The Nonlinearity Effect on the Morphology, Stability, And Regularity of the Diffusion, Dispersion, And Convection Term in the Analytic Solutions of Certain Frontier Nonlinear Partial Differential Equations

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Abstract

In the paper, the Laplace-Adomian Decomposition Algorithm is used to solve several frontier nonlinear partial differential equations analytically. For the analytic or semi-analytic solution of nonlinear partial, integral, and integro-differential equations as well as the system of such equations, the author modified and extended the Laplace Adomian Decomposition Method to the Laplace Adomian Decomposition Algorithm and developed it for boundary value problems as the Shooting-Type Laplace-Adomian Decomposition Algorithm. The author has authored a few related papers, which are included in the bibliography. Analytical solutions are crucial because they provide a physical insight of the phenomenon and can be applied directly or modified appropriately to make them applicable to other phenomena. To obtain an analytic solution, the method does not require linearization, discretisation, perturbation, or initial term guess. Additionally, this approach has certain drawbacks when dealing with singular situations, fractional nonlinearity, trigonometric nonlinearity, and at the poles. When compared to other analytical techniques, the method's convergence is high in differential, polynomial, and exponential cases. Convection, diffusion, and dispersion features of fluid dynamics—particularly gas dynamics, which is the movement of air, gases, or motion of objects, bodies, or materials through the air and its consequences on physical systems—were addressed in the paper. The efficiency, precision, and significance of solving nonlinear equations with initial and boundary conditions are highlighted in this paper.

Keywords: Nonlinear partial differential equations, domain of existence, Adomian polynomials, Dynamic equations, Advection and K(2, 2) equation, Shooting-Type Laplace-Adomian Decomposition Algorithm.

1. Preliminaries

Analytical solutions to nonlinear equations provide a physical knowledge of the phenomenon that numerical solutions do not, they are crucial in applied mathematics. Nonlinear equations are often solved analytically using relatively limited techniques, but numerical approaches that require the discretization of the variables lead to rounding off errors. The study of nonlinear physical and chemical phenomena depends on the accurate solution of nonlinear equations. Solving nonlinear problems is always difficult, especially when attempting to do so analytically. The popular Variational Iteration Methods [2, 13, 14, 15, 17, 22, 23] and perturbation methods [12, 16] rely on the existence of small or large parameters, namely the initial term assumption and the perturbation quantities or variational quantities. These kinds of assumptions are rarely found in nonlinear problems in science and engineering these days. Consequently, some non-perturbative techniques have been developed, including the Adomian Decomposition Method [1, 10], which is independent of small parameters. However, neither perturbative nor non-perturbative methods can readily modify or regulate the convergence zone and the rate of a given approximate series [13].

2. Nonlinear Partial Differential Equation

It is hard to think of any area of applications where its influence is not felt; these are the fundamental domains of applied analysis. Understanding and modelling nonlinear processes or phenomena—which frequently result in partial differential equations—has received a great deal of attention in recent decades. Additionally, one of the most active fields in applied mathematics and analysis is nonlinear partial differential equations [25].

A partial differential equation is an equation involving an unknown function of several variables and its derivatives. Let $\mathbb{R}^m$ be the euclidean space and $\Omega \subset \mathbb{R}^m$ be an open subset and a function $G\colon \Omega \to \mathbb{R}$ of multiple independent variables at $\mathrm{x} = (x_1, x_2, x_3, \ldots., x_i \ldots., x_m) \,\epsilon\, \Omega$ and of their partial derivatives $\frac{\partial \mathrm{G}}{\partial \mathrm{x}_i}$ can be defined as

$$\frac{\partial \mathrm{G(x)}}{\partial \mathrm{x}_i} = \lim_{\Delta x \to 0} \frac{\mathrm{G}(x_1, x_2, x_3, \ldots., x_i + \Delta x \ldots., x_m) - \mathrm{G(x)}}{\Delta x}$$

