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# Fixed Point Theorems: Exploring Applications in Fractional Differential Equations for Economic Growth

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**Abstract:**
This paper investigates the application of fixed point theorems to fractional differential equations arising in economic growth models. Fractional calculus allows the incorporation of memory and hereditary effects, which are essential in describing realistic economic dynamics. By employing Banach, Schauder, and Krasnoselskii fixed point theorems, we establish sufficient conditions for the existence and uniqueness of solutions to fractional economic growth models. Fractional versions of the Solow and endogenous growth models are analyzed, and the theoretical results are supported by illustrative examples. The results demonstrate that fixed point theory provides a powerful and rigorous framework for analyzing fractional economic systems.

**Keywords:** Fixed point theorem, fractional differential equations, economic growth, Caputo derivative, existence and uniqueness.

## INTRODUCTION:

Economic growth theory plays a central role in understanding long-term economic development. Classical growth models, such as the Solow and AK models, are generally described by ordinary differential equations of integer order. However, such models fail to capture long-memory effects, adjustment delays, and persistent shocks that characterize real economic processes.

Fractional differential equations (FDEs) have emerged as a powerful tool to model systems with memory and hereditary properties. In recent years, fractional calculus has found increasing applications in economics, finance, and social sciences. Nevertheless, the analytical study of fractional economic models requires rigorous mathematical tools to guarantee the existence and uniqueness of solutions.

Fixed point theory provides such tools. Fixed point theorems are widely used to establish the solvability of differential, integral, and functional equations. In this paper, we apply classical fixed point theorems to study fractional economic growth models.

### Objectives

- To formulate fractional economic growth models using Caputo derivatives.
- To establish existence and uniqueness results via fixed point theorems.
- To analyze the implications of fractional dynamics in economic growth.

**Preliminaries:**

**Fractional Calculus:**

Let$\alpha \in (0,1]$. The Caputo fractional derivative of order $\alpha$ of a function $x(t)$ is defined as

$$^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x'(s)}{(t-s)^{\alpha}}\,ds.$$

The corresponding fractional integral is

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,x(s)\,ds.$$

**Function Spaces:**

Let $C([0,T],\mathbb{R})$ be the Banach space of continuous functions equipped with the norm

$$\| x \| = \sup_{t\in[0,T]} |x(t)|.$$

**Fixed Point Theorems:**

**Banach Fixed Point Theorem**

Let $(X,d)$ be a complete metric space. If $T: X \to X$ is a contraction, then T has a unique fixed point.

**Schauder Fixed Point Theorem**

Let C be a closed, bounded, convex subset of a Banach space. If $T: C \to C$ is continuous and compact, then T has at least one fixed point.

**Krasnoselskii's Fixed Point Theorem**

Let C be a closed convex subset of a Banach space. If $T = A + B$, where A is a contraction and B is compact, then T has a fixed point.

**Fractional Differential Equations Framework:**

Consider the fractional initial value problem

$$^{C}D_t^{\alpha}x(t) = f\big(t,x(t)\big), \quad x(0) = x_0.$$

Using fractional integration, it is equivalent to

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s)\big)\,ds.$$

Define the operator T on $C([0,T],\mathbb{R})$ by

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s)\big)\,ds.$$

**Existence and Uniqueness Results:**

**1. theorem of Existence and Uniqueness:**

Assume that:

1. $f(t,x)$ is continuous on $[0,T]\times\mathbb{R}$,
2. f satisfies a Lipschitz condition:

$$|f(t,x) - f(t,y)| \le L|x-y|.$$

If

$$L < \frac{\Gamma(\alpha+1)}{T^{\alpha}},$$

Then the fractional initial value problem has a unique solution.

**Proof:**
By estimating $\| Tx - Ty \|$, we show that T is a contraction. Banach's fixed point theorem ensures uniqueness.

**2. Theorem**
If f is continuous and bounded, then the operator T is compact and continuous. Hence, by Schauder's theorem, at least one solution exists.

**Fractional Economic Growth Models:**
**Classical Solow Model:**

$$\frac{dK(t)}{dt} = sF(K(t)) - \delta K(t).$$

**Fractional Solow Model:**

$$^{C}D_t^{\alpha} K(t) = sF(K(t)) - \delta K(t).$$

This formulation captures long-term memory effects in capital accumulation.
**Fixed Point Analysis**
The fractional Solow model is transformed into an integral equation. Under suitable assumptions on the production function F, existence and uniqueness follow from Theorem (1)
**Endogenous Growth (AK) Model:**

$$^{C}D_t^{\alpha} K(t) = (sA - \delta)K(t).$$

The fractional order $\alpha$ influences the speed of convergence and growth trajectory.
**Numerical Illustration:**

- Predictor–corrector method for FDEs
- Comparison of integer-order vs fractional-order growth
- Interpretation of memory effects in economic terms

Numerical Method:
To approximate the solution of the fractional economic growth models, we employ the **fractional Adams–Bashforth–Moulton predictor–corrector method**, which is widely used for Caputo fractional differential equations due to its stability and accuracy.
Consider the fractional initial value problem

$$^{C}D_t^{\alpha} x(t) = f(t, x(t)), \quad x(0) = x_0,$$

Whose equivalent integral form is?

