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Secure Fair Domination in Graphs

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Abstract
Let G be a connected simple graph. A dominating set $S \subseteq V(G)$ is a fair dominating set in G if $S = V(G)$ or if $S \neq V(G)$ and all vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S| > 0$ for every two vertices $u, v \in V(G) \setminus S$. A fair dominating set S of $V(G)$ is a secure fair dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a fair dominating set of G. The minimum cardinality of a secure fair dominating set of G, denoted by $\gamma_{sfd}(G)$, is called the secure fair domination number of G. In this paper, we initiate a study of secure fair domination in graphs and give some important results.

Mathematics Subject Classification: 05C69

Keywords: Dominating Set, Secure Dominating Set, Fair Dominating Set, Secure Fair Dominating Set

1. INTRODUCTION

In [1], Claude Berge and Oystein Ore introduced the domination in graph. Claude Berge, a French mathematician, and Oystein Ore, a Norwegian-American mathematician, are considered pioneers of graph theory, particularly in the area of domination theory. Berge introduced the concept of the "coefficient of external stability" (now called the domination number) in 1958, while Ore formalized "dominating sets" and the domination number in 1962, building on Berge's work. "Towards a Theory of Domination in Graphs" [2], is a seminal 1977 paper by Cockayne and Hedetniemi that laid the groundwork for the study of domination in graphs. It introduced key concepts like dominating sets, the domatic number, and the relationship between domination and graph colorings, providing a foundational framework for later research on network analysis, optimization, and other applications The contributions of Claude Berge and Oystein Ore, Cockayne and Hedetniemi in the area of domination in graphs became an area of study by many researchers [3,4,5,6,7,8,9,10,11,12,13].

Secure domination in graphs was studied and introduced by E.J. Cockayne et.al [14,15]. Accordingly, a dominating set S of $V(G)$ is a secure dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the secure domination number of G. In [16] Enriquez and Canoy, introduced a variant of secure domination in graphs, the concept of secure convex domination in graphs. Some studies on secure domination in graphs were found in the paper [17,18,19,20,21,22,23].

In 2011, Caro, Hansberg and Henning [24] introduced fair domination and k-fair domination in graphs. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinalities $\gamma_{fd}(G)$ is called γ_{fd}-set. Some studies on fair domination in graphs were found in the paper [25,26,27,28,29,30,31,32,33].

For the general concepts, the reader may refer to [34]. Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{ u \in V(G): uv \in E(G) \}$. If $S \subseteq V(G)$, then the open neighborhood of S is the set $N_G(S) = N(S) = \cup_{v \in S} N_G(v)$. The closed neighborhood of S is $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a dominating set of G if for every $v \in (V(G) \setminus S)$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

A fair dominating set S of $V(G)$ is a secure fair dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a fair dominating set of G. The minimum cardinality of a secure fair dominating set of G, denoted by $\gamma_{sfd}(G)$, is called the secure fair domination number of G. In this paper, we initiate a study of secure fair domination in graphs and give some important results.

2. Results

2.1 Realization Problem

The following result implies that the secure fair dominating sets exist in nontrivial connected simple graphs, and the secure fair domination number of a graph G ranges across all positive integers from 1 to $n - 1$.

Theorem 2.1 Let a, b and n be positive integers such that $1 \leq a \leq b \leq n - 1$. Then there exists a connected nontrivial graph G with $|V(G)| = n$ such that $\gamma_{fd}(G) = a$ and $\gamma_{sfd}(G) = b$.

Proof. Let a, b and n be positive integers such that $1 \leq a \leq b \leq n - 1$. Consider the following cases:

Case 1: Suppose that $1 = a = b \leq n - 1$. Consider the graph $G = K_n,\ n \geq 2$. Clearly, $|V(G)| = n$ and $\gamma_{fd}(G) = 1 = a = b = \gamma_{sfd}(G) \leq n - 1$.

Case 2: Suppose that $1 = a < b = n - 1$. Consider the graph $G = S_n$, where $S_n = K_1 + \widetilde{K}_b = K_{1,b}, b \geq 2$, and $V(G) = \{c, v_1, v_2, \dots, v_b\}$. Then the set $A = \{c\}$ is the γ_{fd}-set and the set $B = \{v_1, v_2, \dots, v_b\}$ is the γ_{sfd}-set in G. Thus, $|V(G)| = |A| + |B| = 1 + b = n,\ \gamma_{fd}(G) = |A| = 1 = a$, and $\gamma_{sfd}(G) = |B| = b$. Thus, $1 = a < b = n - 1$.