This function G is a C^2- function or totally differentiable and all its derivatives exists in a neighbourhood of x and are continuous. For general cases, one can define a function

$$G\colon \mathbb{R}^{c_p} \times \mathbb{R}^{c_p - 1} \times \mathbb{R}^{c_p - 2} \times \ldots . \times \mathbb{R}^c \times \mathbb{R} \times \Omega \to \mathbb{R}$$

where $\Omega \subset \mathbb{R}^c$ is an open set for integer p ≥ 1 generates an m^{th} order partial differential equation of the form

$$\mathrm{G}(D^k \mathrm{u(x)}, D^{k-1}\mathrm{u(x)}, D^{k-2}\mathrm{u(x)}, \ldots\ldots, \mathrm{Du(x)}, \mathrm{u(x)}, \mathrm{x}) = 0 \qquad (2.1)$$

with $\mathrm{x} \,\epsilon\, \Omega$ and $u\colon \Omega \to \mathbb{R}$ is an unknown function. Solution of (2.1) means that the functions $u(x)$ satisfies this equation, possibly among those functions satisfying additional boundary conditions on some part of the boundary $\partial\Omega$ of the domain ∂.

A system of partial differential equations is a collection of several PDEs for multiple unknown functions, such as $\mathrm{u(x)}, \mathrm{v(x)}, \mathrm{w(x)}, \ldots.$ This system of PDEs includes several derivatives of these unknown functions with certain boundary conditions.

3. Nature of boundary conditions

In mathematical models, additional information on the boundary $\partial\Omega$ of the domain ∂, depending on time or the spatial dimension or on a portion Γ of the boundary $\partial\Omega$ are needed. These informations are the initial or final conditions, it may be time or dimension and as boundary conditions, it may be space or dimension. If these conditions are specified, then the PDEs are boundary value problems. If the boundary conditions give a value to the domain; for all $x \,\epsilon\, \Gamma$, $\mathrm{u(x)}$ is fixed, then it is a Dirichlet boundary condition. If the boundary condition gives a value to the normal derivative of the problem: for all $x \epsilon \Gamma$,

$$\frac{\partial \mathrm{u(x)}}{\partial \mathrm{v}} = (\nabla u, v)(x)$$

is fixed, where v is the outward normal to Γ, it is a Neumann boundary condition. Different boundary conditions are used on different parts of the boundary $\partial\Omega$ of the domain, then the boundary conditions are mixed. Combination of function values and normal derivatives (Neumann) in a linear way are Robin or Newton boundary conditions. For the majority of PDEs, an exact solution is unknown, and in certain situations, it's even unclear if there is a unique solution. In order to provide precise approximations of the answer in real-world scientific situations, numerical techniques have been developed in conjunction with the examination of simple cases. This is a difficult task, and depending on the problem at hand, even exactly identifying the equation's solution may be difficult. The operator involved determines the character of PDEs and their solutions. The existence of solutions requires the following prerequisites.

3.1 General solution of Partial Differential Equations

The phenomenon or process involved determines the type of PDE solution. The existence of operators and their characteristics are crucial for a thorough analysis of it. Given two Hilbert spaces, X and Y, and a linear operator between them, the following equation is produced:

$$\mathrm{Ly} = \mathrm{g} \qquad (3.1)$$

where $\mathrm{g} \in \mathrm{X}$, is continuous and differentiable, L is bijective linear operator. The existence of a solution to the equation (3.1), the function g is equivalent to the condition $\mathcal{R}(\mathrm{L}) = \mathrm{X}$, the range space of the operator, while the uniqueness of the solution is equivalent to the condition $\mathrm{N(L)} = \{0\}$, the null space.

