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Finite Near Rings and Finite Division Rings

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Abstract
We provide counter examples of finite near ring, which is not a ring. It extends the result to finite division rings. We provide counter example of a finite division ring, which is not a field. **It contradicts the Wedderburn's Little Theorem,** asserts that all finite division rings are commutative and therefore finite field.

Keywords: Near Ring, Finite Ring, Devision Ring, Finite Division Ring

1. Introduction
A near-ring is an algebraic structure similar to a ring but relaxes some of the ring **axioms.** Every finite near ring is not necessarily a ring. However certain conditions can make a finite ring resemble specific types of rings or if a finite near ring satisfies all the axioms of a ring (i-e) Distributivity in directions it becomes a finite ring. In a finite near ring there are only a finite number of subsets. So the number of possible ideals is finite. Lagrange's theorem is also applied to finite near rings.
A non-commutative ring in which the non-zero elements from a group is called a division ring. A commutative division ring is a field. **Wedderburn's Little Theorem,** asserts that all finite division rings are commutative and therefore finite field. We found the counter example of finite division ring, which is not a field. **It contradicts the Wedderburn's Little Theorem**

2. Preliminaries
A Ring is the second algebraic system of the subject of modern algebra. The abstract concept of rings has its origin in the set of integers unlike. The groups have one binary operation the ring has two binary operations, which are usually called addition and multiplication.
Definition2.1: Let "+" and "." be two binary operations on a nonempty R. Then (R, +, .) is said to be a ring, for all $a, b, c \in R$
$R_1: a + b = b + a$
$R_2: (a + b) + c = a + (b + c)$
R_3 there exit $0 \in R$ such that $+(-a) = 0$ for $a \in R$
R_4: There exits $-a \in$ such that $a + (-a) = 0$ for $a \in R$
$R_5: a.(b + c) = a.b + a.c \; and \; (b + c).a = b.a + c.a$
The operation "+" is called the addition and the operation "." is called the multiplication in the ring (R, +, .)
Definition2.2: A near ring is a set N together with two binary operations "+" and "." Such that
(a) (N, +) is a group not necessarily Abelian
(b) (N, .) is a semi-group

(c) For all, $n_1, n_2, n_3 \in N, n_1(n_2 + n_3) = n_1.n_2 + n_1.n_3$ then we call N a left near ring.
Note $(n_1 + n_2).n_3 = n_1.n_3 + n_1.n_3$ then we call N a right near-ring.
Definition2.3: A ring is called a division ring if non zero elements from a group under multiplication.
Definition2.4: A field is a commutative division ring.

3. Finite near Ring

We provide counter examples of finite near ring, which are not a ring.
Example3.1: Let "$+_2$" and "$\times_2$" be two binary operations on a nonempty set of matrices
S = {A, B, C, D}, where
$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
The addition and multiplication tables are as follows:

$+_2$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

Clearly (S, $+_2$) is group.

$\times_2$	A	B	C	D
A	A	A	A	A
B	A	C	B	D
C	A	B	C	D
D	A	B	D	C

Clearly (S, $\times_2$) is a semi group.
Also $A \times_2 (B +_2 C) = (A \times_2 B) +_2 (B \times_2 C)$ for all A, B, C ϵ S
So (S, $+_2$, $\times_2$) is a finite left near ring.
But it is not a finite right near ring because

$$Clearly\ (B +_2 C) \times_2 D = D \times_2 D = C$$

and $(B \times_2 D) +_2 (C \times_2 D) = C +_2 B = D$
$(B +_2 C) \times_2 D \neq (B \times_2 D) +_2 (C \times_2 D)$
So right distributive law not satisfies.
So (S, $+_2$, $\times_2$) is not a ring.
Hence (S, $+_2$, $\times_2$) is a finite near ring but not a finite ring.
Example3.2: Let F = { f, g, h, t} is a set of functions from Z_2 to Z_2, where
f(x) = 0, g(x) = x, h(x) = 1-x, t(x) = 1.
We define addition and multiplication are as follows:
(f+g)(x) = f(x) +g(x) and (fg)(x) = f(g(x)) for all f,g ∈F, x ∈ Z_2.

The addition and multiplication tables are as follows:

+	f	g	h	t
f	f	g	h	t
g	g	f	t	h
h	h	t	f	g
t	t	h	g	f

Clearly (F, +) is group.