Case 3: Suppose that $1 < a < b < n - 1$. Consider the graph $G = P_a \circ K_m$ where $P_a = [v_1, v_2, \dots, v_a]$, $a \geq 2$ and $K_m = [u_1, u_2, \dots, u_m],\ m \geq 2$. Then

$$V(G) = V(P_a \circ K_m) = V(P_a) \cup (\cup_{k=1}^{a} V_k(K_m))$$
$$= V(P_a) \cup (V_1(K_m) \cup V_2(K_m) \cup \ldots \cup V_k(K_m), \ldots, V_a(K_m)).$$

Let $V_k(K_m) = \{u_1^k, u_2^k, \dots, u_m^k\}$. Then
$V(G) = \{v_1, v_2, \dots, v_a\} \cup \{u_1^1, u_2^1, \dots, u_m^1\} \cup \dots \cup \{u_1^a, u_2^a, \dots, u_m^a\}$.
The set $A = V(P_a)$ is the γ_{fd}-set and the set

$$B = V(P_a) \cup \left(\bigcup_{k=1}^{a} \{u_1^k\} \right) = \{v_1, v_2, \dots, v_a\} \cup \{u_1^1, u_1^2, \dots, u_1^a\}$$

is the γ_{sfd}-set in G. Thus, $\gamma_{fd}(G) = a$, $\gamma_{sfd}(G) = a + a = 2a = b$, and $|V(G)| = a + am = n$. That is, $1 < a < b = 2a < (a + am) - 1 = n - 1$. ∎
The following result is an immediate consequence of Theorem 2.1.

Corollary 2.2 The difference $\gamma_{sfd}(G) - \gamma_{fd}(G)$ can be made arbitrarily large.
Proof. Let $G = P_r \circ K_s$ where $P_r = [v_1, v_2, \dots, v_r]$, $r \geq 2$ and $K_s = [u_1, u_2, \dots, u_s]$, $s \geq 2$. Then $\gamma_{fd}(G) = r$ and $\gamma_{sfd}(G) = 2r$ by Theorem 2.1. Thus, $\gamma_{sfd}(G) - \gamma_{fd}(G) = 2r - r = r$. ∎

2.2 Special Graphs

Definition 2.3 A simple path $P_n = [v_1, v_2, \dots, v_n]$ is a special graph of order $n \geq 1$ with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$.
Remark 2.4 Let $G = P_n$ and $2 \leq n \leq 5$. Then

$$\gamma_{sfd}(G) = \begin{cases} 1, & if\ n = 2 \\ 2, & if\ n = 3 \\ 3, & if\ n = 4\ or\ n = 5. \end{cases}$$

The following result is the secure fair domination number for a path P_n of order $n \geq 6$.
Proposition 2.5 Let $G = P_n$ and $n \geq 6$. For all integer $k \geq 1$,

$$\gamma_{sfd}(G) = \begin{cases} \dfrac{3n+5}{5} & if\ n = 5k+5, \\ \dfrac{3n+2}{5} & if\ n = 5k+1, \\ \dfrac{3n+4}{5} & if\ n = 5k+2, \\ \dfrac{3n+6}{5} & if\ n = 5k+3, \\ \dfrac{3n+3}{5} & if\ n = 5k+4. \end{cases}$$

Proof. Let $G = P_n = [v_1, v_2, \dots, v_n]$, $n \geq 6$ and for all integer $k \geq 1$. Consider the following cases.
Case 1. Suppose that $n = 5k + 5$. The set

$$S = \{v_1, v_3, v_4, \dots, v_k, v_{k+2}, v_{k+3}, \dots, v_{n-5}, v_{n-6}, v_{n-4}, v_{n-2}, v_{n-1}, v_n\}$$

is a secure fair dominating set of G.
For $k = 1$, $n = 10$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{10}\}| = 7$.
For $k = 2$, $n = 15$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{15}\}| = 10$.
For $k = 3$, $n = 20$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}, v_{20}\}| = 13$.
By observing the pattern, the order of G is $n = 25, 30, 35, \dots, 5k + 5$ for $k = 4, 5, 6, \dots$ and $|S| = 16, 19, 22, \dots, 3k + 4$ for $k = 4, 5, 6, \dots$. Since $k = \frac{n-5}{5}$, it follows that $|S| = 3\left(\frac{n-5}{5}\right) + 4 = \frac{3n+5}{5}$.

Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+5}{5}$ if $n = 5k + 5$ for all $k \geq 1$.

Case 2. Suppose that $n = 5k + 1$. The set

$$S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-5}, v_{n-3}, v_{n-2}, v_n\}$$

is a secure fair dominating set of G.
For $k = 1, n = 6$ and $|S| = |\{v_1, v_3, v_4, v_6\}| = 4$.
For $k = 2, n = 11$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}\}| = 7$.
For $k = 3, n = 16$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}\}| = 10$.
By observing the pattern, $n = 21,26,31, \ldots, 5k + 1$ for $k = 4,5,6, \ldots$ and $|S| = 13,16,19, \ldots, 3k + 1$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-1}{5}$, it follows that $|S| = 3\left(\frac{n-1}{5}\right) + 1 = \frac{3n+2}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+2}{5}$ if $n = 5k + 1$ for all $k \geq 1$.
Case 3. Suppose that $n = 5k + 2$. The set $S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}$ is a secure fair dominating set of G. *For* $k = 1, n = 7$ and $|S| = |\{v_1, v_3, v_4, v_6, v_7\}| = 5$. For $k = 2$, $n = 12$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{12}\}| = 8$.
For $k = 3$, $n = 17$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{17}\}| = 11$. By observing the pattern, $n = 22,27,32, \ldots, 5k + 2$ for $k = 4,5,6, \ldots$ and $|S| = 14,17,20, \ldots, 3k + 2$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-2}{5}$, it follows that $|S| = 3\left(\frac{n-2}{5}\right) + 2 = \frac{3n+4}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+4}{5}$ if $n = 5k + 2$ for all $k \geq 1$.

Case 4. Suppose that $n = 5k + 3$. The set

$$S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n\}$$

is a secure fair dominating set of G. For $k = 1, n = 8$ and $|S| = |\{v_1, v_3, v_4, v_6, v_7, v_8\}| = 6$. For $k = 2$, $n = 13$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{12}, v_{13}\}| = 9$.
For $k = 3, n = 18$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{17}, v_{18}\}| = 12$.
By observing the pattern, $n = 23,28,33, \ldots, 5k + 3$ for $k = 4,5,6, \ldots$ and $|S| = 15,18,21, \ldots, 3k + 3$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-3}{5}$, it follows that $|S| = 3\left(\frac{n-3}{5}\right) + 3 = \frac{3n+6}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+6}{5}$ *if* $n = 5k + 3$ for all $k \geq 1$.

Case 5. Suppose that $n = 5k + 4$. The set

$$S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-6}, v_{n-5}, v_{n-3}, v_{n-1}, v_n\}$$

is a secure fair dominating set of G. For $k = 1, n = 9$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9\}| = 6$. For $k = 2$, $n = 14$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}\}| = 9$.
For $k = 3, n = 19$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}\}| = 12$.
By observing the pattern, $n = 24,29,34, \ldots, 5k + 4$ for $k = 4,5,6, \ldots$ and $|S| = 15,18,21, \ldots, 3k + 3$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-4}{5}$, it follows that $|S| = 3\left(\frac{n-4}{5}\right) + 3 = \frac{3n+3}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is

not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+3}{5}$ if $n = 5k + 4$ for all $k \geq 1$. ■

Definition 2.6 The cycle $C_n = [v_1v_2, v_2v_3, \ldots, v_nv_1]$ is a special graph with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$ where $n \geq 3$.

Remark 2.7 Let $G = C_n$ and $3 \leq n \leq 5$. Then $\gamma_{sfd}(G) = n - 2,\ if\ n = 3\ or\ n = 4\ or\ n = 5$.

The following result is the secure fair domination number for a cycle C_n of order $n \geq 6$.

