio. Accessed 11 Nov. 2024.
17. Nickolas, Steven. "How Can I Measure Portfolio Variance?" *Investopedia*, 17 Dec. 2023, https://www.investopedia.com/ask/answers/071515/how-can-i-measure-portfolio-variance.asp. Accessed 21 June 2024.
18. Ventelon, Alex. "Rethinking the 60/40 Portfolio | Morgan Stanley." *Morgan Stanley*, 13 Aug. 2024, https://www.morganstanley.com/ideas/60-40-investment-portfolio-future. Accessed 19 Nov. 2024.