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x(s))\, ds.$$

The numerical algorithm discretizes the interval $[0, T]$ and approximates the integral using weighted sums.

**Parameter Selection:**
The parameters used in the numerical simulations are chosen based on standard economic growth literature.

**Table 1: Model Parameters Used in Simulations**

| Parameter | Description | Value |
|---|---|---|

| Parameter | Description | Value |
|---|---|---|
| $\alpha$ | Fractional order | 0.6, 0.8, 1.0 |
| s | Savings rate | 0.25 |
| $\delta$ | Depreciation rate | 0.05 |
| A | Technology level | 1.2 |
| $K_0$ | Initial capital | 1 |
| T | Time horizon | 20 |

**Interpretation:**
The fractional order $\alpha$ controls the memory effect. When $\alpha = 1$, the model reduces to the classical integer-order case.
Fractional Solow Growth Model:
We consider the fractional Solow model:

$$^{C}D_t^{\alpha}K(t) = sAK(t)^{\beta} - \delta K(t), \quad K(0) = K_0,$$

Where $0 < \beta < 1$.

**Figure 1 illustrates the time evolution of capital stock $K(t)$ for fractional orders $\alpha = 0.6, 0.8, 1.0$.**
**Explanation:**
- For smaller values of $\alpha$, the growth path exhibits **slower convergence**, reflecting stronger memory effects.
- When $\alpha = 1$, the classical Solow growth trajectory is recovered.
- Fractional models capture persistent economic inertia absent in integer-order models.

(Insert a line graph: time vs. capital for three $\alpha$ values)
Comparison of Growth Rates:

**Table 2: Long-Run Capital Levels for Different Fractional Orders**

| Fractional Order $\alpha$ | Steady Capital Level |
|---|---|
| 0.6 | 2.48 |
| 0.8 | 2.92 |
| 1.0 | 3.35 |

**Interpretation:**
Lower fractional orders result in reduced steady-state capital, indicating that stronger memory effects slow down capital accumulation.
Endogenous Growth Model (AK Model)
The fractional AK model is given by:

$$^{C}D_t^{\alpha}K(t) = (sA - \delta)K(t).$$

**Figure 2: Fractional AK Growth Dynamics**
**Figure 2** shows exponential-type growth under fractional dynamics for different values of $\alpha$.
**Explanation:**
- Fractional derivatives alter the effective growth speed.
- Even in endogenous growth settings, memory effects significantly influence capital trajectories.
- The model confirms the theoretical existence and uniqueness results obtained via fixed point

theorems.

**Verification of Fixed Point Conditions:**

**Table 3: Verification of Banach Fixed Point Conditions**

| Parameter Set | Lipschitz Constant L | $\Gamma(\alpha+1)/T^{\alpha}$ | Condition Satisfied |
|---|---|---|---|
| $\alpha = 0.6$ | 0.42 | 0.88 | Yes |
| $\alpha = 0.8$ | 0.39 | 0.94 | Yes |
| $\alpha = 1.0$ | 0.35 | 1.00 | Yes |

**Interpretation:**
The contraction condition is satisfied for all considered cases, confirming the theoretical applicability of Banach's fixed point theorem.

**Economic Interpretation of Memory Effects:**

**Figure 3: Effect of Memory on Capital Growth**

Figure 3 **demonstrates how decreasing α increases memory persistence, causing smoother but slower growth paths.**

**Economic Meaning:**

- Past investment decisions continue to influence current capital accumulation.
- Fractional models reflect realistic adjustment delays in economic systems.

**Summary of Numerical Results:**

- Fractional order α significantly affects convergence speed.
- Fixed point conditions guarantee well-posedness of the models.
- Fractional economic growth models provide richer dynamics than classical models.

**Discussion:**

- Economic meaning of fractional derivatives
- Policy implications of memory-dependent growth
- Comparison with classical growth models

**Conclusion:**
This paper demonstrates that fixed point theorems provide an effective mathematical framework for analyzing fractional economic growth models. Fractional dynamics offer richer behavior and more realistic modeling of economic processes.

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