Generalized Mittag-Leffler Function As Its Kernel In Fractional Intergral

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Abstract

In this paper, we introduce fractional integrals and differentials of the extended Mittag–Leffler functions $E_{\alpha,\beta}^{\gamma,\delta\,;c,d}\,(z;p,q)$. In this continuation of the study of fractional calculus.

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1.1 Introduction and preliminaries

The Mittag–Leffler functions appear in special functions as a solution of fractional order integral and differential equations. Some interesting applications of the Mittag Leffler function are considered as follows: studied of kinetic equation, the telegraph equations [5], random walks, Levy flights, supper diffuse transport and complex system.

We begin with the Mittag–Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}\,(z)$ are defined by the following series:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \quad z \in \mathbb{C}; \Re(\alpha) > 0 \qquad (1.1.1)$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \quad z,\beta \in \mathbb{C}; \Re(\alpha) > 0, \qquad (1.1.2)$$

respectively. For further study of Mittag–Leffler function such as generalizations and applications , the readers may refer to the current work of researchers (for example) Džrbašjan [7], Kilbas and Saigo [15], Gorenflo and Mainardi [8] , Gorenflo et al. ([9,11]), Kilbas et al. ([16], Chapter 1] and Saigo and Kilbas [22]. In recent years, the Mittag–Leffler function (1.1.1) and some of its variety of generalizations have been numerically investigated in the complex plane (see [14,24]). A generalization of the Mittag–Leffler function $E_{\alpha,\beta}\,(z)$ of (1.1.2) was introduced by Prabhakar [20] as follows:

$$E_{\alpha,\beta}^{\gamma}\,(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)\; n!} \quad z,\beta \in C; \Re(\alpha) > 0, \qquad (1.1.3)$$

where $(\gamma)_n$ denote the well-known Pochhammer Symbol which is defined by

$$(\gamma)_n = \begin{cases} 1, & when \quad (n = 0, \gamma \neq 0) \\ \gamma\,(\gamma + 1) \ldots (\gamma + n - 1), & when \quad (n \in \mathbb{N}, \gamma \in \mathbb{C}) \end{cases}.$$

Obviously, the following special cases are satisfied:

$$E_{\alpha,\beta}^{1}\,(z) = E_{\alpha,\beta}\,(z) = E_{\alpha,1}^{1}(z) = E_\alpha(z). \qquad (1.1.4)$$

Diaz and Pariguan [see nisar (k,s) fractional calculas] have introduced the Pochhammer p −symbols and p −Gamma function defined as follows

$$(\gamma)_{np} = \begin{cases} 1, & when\ (n = 0, \gamma \neq 0) \\ \gamma\,(\gamma + p)\ldots(\gamma + (n-1)p), & when\ (n \in \mathbb{N}, \gamma \in \mathbb{C}, p > 0) \end{cases}$$

And

$$\Gamma_p(\gamma) = \int_0^\infty t^{\gamma-1} e^{-\frac{t^p}{p}}\, dt$$

In recent times many researchers have investigated the importance and great consideration of Mittag–Leffler function in the theory of special functions for exploring the generalization and some applications. Many extensions for these functions are found in [1–4,10]. Srivastava and Tomovski [25] have defined further generalization of the Mittag–Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ of (3), which is defined as follows:

$$E^{\gamma,\kappa}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\kappa}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.1.5}$$

where $z, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > max\{0, \Re(\kappa) - 1\}; \Re(\kappa) > 0$.

In the same paper, they have used the well-known right-sided Riemann–Liouville fractional integral, derivative and generalized Riemann–Liouville derivative operators (see [12,13,16,23]).Very recently Özarslan and Yilmaz [19] have investigated an extended Mittag–Leffler function $E^{\gamma\,;c}_{\alpha,\beta}(z;\, p)$, which is defined as follows:

$$E^{\gamma,c}_{\alpha,\beta}(z;\, p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + n, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.1.6}$$

where $p \geq 0, \Re(c) > \Re(\gamma) > 0$ and $B_p(x,y)$ is extended beta function defined in [6] as follows:

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t}(1-t)}\, \mathrm{dt} \tag{1.1.7}$$

for $\Re(p) > 0, \Re(x) > 0$ and $R(y) > 0$. In the same paper, they defined the following result for the extended Mittag–Leffler function (1.1.6) as follows:

$$\left(\frac{d}{dz}\right)^n \left(z^{\lambda-1} E^{\gamma\,;c}_{\alpha,\beta}(\omega z^{\alpha};\, p)\right) = z^{\lambda-n-1} E^{\gamma\,;c}_{\alpha,\beta-n}(\omega z^{\alpha};\, p). \tag{1.1.8}$$

