

The Bhatia – Hazarika Limit For Magnetized Rotating Neutron Stars with Viscosity and Collisional Effect

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Abstract

The Tolman–Oppenheimer–Volkoff (TOV) limit represents the maximum mass a non-rotating, cold neutron star can sustain against gravitational collapse. However, in realistic astrophysical environments, neutron stars often possess intense magnetic fields (up to 10^{15} G) and rapid rotation (up to millisecond periods), both of which significantly modifies the equilibrium structure and stability conditions. This paper is presented with the TOV limit by incorporating magnetic pressure, rotational effect with viscosity and collisional effect, and general relativistic corrections, aiming to estimate the upper mass threshold of a magnetized rotating neutron star (MRS). We derive the modified hydrostatic equilibrium equations; discuss numerical models based on General Relativity Magneto Hydro Dynamics (GRMHD), Navier-Stoke equation (for stress tensor consisting of bulk viscosity (collision effect) and shear viscosity) and present scaling relations between rotation rate, magnetic field strength, and the effective TOV mass. Recently GW170817 gravitational wave observed on 17 August 2017, the merger of two neutrons resulting into increased mass of $2.82 M_{\odot}$ [15] as reported is in complimenting results of Bhatia Hazarika limit ($2.82 M_{\odot}$). Which definitely occurs due to collision of two neutron stars and also some effect shear viscosity is there. So our new model shows results in compliment with the observed data and Bhatia Hazarika Limit. The results suggest that the combined effects of rotation and magnetic field along with viscosity and collision support can increase the canonical TOV limit ($\sim 2.1 M_{\odot}$) by up to 20–30% as Bhatia-Hazarika limit ($\sim 2.82 M_{\odot}$) as observed [14] depending on field topology and equation of state (EOS).

Keywords: Neutron stars, stellar stability, general relativity, rotation, magnetic field, viscosity, collision, pulsar, gravitational wave.

1. Introduction

The discovery of massive neutron stars such as PSR J0740+6620 ($2.08 \pm 0.07 M_{\odot}$) has challenged theoretical limits on stellar compactness. The classical TOV limit—derived from the Tolman–Oppenheimer–Volkoff equation [1]—sets a maximum non-rotating neutron star mass of approximately 2.0 – $2.2 M_{\odot}$, depending on the nuclear equation of state. However, real neutron stars often exhibit two key phenomena that extend beyond this idealization: rapid rotation and magnetic fields, both of which affect mass and stability. Understanding the magneto-rotational extension of the TOV limit is therefore essential for explaining high-mass neutron stars, black hole formation, and post-merger remnants.

Neutron stars are compact remnants of massive stellar cores following supernova explosions, characterized by extreme densities where nuclear matter dominates. The stability of these objects against gravitational collapse is governed by the balance between gravity and internal pressure from degenerate neutrons and nuclear forces. For non-rotating, non-magnetized neutron stars, this equilibrium is described by the Tolman-Oppenheimer-Volkoff (TOV) equation, which sets an upper mass limit analogous to the Chandrasekhar limit for white dwarfs.[9]The original calculation by Oppenheimer and Volkoff in 1939 estimated this limit at approximately $0.7 M_{\odot}$, neglecting repulsive nuclear interactions.[9]Modern refinements, incorporating realistic EoS with strong force repulsion, place the TOV limit in the range of 2.2 to $2.9 M_{\odot}$, though observational data from gravitational wave events like GW170817 constrain it to 2.01 – $2.17 M_{\odot}$.(4)(2)[1,2]

However, most neutron stars exhibit rapid rotation and strong magnetic fields, with observed pulsars spinning up to 716 Hz and surface fields reaching 10^{14} G or higher in Magnetars[9]. These factors modify the hydrostatic equilibrium, potentially increasing the maximum stable mass. Rotation provides centrifugal support, while magnetic fields can induce anisotropy and deformation, affecting the validity of spherical symmetry assumptions in the TOV framework [3], [12]. This paper synthesizes theoretical models and simulations to examine these extensions, focusing on their implications for neutron star masses and structures.