3.2 Closed Operator

Given two Banach spaces U and V, an operator $\mathrm{L} : \mathrm{U} \to \mathrm{V}$ is closed if for any sequence $(v_n)_{1 \le n \le \infty} \subset U$, $\mathrm{v}_n \to \mathrm{v}$ and $\mathrm{L}(v_n) \to \omega$, imply that $v \epsilon \mathrm{U}$ and $\omega = Lv$

3.2 Monotonicity

Let V be a Hilbert space and $\mathrm{L} \in \mathcal{L}(\mathrm{V}, \mathrm{V}')$. The operator L is monotone if $\langle \mathrm{Lv}, \mathrm{v} \rangle \ge 0$ for all $v \epsilon \mathrm{V}$, it is strictly monotone if $\langle \mathrm{Lv}, \mathrm{v} \rangle > 0$ for all $\mathrm{v} \ne 0 \in \mathrm{V}$, it is strongly monotone if there exists a constant $\mathrm{C} > 0$ such that $\langle \mathrm{Lv}, \mathrm{v} \rangle \ge \mathrm{C}||v||^2$ for all $v \epsilon \mathrm{V}$. For every $u \epsilon \mathrm{V}$, the element $Lu \epsilon \mathrm{V}'$ is a linear form. The symbol $\langle \mathrm{Lu}, \mathrm{v} \rangle$ represents duality pairing, it means the application of Lu to $v \epsilon \mathrm{V}$. Let V be a Hilbert space, $f \epsilon \mathrm{V}'$ and $L \in \mathcal{L}(\mathrm{V}, \mathrm{V}')$, a strongly monotone linear operator, then, for every $f \epsilon \mathrm{V}'$, the operator equation $\mathrm{Lu} = \mathrm{f}$ has a unique solution $u \epsilon \mathrm{V}$.

4. Laplace-Adomian Decomposition Algorithm

Consider the nonlinear partial differential equation

$$\frac{\partial^k u}{\partial t^k} = f(t, u, u_{1x}, u_{2x}, u_{3x}, \ldots., u_{mx}), \quad k = 1,2,3, \ldots \ldots \qquad (4.1)$$

where u_{mx}; $m = 1, 2, 3, \ldots \ldots$, denote the partial differentiation with respect to x, m times subject to the initial condition $u(x,0) = g_1(x)$, $u_{1x}(x,0) = g_2(x)$, $u_{2x}(x,0) = g_3(x), \ldots \ldots$, $u_{(k-1)x}(x,0) = g_{(k-1)}(x)$ in which $g_i(x); i = 1, 2, 3, \ldots \ldots, (k-1)$ are known functions and $u(x,t)$ be the solution to be determined.

Applying Laplace Transform, the operator denoted by $\mathcal{L}$ in (4.1) and applying initial condition given above, we get

$$\mathcal{L}\big(u(x,t)\big) = \frac{1}{s}g_1(x) + \frac{1}{s^2}g_2(x) + \frac{1}{s^3}g_3(x) + \cdots + \frac{1}{s^{k-1}}g_{(k-1)}(x) + \frac{1}{s^k}\mathcal{L}(f(t,u,u_{1x},u_{2x},u_{3x},\ldots,u_{mx})) \quad (4.2)$$

Assume that $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ be the solution of the nonlinear partial differential equation and the nonlinear terms, say $N(u)$ are decomposed into Adomian polynomials $A_n(x,t)$'s.

$$N(u) = \text{nonlinear term } = f(t,u,u_{1x},u_{2x},u_{3x},\ldots,u_{mx}) = \sum_{n=0}^{\infty} A_n(x,t)$$

(If more than one nonlinear term occurs, we use more Adomian polynomials). The assumed solution and Adomian polynomial into (4.2) and using the iterative algorithm, we obtain the following equations

$$\mathcal{L}\big(u_0(x,t)\big) = \frac{1}{s}g_1(x) + \frac{1}{s^2}g_2(x) + \frac{1}{s^3}g_3(x) + \cdots + \frac{1}{s^{k-1}}g_{(k-1)}(x) \quad (4.3)$$

$$\mathcal{L}\big(u_1(x,t)\big) = \frac{1}{s^k}\mathcal{L}(A_0) \quad (4.4)$$

$$\mathcal{L}\big(u_2(x,t)\big) = \frac{1}{s^k}\mathcal{L}(A_1) \quad (4.5)$$

$$\mathcal{L}\big(u_3(x,t)\big) = \frac{1}{s^k}\mathcal{L}(A_2) \quad (4.6)$$

In general,

$$\mathcal{L}\big(u_{n+1}(x,t)\big) = \frac{1}{s^k}\mathcal{L}(A_n) \quad (4.7)$$