In this continuation of the study on the significance of fractional calculus, we start with the following preliminaries:

In this chapter we investigated an extended Mittag–Leffler function $E^{\gamma,\delta\,;c,d}_{\alpha,\beta}(z; p, q)$, which is defined as follows:

$$E^{\gamma,\delta\,;c,d}_{\alpha,\beta}(z; p, q) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + n, c - \gamma)}{B_p(\gamma, c - \gamma)} \frac{B_q(\delta + n, d - \delta)}{B_q(\delta, d - \delta)} \frac{(c)_{np}(d)_{nq}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.1.9}$$

where $p, q \geq 0, \Re(c) > \Re(\gamma) > 0,\ \Re(d) > \Re(\delta) > 0$ and $B_p(x,y)$ and $B_q(x,y)$ are extended beta function defined as follows,

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(t-1)}}\, dt$$

And

$$B_q(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{q}{t(t-1)}}\, dt$$

For $\Re(p) > 0, \Re(x) > 0$ and $\Re(y) > 0$.

Remark 1.1.1: It should be noted that,

$$E^{1,1\,;c,d}_{\alpha,\beta}(z, 1,1) = E^{1,1}_{\alpha,\beta}(z) = E^{1,c}_{\alpha,\beta}(z;\, 1) = E^{1}_{\alpha,1}(z) = E_{\alpha,1} = E_{\alpha}(z)$$

$$(1.1.9) \Rightarrow (1.1.6) \Rightarrow (1.1.3) \Rightarrow (1.1.2) \Rightarrow (1.1.1.)$$

Or

$$(1.1.9) \Rightarrow (1.1.5) \Rightarrow (1.1.3) \Rightarrow (1.1.2) \Rightarrow (1.1.1)$$

$\mathcal{L}(a,b)$ is the space of Lebesgue measurable of real or complex valued function if it is defined as

$$\mathcal{L}(a,b) = \left(f : \| f \|_1 = \int_a^b f(x)dx < \infty\right) . \qquad (1.1.10)$$

The left- and right-sided Riemann–Liouville fractional integral operators I^{λ}_{a+} and I^{λ}_{b-} are, respectively, defined as (see, e.g., [16]) follows:

$$\left(I^{\lambda}_{a+} f\right)(x) = \frac{1}{\Gamma(\lambda)} \int_a^x \frac{f(\tau)}{(\tau - x)^{1-\lambda}}\, d\tau, (x > a), \qquad (1.1.11)$$

and

$$\left(I^{\lambda}_{b-} f\right)(x) = \frac{1}{\Gamma(\lambda)} \int_x^b \frac{f(\tau)}{(\tau - x)^{1-\lambda}}\, d\tau, (x < b), \qquad (1.1.12)$$

where $f(x) \in \mathcal{L}(a,b), \lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

Similarly, for $f(x) \in \mathcal{L}(a,b), \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ and $n = [R(\lambda)] + 1$, the left- and right-sided Riemann–Liouville fractional differential are defined as follows:

$$\left(D^{\lambda}_{a+} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I^{n-\lambda}_{a+} f\right)(x), \qquad (1.1.13)$$

$$\left(D^{\lambda}_{a-} f\right)(x) = \left(-\frac{d}{dx}\right)^n \left(I^{n-\lambda}_{a-} f\right)(x), \qquad (1.1.14)$$

respectively.

A generalized form of fractional differential operator D^{λ}_{a+} (1.1.13) has been made by investigating the fractional differential operator $D^{\lambda,v}_{a+}$ of order $0 < \lambda < 1$ and type $0 < v < 1$ with respect to x as follows:

$$\left(D^{\lambda,v}_{a+} f\right)(x) = \left(I^{v(1-\lambda)}_{a+} \frac{d}{dx} \left(I^{(1-v)(1-\lambda)}_{a+} f \right)\right)(x) \qquad (1.1.15)$$

(see [23], [16], [21]). Obviously, when $v = 0$ then (1.1.15) reduces to the operator D^{λ}_{a+} defined in (1.1.12).