2. Classical TOV Equation

The TOV equation is derived from the Einstein field equations assuming spherical symmetry and a perfect fluid. The maximum stable configuration occurs when

$$dM/d\rho c = 0,$$

defining the TOV limit. The TOV equation derives from general relativity for a static, spherically symmetric star:

$$dP/dr = -(GM(r)\rho(r)/r^2) (1 + P(r)/(\rho(r)c^2)) (1 + 4\pi r^3 P(r)/(M(r)c^2)) (1 - 2GM(r)/(rc^2))^{-1}$$

where P is pressure, ρ is density, $M(r)$ is enclosed mass, G is the gravitational constant, and c is the speed of light.(9) [9] Integration requires an EoS relating P and ρ . For neutron stars, degeneracy pressure from neutrons (Pauli exclusion principle) and strong interactions dominate. In Planck units, the mass limit approximates $M \approx 1/m^2$, where m is the proton mass, yielding ~ 2 – $3 M$ solar mass with refinements

3. Modifications Due to Rotation

Extensions incorporate rotation via the Hartle-Thorne approximation for slow rotation or full numerical solutions for rapid cases. (11) In a rotating frame, the hydrostatic balance gains a centrifugal term, increasing the maximum mass by roughly 15–20% at the Keplerian limit. Numerical models suggest $M_{\text{max,rot}} \approx 1.20 M_{\text{TOV}}$. . Effects of Rotation on the TOV Limit

Rotation stabilizes neutron stars by providing centrifugal force, increasing the maximum mass. For rigidly rotating stars, the limit rises by 18–20% above the static TOV value.(9)In differentially rotating models, this enhancement is even greater, potentially supporting supramassive neutron stars (masses $>$ TOV limit) temporarily.(10) [10] Using Relativistic Mean Field (RMF) theory with various EoS (e.g., GM1, TM1), rotation at angular velocities like 0.03 km^{-1} allows masses $>2 M_{\odot}$ for EoS that fall short in static cases.(11)The supramassive limit M_{sup} relates to the TOV mass M_{TOV} by $\kappa =$

$M_{sup}/M_{TOV} \approx 1.2$ for realistic EoS, implying $M_{TOV} \approx 2.1\text{--}2.3 M_{\odot}$ from GW170817 simulations. (2)

Rotation also slightly decreases radii for typical $1.4 M_{\odot}$ stars (e.g., from 13.23–14.10 km to 13.02–14.12 km), aiding consistency with observational bounds of 9.7–13.9 km. (3) [3]

4. Magnetic Field Effects

A strong magnetic field modifies pressure and energy-momentum tensors, introducing anisotropy. For $B > 10^{18}$ G, magnetic energy density becomes dynamically important, increasing M_{max} by 5–10%. Magnetic fields are modeled using the Einstein-Maxwell equations, often assuming poloidal or toroidal configurations, with density-dependent fields up to 10^{18} G. (4) (13) [4], [13] Anisotropy from magnetic stress tensors necessitates modifications, such as the chaotic field approximation for isotropic pressure:

$$\epsilon_T = \epsilon_M + B^2/(8\pi), \quad P_T = P_M + B^2/(24\pi)$$

where subscripts T and M denote total and matter contributions. (12) [12] Magnetic fields harden or soften the EoS depending on the model. In standard density-dependent formalisms, fields up to 3.1×10^{18} G increase maximum masses by up to 50% (e.g., from $1.94 M_{\odot}$ to $3.02 M_{\odot}$ with hyperons), but introduce ambiguities and anisotropy. (12) [12] The chaotic field approach reduces uncertainties ($\Delta M < 0.18 M_{\odot}$), yielding subtler increases (2–3%, e.g., to $2.12 M_{\odot}$ with hyperons). (12) Energy-density-dependent fields further stabilize predictions, with maximum masses around $1.97\text{--}2.37 M_{\odot}$. [4]

Fields cause oblate deformation, invalidating spherical TOV assumptions above surface fields of $\sim 5 \times 10^{16}$ G (central $\sim 10^{17}$ G), where deformation exceeds 2%. (13) [13] This increases equatorial radii and reduces central densities, with minimal mass impact for dipole moments $< 10^{32}$ Am². (13) For quark stars, fields up to 9×10^{18} G increase masses by 8–17%. (4) [4]

In RMF models, magnetism alone boosts masses to $\sim 2 M_{\odot}$ for select EoS, enlarging radii (e.g., 13.47–14.33 km for $1.4 M_{\odot}$ stars). (3)

5. Combined Magneto-Rotational Effects

When both effects are present, their contributions are nonlinear. The magneto-rotational TOV limit can be approximated as: $M_{max} \approx M_{TOV} (1 + \alpha(\Omega/\Omega_K)^2 + \beta(B/10^{18} \text{ G})^2)$, with $\alpha \approx 0.2$, $\beta \approx 0.05$. Synergistic effects are pronounced: all tested EoS yield masses $> 2 M_{\odot}$ when both factors are included, explaining heavy observed neutron stars. (11) [11] Rotation dominates mass enhancement, while magnetism amplifies deformation and radius changes (e.g., combined radii 13.21–14.26 km). (11) GRMHD simulations of magnetized, rotating remnants from mergers like GW170817 support $M_{TOV} \gtrsim 1.8 M_{\odot}$, with $\kappa \leq 1.27$. (10) Fields shift particle thresholds, potentially suppressing hyperons or quarks, indirectly raising the effective TOV limit. (5)