Using inverse Laplace transform into (4.3), we get the initial term $u_0(x,t)$. With the help of the initial term, we can find the successive terms $u_1(x,t)$, $u_2(x,t)$, and so on. Hence the analytic/exact solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

and the sum up to $(n-1)$ terms is denoted as $S_n(x,t)$ and its convergence is $u(x,t)$ as $n \to \infty$

5. Some Frontier Nonlinear Partial Differential Equations

The Laplace-Adomian Decomposition Algorithm (LADA), a technique for determining the analytical solution of linear and nonlinear equations, was created and presented by the author [5, 6, 7, 20, 21]. The method is a powerful analytical tool for nonlinear problems that has been successfully applied to a variety of nonlinear problems in research and engineering. The fundamental idea of LADA and STLADA will be discussed in the paper, after which it will be applied to a few frontier problems involving nonlinear partial differential equations that arise in the fields of gas dynamics, fluid mechanics, and solitary waves.

5.1 Nonlinear convection and dispersion in $K(n,n)$ equation

The general form of $K(n,n)$ equation include nonlinear convection and dispersion is the pioneering equation for compactons as

$$u_t + (u^n)_x + (u^n)_{xxx} = 0 \quad (5.1)$$

In solitary wave theory, compactons are defined as solitons with finite wavelengths or solitons free of exponential tails. Compactons are generated as a result of the delicate interaction between nonlinear convection $(u^n)_x$ with the genuine nonlinear dispersion $(u^n)_{xxx}$.

Consider a $K(2,2)$ equation [2, 3, 4, 17, 22]

$$u_t + (u^2)_x + (u^2)_{xxx} = 0 \quad (5.2)$$

with initial condition $u(x,0) = f_1(x)$. This equation have been solved by VIM [2, 22, 23] and HAM [3, 4] for the case $f_1(x) = x$.

Taking Laplace transform on both sides of (5.2), one get

$$\mathcal{L}\big(u(x,t)\big) = \frac{x}{s} - \frac{2}{s}\mathcal{L}(uu_x) - \frac{2}{s}\mathcal{L}(uu_{xxx}) - \frac{6}{s}\mathcal{L}(u_x u_{xx}) \quad (5.3)$$

First three terms of the Adomian polynomials of the nonlinear terms

$N(u) = uu_x = \sum_{n=0}^{\infty} A_n(x,t)$, $M(u) = uu_{xxx} = \sum_{n=0}^{\infty} B_n(x,t)$ and $P(u) = u_x u_{xx} = \sum_{n=0}^{\infty} A_n(x,t)$ are

$$A_0 = u_0 u_{0x}$$
$$A_1 = u_0 u_{1x} + u_1 u_{0x}$$
$$A_2 = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}$$
$$B_0 = u_0 u_{0xxx}$$
$$B_1 = u_0 u_{1xxx} + u_1 u_{0xxx}$$
$$B_2 = u_0 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx}$$
$$C_0 = u_{0x} u_{0xx}$$
$$C_1 = u_{0x} u_{1xx} + u_{1x} u_{0xx}$$
$$C_2 = u_{0x} u_{2xx} + u_{1x} u_{1xx} + u_{2x} u_{0xx}$$

Applying the assumed solution and Adomian polynomial into (5.3), the iterative algorithm yields the initial term $u_0(x,t)$ and the general term as follows.

$$\mathcal{L}\big(u_{n+1}(x,t)\big) = -\frac{2}{s}\mathcal{L}\big(A_n(x,t)\big) - \frac{2}{s}\mathcal{L}\big(B_n(x,t)\big) - \frac{6}{s}\mathcal{L}\big(C_n(x,t)\big) \quad (5.4)$$

Putting $n = 0, 1, 2, 3, \dots \dots$ into (5.4), the successive terms of the solution are

$$u_1(x,t) = -2xt$$
$$u_2(x,t) = 4xt^2$$
$$u_3(x,t) = -8xt^3$$

$u_4(x,t) = 16xt^4$ and so on.