We consider the following basic results for our study:

Theorem 1.1.2 (Mathai and Haubold [17]) *If* $\lambda, \mu \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(\mu) > 0$, *then*

$$I^{\lambda}_{a+}(\tau - a)^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\lambda+\mu)} (x - a)^{\lambda+\mu-1}. \qquad (1.1.16)$$

Theorem 1.1.3 (Srivatava and Manocha [26]) *If a function is* $f(z)$ *is analytic and has a power series expansion such that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *in the disc* $|z| < \Re$, *then*

$$D^{\lambda}_{0,z}\{z^{\mu-1} f(z)\} = \frac{\Gamma(\mu)}{\Gamma(\lambda+\mu)} \sum_{n=0}^{\infty} \frac{a_n (\mu)_n}{(\lambda+\mu)_n} z^n. \qquad (1.1.17)$$

Lemma 1.1.4 (Srivastava and Tomovski [25]) *The following result holds true for fractional derivative operator* $D^{\mu,v}_{a+} f$ *as follows:*

$$D^{\lambda,v}_{a+}\left[(\tau - a)^{\beta-1}\right](x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\lambda)} (x - a)^{\beta-\lambda-1}, \qquad (1.1.18)$$

where $x > a, 0 < \lambda < 1, 0 \le v \le 1, \Re(\beta) > 0$ *and* $\Re(\lambda) > 0$.

2.2 An integral and differential operators involving the extended Mittag–Leffler function

In this section, we introduce fractional integrals and differentials of the extended Mittag–Leffler functions $E^{\gamma,\delta\,;c,d}_{\alpha,\beta}(z;p,q)$. In this continuation of the study of fractional calculus, we define the following integral operator as follows:

Definition 2.1.1 If $\gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$, then

$$\left(\varepsilon_{a+;\alpha,\beta}^{\omega;\gamma,\delta\,;c,d} f\right)(x) = \int_a^x (x-\tau)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(x-\tau)^{\alpha};\, p,q)\, f(\tau)\,d\tau\,, \quad (1.1.19)$$

where $x > \alpha$. Substituting $p = q = 0$, then (1.1.19) reduces to the operator

$$\left(\varepsilon_{a+;\alpha,\beta}^{\omega;\gamma,\delta} f\right)(x) = \int_a^x (x-\tau)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta}(\omega(x-\tau)^{\alpha})\, f(\tau)\, d\tau, \qquad (1.1.20)$$

see [25]. In fact, when $\omega = 0$ then the integral operator in (1.1.20) reduces to the well known Riemann–Liouville fractional integral operator I_{a+}^{λ} defined in (1.1.11).

Theorem 2.2.2 *Suppose* $x > a\big(a \in \mathcal{R}_+ = [0,\infty)\big), \gamma, \delta, \lambda, \beta, \omega \in \mathbb{C},\ \Re(\beta) > 0, \Re(\lambda) > 0$; *then*

$$I_{a+}^{\lambda}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};p,q)\right](x) = (x-a)^{\lambda+\beta-1} E_{\alpha,\beta+\lambda}^{\gamma,\delta\,;c,d}\big((\omega(x-a)^{\alpha})\,;p,q\big)$$

(1.1.21)

$$D_{a+}^{\lambda}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};p,q)\right] = (x-a)^{\beta-\lambda-1} E_{\alpha,\beta-\lambda}^{\gamma,\delta\,;c,d}(\omega(x-a)^{\alpha};p,q)$$

(1.1.22)

and

$$D_{a+}^{\lambda,v}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};p,q)\right](x) = (x-a)^{\beta-\lambda-1} E_{\alpha,\beta-\lambda}^{\gamma,\delta\,;c,d}(\omega(x-a)^{\alpha};p,q).$$

(1.1.23)

Proof

$$I_{a+}^{\lambda}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};p,q)\right] = \frac{1}{\Gamma(\lambda)} \int_a^x \frac{(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};p,q)}{(x-\tau)^{1-\lambda}}\, d\tau$$

$$= \frac{1}{\Gamma(\lambda) B_p(\gamma, c-\gamma) B_q(\delta, d-\delta)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma) B_q(\delta+n, d-\delta)(c)_{np}(d)_{nq}\omega^n}{\Gamma(\alpha n+\beta) n!} \times \int_0^x (\tau-a)^{\beta+\alpha n-1}(x-\tau)^{\lambda-1}\, d\tau$$

$$= \frac{1}{B_p(\gamma, c-\gamma) B_q(\delta, d-\delta)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma) B_q(\delta+n, d-\delta)(c)_{np}(d)_{nq}\omega^n}{\Gamma(\alpha n+\beta) n!} \times \left(I_{a+}^{\lambda}\left[(\tau-a)^{\beta+\alpha n-1}\right]\right).$$