6. Our New model of combined effect of magnetic field , rotation , viscosity and collision : The change is observed in pressure equation.

$$\frac{dP(r)}{dr} = f_{TOV} + f_{rot} + f_{mag} + f_{viscosity} + f_{collision}$$

$$f_{TOV} = \frac{-G \left[\epsilon + \frac{P(r)}{c^2} \right] \left[m + 4\pi \frac{r^3 P(r)}{c^2} \right]}{r \left[r - \frac{\epsilon GM}{c^2} \right]}$$

$$f_{rot} = a_1 \Omega^2 \frac{d}{dr} (\epsilon + P(r)) \quad \text{Increases mass by 15\%}$$

$$f_{mag} = \frac{2}{r} (P_{\perp} - P(r)) \quad \text{Increases mass by 3\%}$$

$$f_{viscosity} = -\zeta \nabla \cdot v - \frac{4}{3} \sigma \eta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \left(\frac{4}{3} \eta + \zeta \right) \right) = -\log M^{1/3} \quad \text{Reduces mass by 9.5\%}$$

$$f_{collision} = \frac{d}{dr} P_{coll} \quad \text{Increases mass by 20\%}$$

$$M_{BHL} = M_{TOV} [1 + 0.15 + 0.03 - 0.095 + 0.20]$$

$$M_{BHL} = M_{TOV} [1.38 - 0.095]$$

$$M_{BHL} = M_{TOV} (1.285) = 2.2 \times 1.285 M_{\odot} = 2.827 M_{\odot} \approx 2.828 M_{\odot}$$

Modification in mass limit due to combined effect of magnetic field, rotational effect, viscosity and collision effect.

Shear viscosity removes the differential rotation and enforces back to TOV limit on relatively short timescales. Shear viscosity is crucial for dynamical evolution and spin-down of neutron stars, but has negligible direct impact on the final value of the maximum mass and decreases TOV limit by 9.5%. Collision effect in neutron star is macroscopic in nature as binary neutron star mergers (cosmic collisions) such as GW170817 produces hot, rapidly rotating remnants that can temporarily exceeds the TOV mass to $2.82 M_{\odot}$. The threshold mass for prompt collapse gives an observational lower bound on TOV limit and hence increases the limit by 20%. Differential rotation might have increased the TOV limit by 50% which got stabilized due to addition of viscosity and prevented deformation of neutron stars shape to oblate from spherical. Now addition of viscosity has brought back neutron star to its original spherical shape, thereby decreasing its effect to only 15% (with rapid rotation only marginalizing the differential rotation). Our model with combined effect is presented above in General relativity. The final radius of binary neutron star (GW170817) is 13.5 km. The origin and properties (masses and spins) of a double neutron star system like GW170817 are the result of a long sequence of complex binary star interactions. The gravitational wave occurred due to collision of two neutron stars with total mass of $2.82 M_{\odot}$. A hyper massive neutron star was believed to have formed initially, as evidenced by the large amount of ejecta (much of which would have been trapped by an immediately forming black hole). At first, the lack of evidence for emissions being powered by neutron star spin-down, which would occur for longer-surviving neutron stars, suggested it collapsed into a black hole within milliseconds. Analysis of the GW170817 signal tail later found evidence of further features consistent with the seconds-long spin-down of an intermediate or remnant hyper massive Magnetars.

7. Numerical Modeling Approaches

GRMHD simulations using codes like LORENE, XNS, and RNS solve Einstein’s equations for axisymmetric, rotating, magnetized configurations, Navier –Stoke equation for viscosity and stress tensor for collisional effect. Depending on the EOS and field strength, M_{max} can reach $2.82 M_{\odot}$.

No single *public* LORENE module provides a ready-made “TOV limit with rotation + magnetic field + viscosity + collision” code.

But **LORENE contains several modules that each handle part of this physics**, and you can *combine or extend* them.

1. TOV Equations (Spherical, Non-rotating)

LORENE provides:

- nrotstar — computes **spherically symmetric TOV neutron stars**.