Hence the analytic/exact solution is $u(x,t) = x - 2xt + 4xt^2 - 8xt^3 + 16xt^4 - \cdots \approx \frac{x}{1+2t}$ valid in the region $|t| < \frac{1}{2}$

5.2 Advection equations

5.2.1 One dimensional Advection problems – Generalization

The fluid within which diffusion takes place is moving in a preferential direction. The advective flux and combine it with diffusive flux, the total flux is

$$q = cu - D\frac{\partial c}{\partial x}$$

At constant values of u and D, this can be formulated as

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2}$$

where $\frac{\partial c}{\partial t}$ is the accumulation, $\frac{\partial c}{\partial x}$ is the advection and $\frac{\partial^2 c}{\partial x^2}$ is the diffusion. Its solution for the prototypical case of an instantaneous at $t = 0$ and at $x = 0$ release is

$$c(x,t) = \frac{M}{\sqrt{4\pi Dt}} e^{-\frac{(x-ut)^2}{4Dt}}$$

For the case of decay, the advection -diffusion-decay leads to the equation

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2} - Kc$$

and its prototypical solution for an instantaneous and localized release is

$$c(x,t) = \frac{M}{\sqrt{4\pi Dt}} e^{-\frac{(x-ut)^2}{4Dt} - Kt}$$

For a special case, consider a one dimensional nonlinear homogeneous advection problem [3, 17, 22, 23]

$$u_t + uu_x = 0 \qquad (5.5)$$

with initial condition $u(x,0) = f_2(x)$. This problem was solved by using Harmonic Analysis Method (HAM) [3, 12] and VIM [22] for the case $f_2(x) = -x$.

The iterative algorithm of LADA yields the following general term.

$$\mathcal{L}\big(u_{n+1}(x,t)\big) = -\frac{1}{s}\mathcal{L}\big(A_n(x,t)\big) \qquad (5.6)$$

Adomian polynomials of the nonlinear term $N(u) = uu_x$ are described in the above example. Successive terms of the solution are

$$u_1(x,t) = -xt$$
$$u_2(x,t) = -xt^2$$
$$u_3(x,t) = -xt^3$$

$u_4(x,t) = -xt^4$ and so on.

Hence the approximate/exact solution is $u(x,t) = -xt - xt^2 - xt^3 - xt^4 - \cdots \approx \frac{x}{t-1}$ valid in the region $|t| < 1$

If the problem defined in the domain $\Omega : [-a, a]$, $a > 0$, the advection equation $u_t + uu_x = 0$ with initial condition $u(0,x) = u_0(x) = x$ for all $x \in \Omega$, is continuous such that $u(\pm a) = \pm a$ and the boundary condition $u(t, \pm a) = \pm a$, $t > 0$. Every solution satisfying $u_t + uu_x = 0$ and the initial condition is constant along the characteristic lines $x_{x_0(t)} = x_0(t+1), x_0 \in \Omega$. Hence, the solution to this problem with initial conditions cannot be constant in time at the end points of Ω. Hence this problem has no solution.

