

Exploration of Smarandachely Product Cordial Labeling Techniques Applied to Christmas Trees

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Abstract

This study investigates the application of Smarandache product cordial labeling techniques to Christmas tree graphs. Graph labeling is a crucial aspect of graph theory, with applications in various fields such as computer science, coding theory, and network design. Smarandache product cordial labeling, in particular, offers a unique approach to labeling graphs while maintaining specific properties. By exploring the application of these techniques to Christmas tree graphs, this research aims to contribute to the understanding of graph labeling and its potential applications in real-world problems. The study presents theoretical results, algorithmic approaches, and potential extensions to related graph structures.

Keywords: Tree, Christmas Tree, Cordial Labeling, Smarandachely product cordial Labeling

1. Introduction

Graph theory was first introduced in the 18th century by the Swiss mathematician (Euler, L. 1736). His work on the famous "Seven Bridges of Königsberg problem," is considered the origin of graph theory. Graph theory is a fundamental concept in graph theory, where labels are assigned to vertices or edges of a graph in a way that satisfies certain conditions. Graph labeling has been an active area of research, with various applications in computer science, networking and other fields. The concept of graph labeling was introduced by (Rosain, (1967)), and since then, numerous researchers have contributed to the field of graph labelings. Smarandache's contributions have led to the development of various mathematical concepts, including Smarandachely product cordial labeling.

Graph study involves various techniques used to solve problems as depending on problem type including Graph traversal, Graph coloring, shortest path algorithms, Graph decomposition, Graph Labeling, Network Flow algorithms, Graph embedding etc. Graph labeling is a fascinating and powerful area of graph theory that involves assigning numbers or labels to the vertices or edges of a graph according to specific rules, (Gallian, J. A., 2020). The objective of Graph labeling is to analyzing graph structure, Identifying patterns and relationship between graphs. Graph labeling not only simplifies complex structures but also improves the performance of algorithm, reduce memory usage and optimize system designs. Over the years, various labeling techniques have been developed, such as graceful labeling, magic labeling and cordial labeling. Each of this method follows unique rules and distinct benefits. Cordial Labeling is introduced by Cahit in their research paper (Cahit I.1987). A type of graph labeling where the vertices of graph are labeled with numbers such that difference between them is 0 and 1 is at most 1.

Smarandachely product cordial labeling is a recent extension of the traditional cordial labeling approach. It is inspired by the Smarandache concepts, which often combine existing mathematical principles to

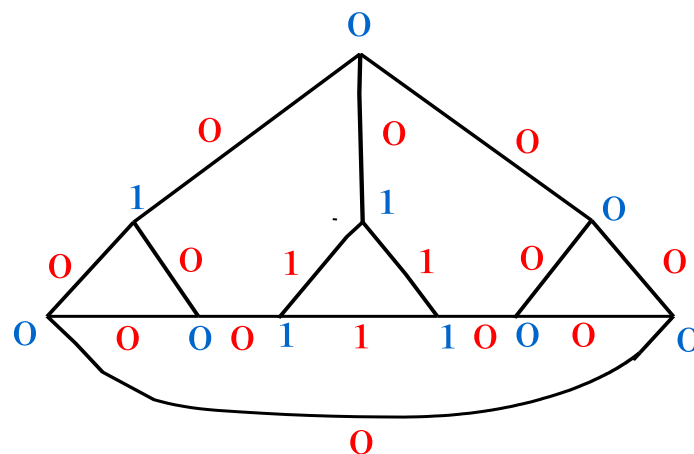
create more generalized and flexible systems. In Smarandachely product cordial labeling, the binary labeling of vertices is retained, but instead of using the sum modulo 2 to label edges, the edge label is determined by the product of the labels of the two incident vertices. That is, if two connected vertices are labeled with 1 and 0 respectively, their connecting edge is labeled as 0. This labeling is considered valid or cordial under the Smarandachely product cordial scheme if the number of vertices labeled 0 and 1 differ by at most one, and the number of edges labeled 0 and 1 also differs by at most one.