Proposition 2.8 Let $G = C_n$ and $n \geq 6$. For all integer $k \geq 1$,

$$\gamma_{sfd}(G) = \begin{cases} \frac{3n+2}{5} & if\ n = 5k+1, \\ \frac{3n+4}{5} & if\ n = 5k+2, \\ \frac{3n+1}{5} & if\ n = 5k+3, \\ \frac{3n+3}{5} & if\ n = 5k+4, \\ \frac{3n}{5} & if\ n = 5k+5. \end{cases}$$

Proof. Let $G = C_n = [v_1, v_2, \ldots, v_n, v_1]$, $n \geq 6$ and for all integer $k \geq 1$. Consider the following cases.

Case 1. Suppose that $n = 5k + 1$. The set $S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-5}, v_{n-3}, v_{n-2}, v_n\}$ is a secure fair dominating set of G. For $k = 1$, $n = 6$ and $|S| = |\{v_1, v_3, v_4, v_6\}| = 4$. For $k = 2$, $n = 11$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}\}| = 7$.

For $k = 3$, $n = 16$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}\}| = 10$.

By observing the pattern, $n = 21,26,31, \ldots, 5k + 1$ for $k = 4,5,6, \ldots$ and $|S| = 13,16,19, \ldots, 3k + 1$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-1}{5}$, it follows that $|S| = 3\left(\frac{n-1}{5}\right) + 1 = \frac{3n+2}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+2}{5}\ if\ n = 5k + 1$ for all $k \geq 1$.

Case 2. Suppose that $n = 5k + 2$. The set $S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}$ is a secure fair dominating set of G. For k =1, $n = 7$ and $|S| = |\{v_1, v_3, v_4, v_6, v_7\}| = 5$. For $k = 2$, $n = 12$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{12}\}| = 8$.

For $k = 3$, $n = 17$ and $S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{17}\}| = 11$.

By observing the pattern, $n = 22,27,32, \ldots, 5k + 2$ for $k = 4,5,6, \ldots$ and $|S| = 14,17,20, \ldots, 3k + 2$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-2}{5}$, it follows that $|S| = 3\left(\frac{n-2}{5}\right) + 2 = \frac{3n+4}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+4}{5}\ if\ n = 5k + 2\ for\ all\ k \geq 1$.

Case 3. Suppose that $n = 5k + 3$. The set $S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-5}, v_{n-4}, v_{n-2}, v_n\}$ is a secure fair dominating set of G.

For $k = 1$, $n = 8$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8\}| = 5$.

For $k = 2$, $n = 13$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}\}| = 8$.

For $k = 3$, $n = 18$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}\}| = 11$.

By observing the pattern, $n = 23,28,33, \ldots, 5k + 3$ for $k = 4,5,6, \ldots$ and $|S| = 14,17,20, \ldots, 3k + 2$ for

$k = 4,5,6,\ldots$. Since $k = \frac{n-3}{5}$, it follows that $|S| = 3\left(\frac{n-3}{5}\right) + 2 = \frac{3n+1}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+1}{5}$ if $n = 5k + 3$ for all $k \geq 1$.

Case 4. Suppose that $n = 5k + 4$. The set

$$S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-6}, v_{n-5}, v_{n-3}, v_{n-1}, v_n\}$$

is a secure fair dominating set of G. For $k = 1$, $n = 9$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9\}| = 6$. For $k = 2$, $n = 14$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}\}| = 9$.
For $k = 3$, $n = 19$ and

$$|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}\}| = 12.$$

By observing the pattern, $n = 24,29,34,\ldots,5k+4$ for $k = 4,5,6,\ldots$ an $|S| = 15,18,21,\ldots,3k+3$ for $k = 4,5,6,\ldots$. Since $k = \frac{n-4}{5}$, it follows that $|S| = 3\left(\frac{n-4}{5}\right) + 3 = \frac{3n+3}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+3}{5}$ if $n = 5k + 4$ for all $k \geq 1$.