By the use of (1.1.16), we have

$$I_{a+}^{\lambda}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};\, p,q)\right]$$

$$= \frac{1}{B_p(\gamma, c-\gamma) B_q(\delta, d-\delta)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma) B_q(\delta+n, d-\delta)(c)_{np}(d)_{nq}\omega^n}{\Gamma(\alpha n+\beta) n!} (x-a)^{\beta+\lambda+\alpha n-1} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\lambda)}$$

$$= (x-a)^{\beta+\lambda-1} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma) B_q(\delta+n, d-\delta)}{B_p(\gamma, c-\gamma) B_q(\delta, d-\delta)} \frac{(c)_{np}(d)_{nq}}{\Gamma(\alpha n+\beta+\lambda)} \frac{[\omega(x-a)^{\alpha}]^n}{n!}$$

$$= (x-a)^{\beta+\lambda-1} E_{\alpha,\beta+\lambda}^{\gamma,\delta\,;c,d}(\omega(x-a)^{\alpha};p,q).$$

This completes the proof of (1.1.21).

Now, we have

$$D_{a+}^{\lambda}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta,p,q}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};p,q)\right] = \left(\frac{d}{dx}\right)^n \left\{I_{a+}^{n-\lambda}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma\,;c}(\omega(\tau-a)^{\alpha};p,q)\right\}\right.$$

and using (1.1.21) this takes the following form:

$$D_{a+}^{\lambda}\left[(\tau-a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-a)^{\alpha};p,q)\right]$$

$$= \left(\frac{d}{dx}\right)^{n}\left\{(x - a)^{\beta-\lambda+n-1} E_{\alpha,\beta-\lambda+n}^{\gamma,\delta\,;c,d}(\omega(x - a)^{\alpha}; p, q)\right\}.$$

Applying (1.1.9), we have

$$D_{a+}^{\lambda}\left[(\tau - a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau - a)^{\alpha};\, c, d)\right](x) = (x - a)^{\beta-\lambda-1} E_{\alpha,\beta-\lambda}^{\gamma,\delta\,;c,d}(\omega(x - a)^{\alpha}; p, q).$$

This completes the desired proof.

To prove (1.1.23), we have

$$\left(D_{a+}^{\lambda,v}\left[(\tau - a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau - a)^{\alpha}; p, q)\right]\right)(x)$$

$$= D_{a+}^{\lambda,v}\left(\left[\Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)B_q(\delta + n, d - \delta)(c)_{np}(d)_{nq}}{B_p(\gamma, c - \gamma)B_q(\delta, d - \delta)\Gamma(\alpha n + \beta)}\frac{\omega^n}{n!}(\tau - a)^{\alpha n+\beta-1}\right]\right)(x)$$

$$= \Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)B_q(\delta + n, d - \delta)(c)_{np}(d)_{nq}}{B_p(\gamma, c - \gamma)B_q(\delta, d - \delta)\Gamma(\alpha n + \beta)}\frac{\omega^n}{n!}\left(D_{a+}^{\lambda,v}\left[(\tau - a)^{\alpha n+\beta-1}\right]\right)(x).$$

By applying (1.1.18), we get

$$\left(D_{a+}^{\lambda,v}\left[(\tau - a)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(t - a)^{\alpha}; p, q)\right]\right)(x)$$

$$= \Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)B_q(\delta + n, d - \delta)(c)_{np}(d)_{nq}}{B_p(\gamma, c - \gamma)B_q(\delta, d - \delta)\Gamma(\alpha n + \beta)}\frac{\omega^n}{n!}\cdot\frac{\Gamma(\alpha n + \beta)}{\Gamma(\alpha n + \beta - \lambda)}(x - a)^{\alpha n+\beta-\lambda-1}$$

$$= (x - a)^{\beta-\lambda-1}\Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)B_q(\delta + n, d - \delta)(c)_{np}(d)_{nq}}{B_p(\gamma, c - \gamma)B_q(\delta, d - \delta)\Gamma(\alpha n + \beta - \lambda)}\frac{[\omega(x - a)^{\alpha}]^n}{n!}$$

$$= (x - a)^{\beta-\lambda-1} E_{\alpha,\beta-\mu}^{\gamma,\delta\,;c,d}(\omega(x - a)^{\alpha}; p, q),$$

which completes the required proof.