The Christmas tree industry has experienced significant growth in recent years, driven by increasing consumer demand for festive decorations and sustainable products. As consumers become more environmentally conscious and discerning in their purchasing decisions, companies are seeking innovative ways to communicate product value, sustainability, and authenticity. One approach gaining attention is Smarandachely product cordial labeling, a technique that emphasizes transparency, clarity, and consumer engagement. This study aims to analyze the application of Smarandachely product cordial labeling techniques in the Christmas tree industry, exploring their impact on consumer perceptions, preferences, and purchasing decisions.

2. Theorem 1: The Christmas tree CT (n) admits Smarandachely product cordial labeling where $n \geq 2$.

Proof: - Case1: When n is even.

1. Step to construct a tree $T = CT(n)$
 2. First, we join the root vertex u to the entire left l vertex and then to right r vertices and construct the Christmas tree.
 3. Label the tree $T = CT(n)$ when $n = 0, 2, 4, 6, \dots$ in the given step.
- (i) **Label all the even order of vertices 1, 1, 0, 0, 1, 1, 0, 0, respectively**



Graph 1.1: Smarandachely Product Cordial Labeling of CT (2)

Formula: $|v_f(0) - v_f(1)| \geq 2$

$|e_f(0) - e_f(1)| \geq 2$

(a) Generalization:

Where $n = 2$

Where $n = 4$

Where $n = 6$

Where $n = 8$

Where $n = 10$

Truth Table:

n	$v_f(0)$	$v_f(1)$	$\sum v_f(0) - v_f(1) \geq 2$
2	6	4	2
4	24	22	2
6	96	94	2
8	384	382	2
10	1556	1554	2

So on

Generalization:

$$v_f(0): a_n = 6 \times 4^{n-1} \quad \{n = 1, 2, 3, 4, \dots\}$$

$$6 \times 4^{1-1} = 6$$

$$6 \times 4^{2-1} = 24$$

$$6 \times 4^{3-1} = 96$$

$$6 \times 4^{4-1} = 384$$

$$6 \times 4^{5-1} = 1556$$

$$V_f(1): a_n = 27n^3 - 135n^2 + 234n - 122 \quad \{n = 1, 2, 3, 4, 5, \dots\}$$

$$27(1)^3 - 135(1)^2 + 234(1) - 122 = 4$$

$$27(2)^3 - 135(2)^2 + 234(2) - 122 = 22$$

$$27(3)^3 - 135(3)^2 + 234(3) - 122 = 94$$

$$27(4)^3 - 135(4)^2 + 234(4) - 122 = 382$$

Truth Table:-

So on.....

CT(n)	$e_f(0)$	$e_f(1)$	$\sum e_f(0) - e_f(1) \geq 2$
2	12	3	9
4	53	16	37
6	216	70	146
8	870	286	584
10	3485	574	2912

Generalization:

$$e_f(0): a_n = 61.5(n)^3 - 308(n)^2 + 534.5n - 276 \quad \{n=1,2,3,4,\dots\}$$

$$61.5(1)^3 - 308(1)^2 + 534.5(1) - 276 = 12$$

$$61.5(2) - 308(2)^2 + 534.5(2) - 276 = 53$$

$$61.5(3) - 308(3) + 534.5(3) - 276 = 216$$

$$61.5(4) - 308(4) + 534.5(4) - 276 = 870$$

$$61.5(5) - 308(5) + 534.5(5) - 276 = 3485$$

$$e_f(1): a_n = \frac{121}{6} n^3 - \frac{201}{2} n^2 + \frac{520}{3} n - 90 \quad \{n = 1, 2, 3, 4, \dots\}$$