Case 5. Suppose that $n = 5k + 5$. The set

$$S = \{v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-7}, v_{n-6}, v_{n-4}, v_{n-2}, v_{n-1}\}$$

is a secure fair dominating set of G.
For $k = 1$, $n = 10$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9\}| = 6$.
For $k = 2$, $n = 15$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}\}| = 9$.
For $k = 3$, $n = 20$ and $|S| = |\{v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}\}| = 12$.
By observing the pattern, the order of G is $n = 25,30,35,\ldots,5k+5$ for $k = 4,5,6,\ldots$ and $|S| = 15,18,21,\ldots,3k+3$ for $k = 4,5,6,\ldots$. Since $k = \frac{n-5}{5}$, it follows that $|S| = 3\left(\frac{n-5}{5}\right) + 3 = \frac{3n}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n}{5}$ if $n = 5k + 5$ for all $k \geq 1$. ■

Definition 2.9 The complete graph K_n is a special graph of order n where every pair of vertices is adjacent. The following remark gives the secure fair domination number of the complete graph K_n of order n.
Remark 2.10 Let $G = K_n$ for all $n \geq 2$. Then $\gamma_{sfd}(G) = 1$.
Definition 2.11 The fan F_n is the special graph of order $n + 1$ with $V(F_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ and $E(F_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{v_0v_i : i = 1,2,\ldots,n\}$.
Remark 2.12 Let $G = F_n$ of order $n + 1$. Then

$$\gamma_{sfd}(G) = \begin{cases} 1, & if\ n = 2 \\ 3, & if\ n = 3 \\ 4, & if\ n = 5. \end{cases}$$

The following result shows the secure fair domination number of a fan F_n of order $n + 1$.

Proposition 2.13 Let $G = F_n$ and $n = 4$ or $n \geq 6$. For all integer $k \geq 1$,

$$\gamma_{sfd}(G) = \begin{cases} \frac{3n+8}{5} & if\ n = 5k-1, \\ \frac{3n+7}{5} & if\ n = 5k+1, \\ \frac{3n+9}{5} & if\ n = 5k+2, \\ \frac{3n+11}{5} & if\ n = 5k+3, \\ \frac{3n+10}{5} & if\ n = 5k+5. \end{cases}$$

Proof. Let $G = F_n = [v_0, v_1, v_2, \ldots, v_n]$, $n = 4$ or $n \geq 6$ and for all integer $k \geq 1$. Consider the following cases.

Case 1. Suppose that $n = 5k - 1$. The set $S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-3}, v_{n-1}, v_n\}$ is a secure fair dominating set of G. For $k = 1, n = 4$ and $|S| = |\{v_0, v_1, v_3, v_4\}| = 4$. For $k = 2, n = 9$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9\}| = 7$.

For $k = 3, n = 14$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}\}| = 10$.

By observing the pattern, $n = 19,24,29, \ldots, 5k - 1$ for $k = 4,5,6, \ldots$ and $|S| = 13,16,19, \ldots, 3k + 1$ for $k = 4,5,6, \ldots$. Since $k = \frac{n+1}{5}$, it follows that $|S| = 3\left(\frac{n+1}{5}\right) + 1 = \frac{3n+8}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd} = |S| = \frac{3n+8}{5}$ if $n = 5k - 1$ for all $k \geq 1$.

Case 2. Suppose that $n = 5k + 1$. The set $S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-3}, v_{n-2}, v_n\}$ is a secure fair dominating set of G. For $k = 1, n = 6$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6\}| = 5$. For $k = 2, n = 11$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}\}| = 8$.

For $k = 3, n = 16$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}\}| = 11$.

By observing the pattern, $n = 21,26,31, \ldots, 5k + 1\ for\ k = 4,5,6, \ldots$ and $|S| = 14,17,20, \ldots, 3k + 2$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-1}{5}$, it follows that $|S| = 3\left(\frac{n-1}{5}\right) + 2 = \frac{3n+7}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+7}{5}$ if $n = 5k + 1$ for all $k \geq 1$.

Case 3. Suppose that $n = 5k + 2$. The set $S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}$ is a secure fair dominating set of G. For $k = 1, n = 7$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_7\}| = 6$. For $k = 2$, $n = 12$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{12}\}| = 9$.

For $k = 3, n = 17$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{17}\}| = 12$.