2.3 Some properties of the operator $\left(\boldsymbol{\varepsilon}_{a+;\alpha,\beta}^{\omega;\gamma,\delta\,;c,d}\boldsymbol{f}\right)(\boldsymbol{x})$

Theorem 2.3.1 *If* $\gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0$ *and* $\Re(\beta) > 0$, *then*

$$\left(\boldsymbol{\varepsilon}_{a+;\alpha,\beta}^{\omega;\gamma,\delta\,;c,d}[(\tau - a)^{\mu-1}]\right)(x) = (x - a)^{(\mu+\beta-1)}\Gamma(\mu) E_{\alpha,\beta+\mu}^{\gamma,\delta\,;c,d}(\omega(x - a)^{\mu}\,; p, q)\, f\,(\tau\,)d\tau.$$

(1.1.24)

Proof From (1.1.19)

$$\left(\boldsymbol{\varepsilon}_{a+;\alpha,\beta}^{\omega;\gamma,\delta\,;c,d} f\right)(x) = \int_{\alpha}^{x}(x - \tau)^{\beta-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(x - \tau)^{\alpha}; p, q)\, f\,(\tau\,)\, d\tau.$$

Therefore, we have

$$\left(\boldsymbol{\varepsilon}_{a+;\alpha,\beta}^{\omega;\gamma,\delta\,;c,d}[(\tau - a)^{\mu-1}]\right)(x) = \int_{\alpha}^{x}(x - \tau)^{\beta-1}(\tau - a)^{\mu-1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(x - \tau)^{\alpha}; p, q)\, d\tau$$

$$= \Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)B_q(\delta + n, d - \delta)(c)_{np}(d)_{nq}}{B_p(\gamma, c - \gamma)B_q(\delta, d - \delta)\Gamma(\alpha n + \beta)}\frac{\omega^n}{n!}\left(\int_{\alpha}^{x}(\tau - a)^{\mu-1}(x - \tau)^{\beta+\alpha n-1}\, d\tau\right)$$

$$\Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)B_q(\delta + n, d - \delta)(c)_{np}(d)_{nq}}{B_p(\gamma, c - \gamma)B_q(\delta, d - \delta)\Gamma(\alpha n + \beta)}\frac{\omega^n}{n!} I_{a+}^{\alpha n+\beta}\left[(\tau - a)^{\mu-1}\right]$$

$$= (x - a)^{\beta+\mu-1}\,\Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)B_q(\delta + n, d - \delta)(c)_{np}(d)_{nq}}{B_p(\gamma, c - \gamma)B_q(\delta, d - \delta)\Gamma(\alpha n + \beta)}\frac{[\omega(x - a)^{\alpha}]^n}{n!}\frac{\Gamma(\mu)\Gamma(\alpha n + \beta)}{\Gamma(\alpha n + \beta + \mu)}$$

$$= (x - a)^{\beta+\mu-1}\Gamma(\mu)\, E_{\alpha,\beta+\mu}^{\gamma,\delta\,;c,d}(\omega(x - a)^{\alpha}; p, q).$$

This completes the desired proof. _

Theorem 2.3.2 *If* $\gamma, \alpha, \beta, \omega, c \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(c) > 0$, *then* $\left\| \boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} \varphi \right\|_1 \leq B\|\varphi\|_1$,

where

$$B = (b - a)^{R(\beta)} \Sigma_{n=0}^{\infty} \frac{B_p(\gamma + n, c - \gamma) B_q(\delta + n, d - \delta) |(c)_{np}| |(d)_{nq}|}{B_p(\gamma, c - \gamma) B_q(\delta, d - \delta) \Gamma(\alpha n + \beta)(R(\beta) + R(\alpha) n)} \frac{|\omega (b - a)^{\alpha}|^n}{n!} \qquad (1.1.25)$$

Proof From (1.1.19) and (1.1.10) and by changing the order of integration by using the Dirichlet formula [18], we have

$$\left\| \boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} \varphi \right\|_1 = \int_a^b \left| \int_a^x (x - \tau)^{\beta - 1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d} (\omega (x - \tau)^{\alpha}; p, q) \varphi(\tau)\, d\tau \right| dx$$

$$\leq \int_a^b \left[\int_\tau^b (x - \tau)^{R(\beta) - 1} \left| E_{\alpha,\beta}^{\gamma,\delta\,;c,d} (\omega (x - \tau)^{\alpha}; p, q) \right| dx \right] |\varphi(\tau)| d\tau.$$