$$\frac{121}{6}(1)^3 - \frac{201}{2}(1)^2 + \frac{520}{3}(1) - 90 = 3$$

$$\frac{121}{6}(2)^3 - \frac{201}{2}(2)^2 + \frac{520}{3}(2) - 90 = 16$$

$$\frac{121}{6}(3)^3 - \frac{201}{2}(3)^2 + \frac{520}{3}(3) - 90 = 70$$

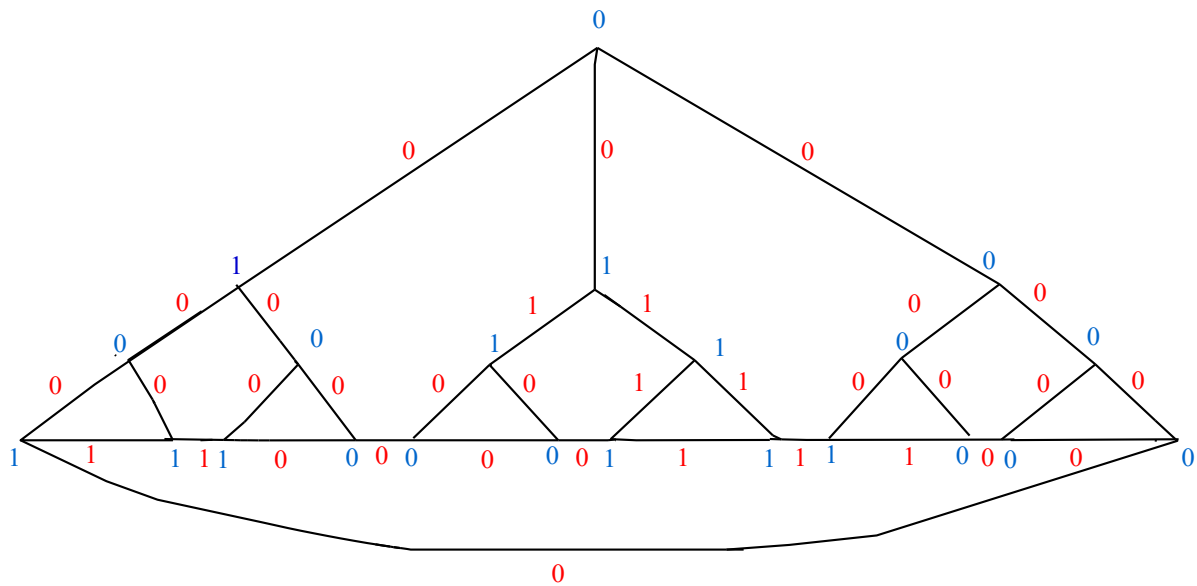
$$\frac{121}{6}(4)^3 - \frac{201}{2}(4)^2 + \frac{520}{3}(4) - 90 = 286$$

$$\frac{121}{6}(5)^3 - \frac{201}{2}(5)^2 + \frac{520}{3}(5) - 90 = 574$$

$$\sum |(61.5n^3 - 308n^2 + 534.5n - 276) - (\frac{121}{6}n^3 - \frac{201}{2}n^2 + \frac{520}{3}n - 90)| \geq 2 \quad (\text{approx})$$

Case 2: when n is odd

1. Step to Construct a tree T = CT (n).
 2. First, we join the root vertex u to all left l vertices and then to right r vertices and construct the Christmas tree.
 3. Label the tree T = CT (n) when n = 1, 3, 5, 7..... in the given step.
- (i) Label all the vertices with 1, 1, 0, 0, 1, 1, respectively.



Graph 1.2: Smarandachely product cordial labeling of CT (3)

Generalization:

- When n = 3
- When n = 5
- When n = 7
- When n = 9
- When n = 11

Truth table:-

CT (n)	v _f (0)	v _f (1)	∑ v _f (0) - v _f (1) ≥ 2
3	12	10	2
5	48	46	2
7	192	190	2

9	768	766	2
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So on.....

Generalization:

$$v_f(0): a_n = 12 \times 4^{n-1} \quad \{n = 1, 2, 3, 4, \dots\}$$

$$12 \times 4^{1-1} = 12$$

$$12 \times 4^{2-1} = 48$$

$$12 \times 4^{3-1} = 192$$

$$12 \times 4^{4-1} = 768$$

$$v_f(1): a_n = 3 \times 4^n - 2 \quad \{n = 1, 2, 3, 4, \dots\}$$

$$3 \times 4^1 - 2 = 10$$

$$3 \times 4^2 - 2 = 46$$

$$3 \times 4^3 - 2 = 190$$

$$3 \times 4^4 - 2 = 766$$

$$\sum | (12 \times 4^{n-1}) - (3 \times 4^n - 2) | \geq 2$$

Truth Table:

CT (n)	$e_f(0)$	$e_f(1)$	$\sum e_f(0) - e_f(1) \geq 2$
3	26	7	19
5	107	34	73
7	431	142	289
9	1742	574	1168