By observing the pattern, $n = 22,27,32, \ldots, 5k + 2$ for $k = 4,5,6, \ldots$ and $|S| = 15,18,21, \ldots, 3k + 3$ for $k = 4,5,6, \ldots$. Since $k = \frac{n-2}{5}$, it follows that $|S| = 3\left(\frac{n-2}{5}\right) + 3 = \frac{3n+9}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+9}{5}$ if $n = 5k + 2$ for all $k \geq 1$.

Case 4. Suppose that $n = 5k + 3$. The set

$$S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n\}$$

is a secure fair dominating set of G. For $k = 1, n = 8$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_7, v_8\}| = 7$. For $k = 2, n = 13$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{12}, v_{13}\}| = 10$.

For $k = 3$, $n = 18$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{17}, v_{18}\}| = 13$.
By observing the pattern, $n = 23,28,33,\dots,5k+3$ for $k = 4,5,6,\dots$ and $|S| = 16,19,22,\dots,3k+4$ for $k = 4,5,6,\dots$. Since $k = \frac{n-3}{5}$, it follows that $|S| = 3\left(\frac{n-3}{5}\right) + 4 = \frac{3n+11}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+11}{5}$ if $n = 5k + 3$ for all $k \geq 1$.
Case 5. Suppose that $n = 5k + 5$. The set $S = \{v_0, v_1, v_3, v_4, \dots, v_k, v_{k+2}, v_{k+3}, \dots, v_{n-4}, v_{n-2}, v_{n-1}, v_n\}$ is a secure fair dominating set of G. For $k = 1$, $n = 10$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{10}\}| = 8$. For $k = 2$, $n = 15$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{15}\}| = 11$.
For $k = 3$, $n = 20$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}, v_{20}\}| = 14$.
By observing the pattern, $n = 25,30,35,\dots$,5k+5 for $k = 4,5,6,\dots$ and $|S| = 17,20,23,\dots,3k+5$ for $k = 4,5,6,\dots$. Since $k = \frac{n-5}{5}$, it follows that $|S| = 3\left(\frac{n-5}{5}\right) + 5 = \frac{3n+10}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+10}{5}$ if $n = 5k + 5$ for all $k \geq 1$. ∎

Definition 2.14 The wheel W_n is the special graph of order $n + 1$ with $V(W_n) = \{v_0, v_1, v_2, \dots, v_n\}$ and $E(W_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\} \cup \{v_0v_i : i = 1,2,\dots,n\}$ where $n \geq 3$. The following result shows the secure fair domination number of a wheel W_n where $n \geq 3$.
Remark 2.15 Let $G = W_n$ of order $n + 1$ and $3 \leq n \leq 5$. Then

$$\gamma_{sfd}(G) = \begin{cases} 1, & if\ n = 3 \\ 3, & if\ n = 4 \\ 4, & if\ n = 5. \end{cases}$$

The following result shows the secure fair domination number of a wheel W_n of order $n + 1$.

Proposition 2.16 Let $G = W_n$ and $n \geq 6$. For all integer $k \geq 1$,

$$\gamma_{sfd}(G) = \begin{cases} \frac{3n+7}{5} & if\ n = 5k+1, \\ \frac{3n+9}{5} & if\ n = 5k+2, \\ \frac{3n+6}{5} & if\ n = 5k+3, \\ \frac{3n+8}{5} & if\ n = 5k+4, \\ \frac{3n+5}{5} & if\ n = 5k+5. \end{cases}$$

Proof. Let $G = W_n$, where $V(G) = \{v_0, v_1, v_2, \dots, v_n\}$, $n \geq 6$. For all integer $k \geq 1$, consider the following cases.
Case 1. Suppose that $n = 5k + 1$. The set $S = \{v_0, v_1, v_3, v_4, \dots, v_k, v_{k+2}, v_{k+3}, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_n\}$ is a secure fair dominating set of G. For $k = 1$, $n = 6$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6\}| = 5$. For $k = 2$, $n = 11$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}\}| = 8$.
For $k = 3$, $n = 16$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}\}| = 11$.
By observing the pattern, $n = 21,26,31,\dots,5k+1$ for $k = 4,5,6,\dots$ and $|S| = 14,17,20,\dots,3k+2$ for

$k = 4,5,6,\ldots$. Since $k = \frac{n-1}{5}$, it follows that $|S| = 3\left(\frac{n-1}{5}\right) + 2 = \frac{3n+7}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+7}{5}$ if $n = 5k+1$ for all $k \geq 1$.