.	f	g	h	t
f	f	f	f	f
g	f	g	h	t
h	f	h	g	t
t	f	t	t	t

Clearly (F, .) is a semi group.
(F, + , .) is a finite right near ring but it is not a finite left near ring because
h.(g+t) ≠ h.g+ h.t
Hence (F, + , .) is a finite near ring but not a finite ring.
Example3.3: Let Q = $\{\propto_0 + \propto_1 i + \propto_2 j + \propto_3 k \,/ \propto_0, \propto_1, \propto_2, \propto_3 \in Z_2\}$
Where i, j, k are quarter units.
Let $X, Y \in$ Q such that $X = \propto_0 + \propto_1 i + \propto_2 j + \propto_3 k = (\propto_0 \propto_1 \propto_2 \propto_3)$
$Y = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k = (\beta_0 \beta_1 \beta_2 \beta_3)$
We define addition (+) and multiplication (×) are as follows:

$$X + Y = (\propto_0 + \beta_0) + (\propto_1 + \beta_1)\bar{\imath} + (\propto_2 + \beta_2)\bar{\jmath} + (\propto_3 + \beta_3)\bar{k}$$

$$X \times Y = (\propto_0.\beta_0 - \propto_1.\beta_1 - \propto_2.\beta_2 - \propto_3.\beta_3) + (\propto_0.\beta_1 + \propto_1.\beta_0 + \propto_2.\beta_3 - \propto_3.\beta_2)\bar{\imath} + (\propto_0.\beta_2 + \propto_2.\beta_0 + \propto_3.\beta_1 - \propto_1.\beta_3)\bar{\jmath} + (\propto_0.\beta_3 - \propto_3.\beta_0 - \propto_1.\beta_2 - \propto_2.\beta_1)\bar{k}$$

Let a=(0000), b=(0001), c=(0010), d=(0011), e=(0100), f=(0101), g=(0110), h=(0111)
i=(1000), j=(1001), k=(1010), l=(1011), m=(1100), n=(1101), o=(1110), p=(1111)
The addition and multiplication tables are as follows

+	**a**	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**	**j**	**k**	**l**	**m**	**n**	**o**	**p**
a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
b	b	a	d	c	f	e	h	g	j	i	l	k	n	m	p	o
c	c	d	a	b	g	h	e	f	k	l	i	j	o	p	m	n
d	d	c	b	a	h	g	f	e	l	k	j	i	p	o	n	m
e	e	f	g	h	a	b	c	d	m	n	o	p	i	j	k	l
f	f	e	h	g	b	a	d	c	n	m	p	o	j	i	l	k
g	g	h	e	f	c	d	a	b	o	p	m	n	k	l	i	j
h	h	g	f	e	d	c	b	a	p	o	n	m	l	k	j	i
i	i	j	k	l	m	n	o	p	a	b	c	d	e	f	g	h
j	j	i	l	k	n	m	p	o	b	a	d	c	f	e	h	g
k	k	l	i	j	o	p	m	n	c	d	a	b	g	h	e	f

l	l	k	j	i	p	o	n	m	d	c	b	a	h	g	f	e
m	m	n	o	p	i	j	k	l	e	f	g	h	a	b	c	d
n	n	m	p	o	j	i	j	k	f	e	h	g	b	a	d	c
o	o	p	m	n	k	l	i	j	g	h	e	f	c	d	a	b
p	p	o	n	m	l	k	j	i	h	g	f	e	d	c	b	a

×	**a**	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**	**j**	**k**	**l**	**m**	**n**	**o**	**p**
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	a	i	b	j	c	k	d	l	b	i	j	d	k	c	l	d
c	a	b	i	i	k	c	l	d	c	j	i	b	g	o	e	m
d	a	j	j	a	l	d	c	k	d	b	b	c	h	p	f	n
e	a	c	k	l	i	j	b	d	e	g	f	h	m	n	o	p
f	a	k	c	d	j	i	d	b	f	h	g	e	n	m	p	o
g	a	d	l	c	b	d	i	j	g	f	h	e	o	p	m	n
h	a	l	d	l	d	b	j	i	h	g	e	f	p	o	n	m
i	a	b	c	d	e	f	g	h	i	i	k	l	m	n	o	p
j	a	i	j	b	g	h	e	f	j	i	d	k	n	m	p	o
k	a	j	i	b	f	g	h	e	k	d	i	b	o	p	m	n
l	a	d	b	j	h	f	e	g	l	k	d	i	p	o	n	m
m	a	c	g	h	m	n	o	p	m	n	o	f	i	i	k	l
n	a	k	g	i	n	m	p	o	n	m	p	o	j	j	l	k
o	a	l	e	f	o	p	m	n	o	p	m	n	k	k	i	j
p	a	d	f	e	p	o	n	m	p	o	n	m	i	l	j	i

Clearly (Q,+) is group and (Q,×) is a semi group.
Also $m \times (k + l) = (m \times k)$ + (m× l) for all m, k, l ϵ Q
So (Q, $+_2$, $\times_2$) is a finite left near ring but it is not a finite right near ring because
$(f + c) \times j = p$ ≠m= (f×j) + (c×j)
Hence (Q, + , .) is a finite near ring but not a finite ring.