So on.....

$$e_f(0): a_n = 6.75 \times 4^n - 1 \quad \{n = 1, 2, 3, 4, \dots\}$$

$$6.75 \times 4^1 - 1 = 26$$

$$6.75 \times 4^2 - 1 = 107$$

$$6.75 \times 4^3 - 1 = 431$$

$$6.75 \times 4^4 - 1 = 1742$$

$$e_f(1): a_n = 2.25 \times 4^n - 2 \quad \{n = 1, 2, 3, 4, \dots\}$$

$$2.25 \times 4^1 - 2 = 7$$

$$2.25 \times 4^2 - 2 = 34$$

$$2.25 \times 4^3 - 2 = 142$$

$$2.25 \times 4^4 - 2 = 574$$

$$\sum | (6.75 \times 4^n - 1) - (2.25 \times 4^n - 2) | \geq 2 \quad (\text{Approx}).$$

Therefore, $|\sum v_f(0) - \sum v_f(1)| \geq 2$ and $|\sum e_f(0) - \sum e_f(1)| \geq 2$

Hence the Christmas tree CT (n) admits Smarandachely product cordial labeling where $n \geq 2$.

3. Properties of Smarandachely Product Cordial Labeling:

- 1. Cordiality:** The labeling satisfies the cordial condition, where the difference between the number of vertices with label 0 and label 1 is at most 1.
- 2. Product Cordiality:** The labeling is product cordial if the product of the labels of adjacent vertices satisfies certain conditions.
- 3. Smarandachely Property:** The labeling exhibits the Smarandachely property, which is a specific characteristic of the labeling.

- 4. Graph Structure:** Smarandachely product cordial labeling can be applied to various graph structures, including trees, cycles, and more.
- 5. Labeling Pattern:** The labeling pattern follows specific rules and conditions, ensuring the Smarandachely product cordial property.

4. Key Aspects:

Vertex Labeling: Assigning labels to vertices while satisfying the Smarandachely product cordial condition.

Edge Labeling: Deriving edge labels from the product of adjacent vertex labels.

Graph Properties: Analyzing the properties of graphs that admit Smarandachely product cordial labeling.

5. Research Significance:

Graph Theory: Smarandachely product cordial labeling contributes to the advancement of graph theory and its applications.

Network Analysis: Insights from Smarandachely product cordial labeling can inform network analysis and design.

Cryptography: Graph labeling techniques can be used to develop secure cryptographic protocols.

6. Applications:

- Graph Theory and Network Analysis:** Insights from this research can contribute to the development of new graph theoretical models and algorithms for network analysis.
- Computer Science and Data Structures:** The findings of this study can inform the design of efficient data structures and algorithms for complex networks.
- Cryptography and Network Security:** Graph labeling techniques can be used to develop secure cryptographic protocols and encryption methods for network security.
- Optimization and Resource Allocation:** Smarandachely product cordial labeling can be applied to optimize resource allocation and scheduling in complex systems.

7. Potential Impact:

- Advancements in Graph Theory:** This research can contribute to the advancement of graph theory, enabling new discoveries and a deeper understanding of complex networks.
- Improved Network Efficiency:** The application of Smarandachely product cordial labeling can result in more efficient and robust network designs.
- Enhanced Data Security:** Graph labeling techniques can be used to develop secure cryptographic protocols, enhancing data security and protection.

8. Future Research Directions:

- Exploring New Applications:** Further research can explore new applications of Smarandachely product cordial labeling in various fields, such as biology, physics, and social sciences.
- Developing Efficient Algorithms:** Developing efficient algorithms for Smarandachely product cordial labeling can facilitate its practical applications.
- Investigating Theoretical Foundations:** Investigating the theoretical foundations of Smarandachely product cordial labeling can lead to new insights and discoveries.

9. Conclusion

This research paper has presented a comprehensive analysis of Smarandachely product cordial labeling techniques applied to Christmas trees. The study has explored the theoretical foundations of these labeling techniques and their applications in graph theory. The findings of this research contribute to the advancement of knowledge in the field of graph labeling, providing insights into the structural properties of Christmas trees and their labeling patterns. The results of this study can be used to inform future research in graph theory and its applications.

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