Case 2. Suppose that $n = 5k+2$. The set $S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-4}, v_{n-3}, v_{n-1}, v_n\}$ is a secure fair dominating set of G. For $k = 1$, $n = 7$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_7\}| = 6$. For $k = 2$, $n = 12$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{12}\}| = 9$.

For $k = 3$, $n = 17$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{17}\}| = 12$.

By observing the pattern, $n = 22,27,32,\ldots,5k+2$ for $k = 4,5,6,\ldots$ and $|S| = 15,18,21,\ldots,3k+3$ for $k = 4,5,6,\ldots$. Since $k = \frac{n-2}{5}$, it follows that $|S| = 3\left(\frac{n-2}{5}\right) + 3 = \frac{3n+9}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+9}{5}$ if $n = 5k+2$ for all $k \geq 1$.

Case 3. Suppose that $n = 5k+3$. The set $S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-5}, v_{n-4}, v_{n-2}, v_n\}$ is a secure fair dominating set of G. For $k = 1$, $n = 8$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8\}| = 6$. For $k = 2$, $n = 13$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}\}| = 9$.

For $k = 3$, $n = 18$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}\}| = 12$.

By observing the pattern, $n = 23,28,33,\ldots,5k+3$ for $k = 4,5,6,\ldots$ and $|S| = 15,18,21,\ldots,3k+3$ for $k = 4,5,6,\ldots$. Since $k = \frac{n-3}{5}$, it follows that $|S| = 3\left(\frac{n-3}{5}\right) + 3 = \frac{3n+6}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+6}{5}$ if $n = 5k+3$ for all $k \geq 1$.

Case 4. Suppose that $n = 5k+4$. The set $S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-5}, v_{n-3}, v_{n-1}, v_n\}$ is a secure fair dominating set of G. For $k = 1$, $n = 9$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9\}| = 7$. For $k = 2$, $n = 14$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}\}| = 10$.

or $k = 3$, $n = 19$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}\}| = 13$.

By observing the pattern, $n = 24,29,34,\ldots,5k+4$ for $k = 4,5,6,\ldots$ and $|S| = 16,19,22,\ldots,3k+4$ for $k = 4,5,6,\ldots$. Since $k = \frac{n-4}{5}$, it follows that $|S| = 3\left(\frac{n-4}{5}\right) + 4 = \frac{3n+8}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+8}{5}$ if $n = 5k+4$ for all $k \geq 1$.

Case 5. Suppose that $n = 5k+5$. The set $S = \{v_0, v_1, v_3, v_4, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{n-4}, v_{n-2}, v_{n-1}\}$ is a secure fair dominating set of G. For $k = 1$, $n = 10$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9\}| = 7$. For $k = 2$, $n = 15$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}\}| = 10$.

For $k = 3$, $n = 20$ and $|S| = |\{v_0, v_1, v_3, v_4, v_6, v_8, v_9, v_{11}, v_{13}, v_{14}, v_{16}, v_{18}, v_{19}\}| = 13$.

By observing the pattern, $n = 25,30,35,\ldots,5k+5$ for $k = 4,5,6,\ldots$ and $|S| = 16,19,22,\ldots,3k+4$ for $k = 4,5,6,\ldots$. Since $k = \frac{n-5}{5}$, it follows that $|S| = 3\left(\frac{n-5}{5}\right) + 4 = \frac{3n+5}{5}$. Observe that if $x \in S$, $S \setminus \{x\}$ is not a secure fair dominating set of G. Thus, S must be a minimum secure fair dominating set of G, that is, $\gamma_{sfd}(G) = |S| = \frac{3n+5}{5}$ if $n = 5k+5$ for all $k \geq 1$. ■

Conclusion and Recommendations

In this paper, we introduced a new parameter of domination in graphs - the secure fair domination in graphs. The existence of the secure fair domination in graph was explored. Next, the secure fair domination number resulting of some special graphs were presented and computed. This study will pave a way to new researches such as bounds and binary operations of two connected graphs. Other parameters relating secure fair domination in graphs may also be explored. Finally, the characterization of a secure fair domination in graphs is a promising extension of this study.

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