Substituting $(x - \tau) = u$, we obtain

$$= \int_a^b \left[\int_0^{b - \tau} (u)^{R(\beta) - 1} \left| E_{\alpha,\beta}^{\gamma,\delta\,;c,d} (\omega (u)^{\alpha}; p, q) \right| du \right] |\varphi(\tau)| d\tau.$$

After simplification, we have

$$\left\| \boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} \varphi \right\|_1$$

$$\leq \int_a^b \left[\Sigma_{n=0}^{\infty} \frac{B_p(\gamma + n, c - \gamma) B_q(\delta + n, d - \delta) |(c)_{np}| |(d)_{nq}|}{B_p(\gamma, c - \gamma) B_q(\delta, d - \delta) \Gamma(\alpha n + \beta)} \frac{|\omega^n|}{n!} \left(\frac{(u)^{\boldsymbol{Re}(\beta) + R(\alpha) n}}{R(\beta) + R(\alpha) n} \right)_0^{b - a} \right] |\varphi(\tau)|\, d\tau.$$

This can also be written as

$$\left\| \boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} \varphi \right\|_1 \leq$$

$$(b - a)^{R(\beta)} \Sigma_{n=0}^{\infty} \frac{B_p(\gamma + n, c - \gamma) B_q(\delta + n, d - \delta) |(c)_{np}| |(d)_{nq}|}{B_p(\gamma, c - \gamma) B_q(\delta, d - \delta) \Gamma(\alpha n + \beta)(R(\beta) + R(\alpha) n)} \frac{|\omega (b - a)^{\alpha}|^n}{n!} \cdot \int_a^b |\varphi(\tau)|\, d\tau$$

$$= B ||\varphi||_1,$$

where

$$B = (b - a)^{R(\beta)} \Sigma_{n=0}^{\infty} \frac{B_p(\gamma + n, c - \gamma) B_q(\delta + n, d - \delta) |(c)_{np}| |(d)_{nq}|}{B_p(\gamma, c - \gamma) B_q(\delta, d - \delta) \Gamma(\alpha n + \beta)(R(\beta) + R(\alpha) n)} \frac{|\omega (b - a)^{\alpha}|^n}{n!}. (26)$$

This completes the desired proof.

Theorem 2.3.3 *If* $\lambda, \gamma, \alpha, \beta \omega \in \mathbb{C}$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0$ *and* $x > a$, *then*

$$\left(I_{a+}^{\lambda} \left[\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f \right] \right)(x) = \left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta+\mu}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f \right)(x) = \left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} [I_{\alpha+}^{\lambda} f] \right)(x) \qquad (1.1.27)$$

holds for any function $f \in L(\alpha, \beta)$.

Proof From (1.1.19) and (1.1.11), we have

$$\left(I_{a+}^{\lambda} \left[\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f \right] \right)(x) = \frac{1}{\Gamma(\lambda)} \int_a^x \frac{\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f)(\tau)}{(x - \tau)^{1 - \mu}} d\tau$$

$$= \frac{1}{\Gamma(\lambda)} \int_a^x (x - \tau)^{\lambda - 1} \times \left\{ \int_\alpha^\tau (\tau - u)^{\beta - 1} E_{\alpha,\beta}^{\gamma,\delta\,;c,d} (\omega (\tau - u)^{\alpha}; p, q) f(u)\, du \right\} d\tau.$$

By interchanging the order of integration and applying the Dirichlet formula [19], we have

$$\left(I_{a+}^{\lambda}\left[\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f\right]\right)(x)$$
$$= \int_a^x \left[\frac{1}{\Gamma(\lambda)}\int_u^x (x-\tau)^{(\lambda-1)}(\tau-u)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\tau-u)^{\alpha};p,q)\,d\tau\right] f(u)du.$$

Substituting $(\tau - u) = \rho$, we obtain

$$\left(I_{a+}^{\lambda}\left[k\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f\right]\right)(x) = \int_a^x\left[\frac{1}{\Gamma(\lambda)}\int_0^{x-u}(x-u-\rho)^{\lambda-1}(\rho)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\rho)^{\alpha};p,q)\,d\rho\right] f(u)du$$
$$= \int_a^x\left[\frac{1}{\Gamma(\lambda)}\int_0^{x-u}(\rho)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\rho)^{\alpha};p,q)(x-u-\rho)^{1-\lambda}\,d\rho\right] f(u)du. \qquad (1.1.28)$$