5.2.2 Two dimensional advection problems

Consider a two dimensional nonlinear nonhomogeneous advection problem [3, 22, 23],

$$u_t + uu_x = 2t + x + t^3 + xt^2 \tag{5.7}$$

with initial condition $u(x,0) = f_3(x)$. This problem was solved using VIM and ADM [23, 24] for the case $f_3(x) = 0$. Exact solution of the problem is $u(x,t) = xt + t^2$. By Laplce Adomian Decomposition Algorithm, (5.7) yields

$$\mathcal{L}\big(u(x,t)\big) = \frac{2}{s^3} + \frac{x}{s^3} + \frac{6}{s^5} + \frac{2x}{s^4} - \frac{2}{s}\mathcal{L}(uu_x) \tag{5.8}$$

By iterative scheme, the initial term is

$$u_0(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{4}xt^3$$

with $\frac{1}{4}t^4 + \frac{1}{4}xt^3$ as the noise term. The successive terms are as follows.

$$u_1(x,t) = -\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{2}{15}xt^5 - \frac{7}{32}t^6 - \frac{1}{63}xt^7 - \frac{1}{96}t^8$$

$$u_2(x,t) = \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} + \frac{38}{2835}xt^9 + \frac{143}{2880}t^8 + \frac{22}{315}xt^7 + \frac{7}{72}t^6 + \frac{2}{15}xt^5$$

$$u_3(x,t) = -\frac{163}{4257792}t^{16} - \frac{13}{218295}xt^{15} - \frac{218201}{279417600}t^{14} - \frac{1412}{1216215}xt^{13} - \frac{4991}{777600}t^{12} - \frac{206}{22275}xt^{11} - \frac{15179}{604800}t^{10} - \frac{20}{567}xt^9 - \frac{113}{2880}t^8 - \frac{17}{315}xt^7$$

The noise terms appear between the two components u_0 and u_1. The noise terms are identified as the identical terms with opposite signs. We then cancel the noise terms $\pm\frac{1}{4}t^4 \pm \frac{1}{4}xt^3$ between the components u_0 and u_1, and justify the remaining terms of u_0 satisfy the equation. For nonhomogeneous equations, the appearance of noise terms, if the criterion set in [24] for this appearance exists will facilitate the calculations. However, the existence of noise term requires necessary conditions that are not always available. The successive sum up to n terms are denoted as

$$S_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_n(x,t)$$

and the analytic solution converges to $u(x,t)$ as $n \to \infty$. The following table shows its convergence and is compared with exact solution at $x = 0$. The error estimate up to $S_3(x,t)$ as shown in table 1. Fig.1 shows the exact solution and Fig.2, Fig.3 and Fig.4 compares the convergence of the first three approximation of the solution in 3-Dimension using LADA.

$$S_1(x,t) = t^2 + tx - \frac{1}{96}t^8 - \frac{1}{63}xt^7 - \frac{7}{72}t^6 - \frac{2}{15}xt^5$$

$$S_2(x,t) = t^2 + tx + \frac{113}{2880}t^8 + \frac{17}{315}xt^7 + \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} + \frac{38}{2835}xt^9$$

$$S_3(x,t) = t^2 + tx - \frac{15781}{2721600}t^{12} + \frac{1292}{155925}xt^{11} - \frac{9613}{604800}t^{10} - \frac{62}{2835}xt^9 - \frac{163}{4257792}t^{16} - \frac{13}{218295}xt^{15} - \frac{218201}{279417600}t^{14} - \frac{1412}{1216215}xt^{13}$$