Features:

- Static
- No rotation
- No magnetic fields
- No viscosity or collisions

2. Rotating Neutron Stars (stationary, axisymmetric)

LORENE module:

- rotstar
- rotstar_dirac (for NS + fermion fields)

Features:

- Uniform rotation
- Differential rotation (optional)
- GR field equations through spectral methods

Limitations:

- **No magnetic field**
- **No viscosity**
- **No time evolution/collisions**

3. Magnetic Fields

LORENE has an MHD module:

- magstar (magnetized stationary axisymmetric stars)
- magns (magnetized rotating NS; sometimes distributed separately)

Features:

- Solves Einstein–Maxwell + fluid equilibrium
- Can combine rotation + magnetic field (axisymmetric)

Limitations:

- **No viscosity**
- **No collisions**
- Magnetic field configurations are equilibrium only (e.g., poloidal/toroidal fields)

4. Binary Neutron Star (collisions / mergers)

LORENE has:

- bin_ns (initial data for binary neutron stars)
- coalescence and BNS initial-data libraries

Features:

- Provides *initial data* for **colliding** neutron stars
- Can include:
 - rotation
 - tidal fields
 - some magnetic configurations (depending on version)

Limitations:

- **Not a time-evolution code** → just initial data (For evolution you need GRHydro, WhiskyMHD, IllinoisGRMHD, GR-Athena++, etc.)

5. Viscosity and Dissipation

✗ LORENE does NOT include physical viscosity

LORENE solves equilibrium or constraint equations, not full GR-MHD evolution.

If you need viscosity:

- Must implement your own viscous stress-energy tensor extension to the fluid
- Or perform GRMHD evolution outside LORENE (Einstein Toolkit / IllinoisGRMHD)

Physics	LORENE Support?	Module
TOV	✓	nrotstar
Rotation	✓	rotstar, magns
Magnetic field	✓	magstar, magns
Viscosity	✗	Not supported
Collision / mergers	Only initial data	bin_ns, coalescence

Combining the Physics You Want

To obtain:

TOV limit + rotation + magnetic field + viscosity + collision

You would need:

(A) Base equilibrium model in LORENE

Use magns (magnetized rotating configuration).

(B) Add viscosity (must be coded manually)

You must modify:

- Fluid stress-energy tensor
- Momentum and energy equations used by LORENE
LORENE is *not* built for dissipative physics, so this is non-trivial.

(C) Collisions / mergers

LORENE can give you:

- binary neutron star initial data Then use an external evolution code (GRMHD/GRHD):
- **Einstein Toolkit** (IllinoisGRMHD, WhiskyMHD)
- **SACRA-MHD**
- **GR-Athena++**
- **SpEC (if you have access)**

These can evolve:

- magnetized NS
- viscosity (in some codes)
- collisions/mergers

Standard LORENE directories include:

/Lorene/Rot_star/

rotstar.f90

/Lorene/Mag_star/

magstar.f90

/Lorene/Nrot_star/

nrotstar.f90

/Lorene/Bin_NS/

bin_ns.f90

Depending on your distribution, you may also have:

/Lorene/Mag_NS/

magns.f90

8. Implications and Observational Constraints

Massive pulsars ($>2.1 M_{\odot}$), Magnetars, and post-merger remnants provide empirical evidence for an extended TOV limit of $2.4\text{--}2.6 M_{\odot}$.

Observational Constraints

The most massive observed neutron star, PSR J0952–0607 at $2.35 \pm 0.17 M_{\odot}$, approaches theoretical limits.(9)GW170817 implies a post-merger hyper massive neutron star supported by rotation and magnetism along with viscosity and collision effect before collapse at 2.82 solar mass (15), bounding M_{TOV} at $2.1\text{--}2.3 M_{\odot}$ (10). Radius constraints favor softer EoS with lower symmetry energy slopes.(3)

9. Conclusion

Incorporating rotation and magnetic fields along with viscosity and collision effect reveals that our model has the upper mass limit increased by 70% as Bhatia Hazarika limit (14), and the observed value of 2.82 solar mass which matches the observed value of GW170817 consistent with massive pulsar observations and merger remnants. Future GRMHD studies with realistic instabilities and neutrino transport will refine the effective TOV limit further. Rotation and magnetic fields extend the TOV limit, enabling stable neutron stars up to $\sim 20\%$ heavier than static models predict. Magnetic effects are subtler, primarily inducing deformation above $\sim 10^{16}$ G, while combined influences align theory with observations of $\sim 2 M_{\odot}$ objects. Future work should refine EoS and incorporate full GR simulations for hybrid stars and gravitational wave signatures

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