By the use of (1.1.11) and applying (1.1.21), we get

$$\left(I_{a+}^{\lambda}\left[\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f\right]\right)(x) = \int_a^x\left[(\rho)^{\lambda+\beta-1}E_{\alpha,\beta+\lambda}^{\gamma,\delta\,;c,d}(\omega(\rho)^{\alpha};p,q)\right] f(u)du$$
$$= \int_a^x (x-u)^{\lambda+\beta-1}E_{\alpha,\beta+\lambda}^{\gamma,\delta\,;c,d}(\omega(x-u)^{\alpha};p,q)\,f(u)du\ ;$$

thus, we get

$$\left(I_{a+}^{\lambda}\left[\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f\right]\right)(x) = \left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta+\lambda}^{\boldsymbol{\omega};\gamma,\delta\,;c,d} f\right)(x), \qquad (1.1.29)$$

this is the required proof of (1.1.27).

To prove the second part, we begin from the right-hand side of (1.1.27) and using (1.1.19), we have

$$\left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d}\left[I_{a+}^{\lambda} f\right]\right)(x) = \int_a^x (x-\tau)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(x-\tau)^{\alpha};p,q)[I_{a+}^{\lambda} f](\tau)\,d\tau$$
$$= \int_a^x E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(x-\tau)^{\alpha};p,q)\left(\frac{1}{\Gamma(\lambda)}\int_a^{\tau}\frac{f(u)}{(\tau-u)^{1-\mu}}\,du\right)d\tau.$$

By interchanging the order of integration and using the Dirichlet formula [18], we obtain

$$\left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d}\left[I_{a+}^{\lambda} f\right]\right)(x) = \int_a^x \frac{1}{\Gamma(\lambda)}\left[\int_u^x (x-\tau)^{\beta-1}(\tau-u)^{\lambda-1}E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(x-\tau)^{\alpha};p,q)\,d\tau\right]\times f(u)du.$$

Substituting $(x - \tau) = \rho$

$$(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d}[I_{a}^{\lambda}+)\,f\,])(x)$$
$$= \int_a^x \frac{1}{\Gamma(\lambda)}\left[\int_{x-u}^{0}(\rho)^{\beta-1}(x-\rho-u)^{\lambda-1}E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\rho)^{\alpha};p,q)(-d\rho)\right]\times f(u)du$$
$$= \int_a^x \frac{1}{\Gamma(\lambda)}\left[\int_0^{x-u}(\rho)^{\beta-1}(x-\rho-u)^{\lambda-1}E_{\alpha,\beta}^{\gamma,\delta\,;c,d}(\omega(\rho)^{\alpha};p,q)d\rho\right]\times f(u)du.$$

Again by making use of (1.1.11) and applying (1.1.21), we get

$$\left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma,\delta\,;c,d}\left[I_{a+}^{\lambda} f\right]\right)(x) = \left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta+\lambda}^{\boldsymbol{\omega};\gamma,\delta\,;c,d}\right) f(x). \qquad (1.1.30)$$

Thus (1.1.29) and (1.1.30) complete the desired proof of (1.1.27).

2.4. Special Cases

If we put $\delta = q = 1$ in Theorem **2.** 2.2 then we get following result.

Theorem 2.4.1 *Suppose* $x > \alpha\big(\alpha \in \mathcal{R}_+ = [0,\infty)\big), \gamma, \lambda, \beta, \omega \in \mathbb{C},\ \Re(\beta) > 0, \Re(\lambda) > 0;$ *then*

$$I_{a+}^{\lambda}\left[(\tau-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c}(\omega(\tau-a)^{\alpha};p)\right](x) = (x-a)^{\lambda+\beta-1}E_{\alpha,\beta+\lambda}^{\gamma,c}(\omega(x-a)^{\alpha})\,;p$$
$$D_{a+}^{\lambda}\left[(\tau-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c}(\omega(\tau-a)^{\alpha};p)\right] = (x-a)^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}^{\gamma,c}(\omega(x-a)^{\alpha};p)$$

and

$$D_{a+}^{\lambda,\nu}\left[(\tau-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c}(\omega(\tau-a)^{\alpha};p)\right](x) = (x-a)^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}^{\gamma,c}(\omega(x-a)^{\alpha};p).$$

If we put $\gamma = p = \delta = q = 1$ in theorem **2.** 2.2 then we get following result.