Remark3.4: Language's theorem can be applicable to finite near rings also.
Lemma3.5: If a subset of a finite near ring is closed under addition, and multiplication then it is a finite near ring itself.
Lemma3.6: Addition of sub near rings is also a near-ring.
For example S_1 = {0,2} and S_2 = {0,4} are two sub near rings of N=(Z_6, $+_2$, $\times_2$).
$S_1 + S_2 = \{ S_1 + S_2 / \ S_1 \in S_1, S_1 \in S_1 \}$.
Clearly $S_1 + S_2$ is closed under addition module 6 and multiplication module 6.
So $S_1 + S_2 = \{ S_1 + S_2 / \ S_1 \in S_1, S_1 \in S_1 \}$ is also a near-ring.
Lemma3.7: Intersection of sub near rings is also a near-ring.
Remark3.8: The union of sub near rings need not be a near ring.
For example $S_1 = \{0,3\}, S_1 = \{0,2,4\}$ are two sub near rings of N=(Z_6, $+_2$, $\times_2$).

$S_1 \cup S_2 = \{0,2,3,4\}$ is not a near ring because $2 + 3 = 5 \notin S_1 \cup S_2$
So union of sub near rings need not be a near the ring.

4. Finite division Ring:

We provide counter examples of finite division ring, which are not a field. **It contradicts the Wedderburn's Little Theorem,** asserts that all finite division rings are commutative and therefore finite field

Example4.1: Consider the set of upper triangular Heisenberg matrices over Z_2.
Since $Z_2=\{0,1\}$, there are $2^3 = 8$ matrices, making it a finite set. The matrices are as follows:

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad B = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad C = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad D = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

$$E = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad F = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad G = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad H = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

So $H_2(Z_2) = \{A, B, C, D, E, F, G, H\}$
Addition defined as follows, Let $A, B \in H_2(Z_2)$ then $A + B = A + B +_2 I_3$,

where $I_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

The addition table is as follows:

$+_2$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	D	C	F	E	H	G
C	C	D	A	B	G	H	E	F
D	D	C	B	A	H	G	F	E
E	E	F	G	H	A	B	G	D
F	F	E	H	G	B	A	D	G
G	G	H	E	F	G	D	A	B
H	H	G	F	E	D	G	B	A

From the above table $H_2(Z_2)$ is a commutative group with respect to addition.
Clearly $H_2(Z_2)$ is a non-commutative group under matrix multiplication. Further it is distributive with respect to addition.
Hence $H_2(Z_2)$ is a finite division ring.
Remark: As per **Wedderburn's Little Theorem,** that all finite division rings are commutative and therefore finite field. But in the example 4.1, $H_2(Z_2)$ is a finite division ring and it is not commutative. So **it contradicts the Wedderburn's Little Theorem.**

Example4.1: Consider the set of upper triangular Heisenberg matrices over Z_3.

$$H_3(Z_3) = \begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix} \; a, b, c \in z_3$$

Since $Z_3=\{0,1,2\}$, there are $3^3 = 27$ matrices, making it a finite set. The matrices are as follows:

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad B = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad C = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \quad D = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad E = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

$$F = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \quad G = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad H = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad I = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \quad J = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$K = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad l = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \quad m = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad n = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad o = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$$

$$P = \begin{bmatrix}1 & P & 2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad Q = \begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad R = \begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \quad S = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad T = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

$$u = \begin{bmatrix}1 & 2 & 0\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \quad v = \begin{bmatrix}1 & 2 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad w = \begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad X = \begin{bmatrix}1 & 2 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \quad Y = \begin{bmatrix}1 & 2 & 2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$z = \begin{bmatrix}1 & 2 & 2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad \alpha = \begin{bmatrix}1 & 2 & 2\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$$

Addition defined as follows, Let A, $B \in H_3(Z_3)$ then $A + B = A + B -_3 I_3$,

where $I_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

The addition table is as follows:

+	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	α
A	**A**	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	α
B	B	C	**A**	E	F	D	H	I	G	K	L	J	N	O	M	Q	R	P	T	U	D	W	X	V	Z	α	Y
C	C	**A**	B	F	D	E	I	G	H	L	J	K	O	M	N	R	P	Q	U	S	T	X	V	W	α	Y	Z
D	D	E	F	G	H	I	**A**	B	C	M	N	O	P	Q	R	I	K	L	V	W	X	Y	Z	α	S	T	U
E	E	F	D	H	I	G	B	C	**A**	N	O	M	Q	R	P	K	L	J	W	X	V	Z	α	Y	T	*U*	S
F	F	D	E	I	G	H	C	**A**	B	O	M	N	R	P	Q	L	J	K	X	V	W	α	Y	Z	U	S	T
G	G	H	I	**A**	B	C	D	E	F	P	Q	R	J	K	L	M	N	O	Y	Z	α	S	T	U	V	W	X
H	H	I	G	B	C	**A**	E	F	D	Q	R	P	K	L	J	N	O	M	Z	α	Y	T	U	S	W	X	V
I	I	G	H	C	**A**	B	F	D	E	R	P	Q	L	J	K	O	M	N	α	Y	Z	U	S	T	X	V	W
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	α	**A**	B	C	D	E	F	G	H	I
K	K	L	J	N	O	M	Q	R	P	T	U	J	W	X	V	Z	α	Y	B	C	A	E	F	D	H	I	G
L	L	J	K	O	M	N	R	P	Q	U	J	T	X	V	W	α	Y	Z	*C*	**A**	B	F	D	E	I	G	H
m	M	N	O	P	Q	R	J	K	L	V	W	X	Y	Z	α	S	T	U	D	E	F	G	H	I	**A**	B	C
n	N	O	M	Q	R	P	K	L	J	W	X	V	Z	α	Y	T	U	S	E	F	D	H	I	G	B	C	**A**
O	O	M	N	R	P	Q	L	J	K	X	V	W	α	Y	Z	U	S	T	F	D	E	I	G	H	C	**A**	B
P	P	Q	R	I	K	L	M	N	O	Y	Z	α	S	T	U	V	W	X	G	H	I	**A**	B	C	D	E	F
Q	Q	R	P	K	L	J	N	O	M	N	α	Y	T	U	S	W	X	M	H	I	G	B	C	**A**	E	F	D
R	R	P	Q	L	J	K	O	M	N	α	Y	Z	U	S	T	X	M	W	I	G	H	C	**A**	B	F	D	E

S	S	T	U	V	W	X	Y	Z	α	**A**	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	S	W	X	V	Z	α	Y	B	C	**A**	E	F	D	H	I	G	K	L	J	N	O	M	Q	R	P
U	U	S	T	X	V	W	α	Y	Z	C	**A**	B	F	D	E	I	G	H	L	J	K	O	M	N	R	P	Q
V	V	W	X	Y	Z	α	S	T	U	D	E	F	G	H	I	**A**	B	C	M	N	O	P	Q	R	J	K	L
W	W	X	V	Z	α	Y	T	U	S	E	F	D	H	I	G	B	C	**A**	N	O	M	Q	R	P	K	L	J
X	X	V	W	α	T	Z	U	S	T	F	D	E	I	G	H	C	**A**	B	O	M	N	R	P	Q	L	J	K
Y	Y	Z	α	S	T	U	V	W	X	G	H	I	**A**	B	C	D	E	F	P	Q	R	J	K	L	M	N	O
Z	Z	α	Y	T	U	S	W	X	V	H	I	G	B	C	**A**	E	F	D	Q	R	P	K	L	J	N	O	M
α	α	Y	Z	U	S	T	X	V	W	I	G	H	C	**A**	B	F	D	E	R	P	Q	L	J	K	O	M	N

From the above table $H_3(Z_3)$ is a commutative group with respect to addition. Clearly $H_3(Z_3)$ is a non-commutative group under matrix multiplication. Further it is distributive with respect to addition. Hence $H_3(Z_3)$ is a finite division ring. But it is not a field

Remark: As per **Wedderburn's Little Theorem,** that all finite division rings are commutative and therefore finite field. But in the example 4.2, $H_3(Z_3)$ is a finite division ring and it is not commutative. So, **it contradicts the Wedderburn's Little Theorem.**

5. Conclusion

In this paper I found examples of finite near rings and finite division rings (skew fields) which are not fields. This is a contradiction to **Wedderburn's Little Theorem**. He told “every finite division ring is a field” It is hoped that in future mathematicians found a lot of examples of finite division rings which are not fields. Similarly, mathematicians found a lot of examples of finite near rings.

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