Theorem 2.4.2 *Suppose* $x > \alpha\left(\alpha \in \mathcal{R}_+ = [0,\infty)\right), \gamma, \lambda, \beta, \omega \in \mathbb{C},\ \Re(\beta) > 0, \Re(\lambda) > 0$; *then*

$$I_{a+}^{\lambda}\left[(\tau - a)^{\beta-1}E_{\alpha,\beta}(\omega(\tau - a)^{\alpha})\right](x) = (x - a)^{\lambda+\beta-1}E_{\alpha,\beta+\lambda}(\omega(x - a)^{\alpha})$$

$$D_{a+}^{\lambda}\left[(\tau - a)^{\beta-1}E_{\alpha,\beta}(\omega(\tau - a)^{\alpha})\right] = (x - a)^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}(\omega(x - a)^{\alpha})$$

and

$$D_{a+}^{\lambda,v}\left[(\tau - a)^{\beta-1}E_{\alpha,\beta}(\omega(\tau - a)^{\alpha})\right](x) = (x - a)^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}(\omega(x - a)^{\alpha}).$$

If we take $\delta = d = 1$ in Definition **2.**1.1then we get the following definition

Definition 2.4.3 If $\gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$, then

$$\left(\varepsilon_{a+;\alpha,\beta}^{\omega;\gamma;c} f\right)(x) = \int_a^x (x - \tau)^{\beta-1}E_{\alpha,\beta}^{\gamma;c}(\omega(x - \tau)^{\alpha}; p)\, f(\tau)d\tau,$$

where $x > \alpha$. Substituting $p = 0$, then (19) reduces to the operator

$$\left(\varepsilon_{a+;\alpha,\beta}^{\omega;\gamma} f\right)(x) = \int_a^x (x - \tau)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta}(\omega(x - \tau)^{\alpha})\, f(\tau)\, d\tau,$$

If we take $\gamma = c = \delta = d = 1$ in Definition **2.**1.1then we get the following definition

Definition 2.4.4 If $\gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$, then

$$\left(\varepsilon_{a+;\alpha,\beta}^{\omega} f\right)(x) = \int_a^x (x - \tau)^{\beta-1}E_{\alpha,\beta}(\omega(x - \tau)^{\alpha})\, f(\tau)d\tau$$

If we take $\delta = d = 1$ in Definition **2.**3.1 then we get the following results

Theorem 2.4.5 *If* $\gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0$ *and* $\Re(\beta) > 0$, *then*

$$\left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma;c}[(\tau - a)^{\mu-1}]\right)(x) = (x - a)^{(\mu+\beta-1)}\Gamma(\mu)E_{\alpha,\beta+\mu}^{\gamma;c}(\omega(x - a)^{\mu}; p)\, f(\tau)d\tau.$$

If we take $\gamma = c = \delta = d = 1$ in Theorem **2.**3.2 then we get the following results

Theorem 2.4.6 *If* $\gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0$ *and* $\Re(\beta) > 0$, *then*

$$\left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega}}[(\tau - a)^{\mu-1}]\right)(x) = (x - a)^{(\mu+\beta-1)}\Gamma(\mu)E_{\alpha,\beta+\mu}(\omega(x - a)^{\mu})\, f(\tau)d\tau.$$

If we take $\delta = d = 1$ in Theorem **2.**3.2 then we get the following results

Theorem 2.4.7 *If* $\gamma, \alpha, \beta, \omega, c \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(c) > 0$, *then* $\left\| \boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega};\gamma;c}\varphi \right\|_1 \leq B\|\varphi\|_1$,

where

$$B = (b - a)^{R(\beta)}\Sigma_{n=0}^{\infty}\frac{B_p(\gamma + n, c - \gamma)\left|(c)_{np}\right|}{B_p(\gamma, c - \gamma)\Gamma(\alpha n + \beta)(R(\beta) + R(\alpha)n)}\frac{|\omega(b - a)^{\alpha}|^n}{n!}.$$

If we take $\gamma = c = \delta = d = 1$ in Theorem **2.**3.2 then we get the following results

Theorem 2.4.8 *If* $\lambda, \gamma, \alpha, \beta, \omega \in \mathbb{C},\ \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\lambda) > 0$ *and* $x > a$, *then*

$$\left(I_{a+}^{\lambda}\left[\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega}} f\right]\right)(x) = \left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta+\mu}^{\boldsymbol{\omega}} f\right)(x) = \left(\boldsymbol{\varepsilon}_{\boldsymbol{a}+;\alpha,\beta}^{\boldsymbol{\omega}}\left[I_{\alpha+}^{\lambda} f\right]\right)(x)$$

holds for any function $f \in L(\alpha, \beta)$.

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