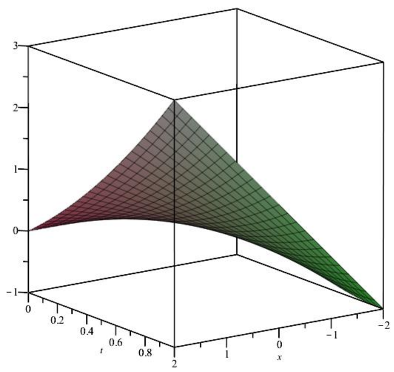

Fig.1 Exact solution in 3-D

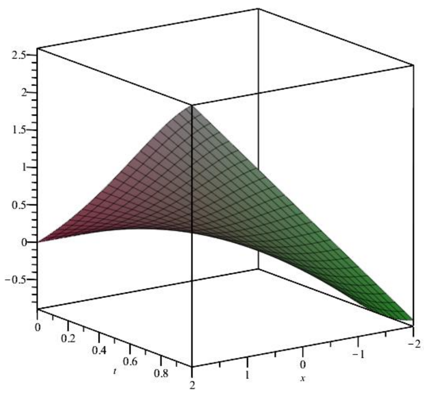

Fig.2 $S_1(x,t)$ in 3-D

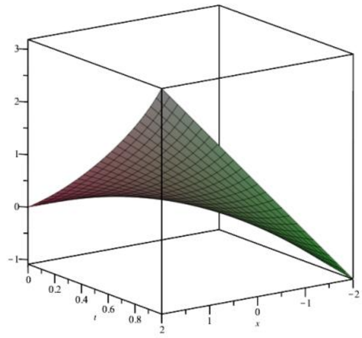

Fig.3 $S_2(x,t)$ in 3 − D

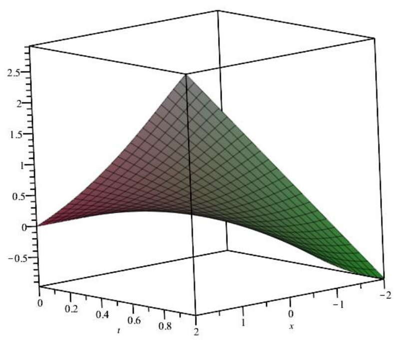

Fig.4 $S_3(x,t)$ in 3 − D

t	S_1	S_2	S_3	Exact	Absolute Error
0.0	0.000000000	0.000000000	0.000000000	0.00	0.00000000
0.1	0.009999902	0.010000000	0.009999999	0.01	1.0000E-08
0.2	0.039993751	0.040000101	0.039999998	0.04	4.1250E-08
0.3	0.089928441	0.090002629	0.089999903	0.09	1.0776E-08
0.4	0.159594951	0.160026689	0.159998233	0.16	1.1038E-08
0.5	0.248440212	0.250162405	0.249983014	0.25	6.7948E-08

Table 1: Convergence, comparison with exact and error estimates of the solution up to S_3

5.2.3 Higher dimensional advection problems

At two dimensional with velocity vector (u, v) along axis directions x and y

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right)$$

At three dimensional with velocity vector (u, v, w) along axis directions x, y and z

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + wv\frac{\partial c}{\partial y} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right)$$

An advection direction may not be active at the same time as diffusion in the same direction. Change the scales u, c and x respectively as U, C and L, the derivative $\frac{\partial c}{\partial x}$ is expressing, after all, a difference in concentration over a distance, we can estimate it to be approximately $\frac{C}{L}$, and the advection term scale as

$$u\frac{\partial c}{\partial x} \approx U\frac{C}{L}$$

Similarly, the second derivative $\frac{\partial^2 c}{\partial x^2}$ represents a difference of the gradient over a distance and is estimated at $\frac{\frac{C}{L}}{L} = \frac{C}{L^2}$ and the diffusion term scales as

$$D\frac{\partial^2 c}{\partial x^2} \approx D\frac{C}{L^2}$$

Equipped with these estimates, we can then compare the two processes by forming the ratio of their scales

$\frac{\text{advection}}{\text{diffusion}} \approx \frac{\frac{UC}{L}}{\frac{DC}{L^2}} = \frac{UL}{D}$, it is dimensionless

This ratio is called **Peclet number** denoted by Pe. That is, $Pe = \frac{UL}{D}$. If $Pe \ll 1$, may be $Pe < 0.1$, the advection term is significantly smaller than the diffusion term. Physically, diffusion dominates and advection is negligible. If $Pe \gg 1$, may be $Pe \gg 10$, the advection term is significantly bigger than the diffusion term. Physically, advection dominates and diffusion is negligible and is almost inexistent, with the patch of pollutant being simply moved along by the flow. If $Pe \approx 1$, may be $0.1 < Pe < 10$, the advection and diffusion terms are not significantly different and neither process dominates over the other. No approximation to the equation can be justified, and the full equation must be utilized.

5.3 Nonlinear Gas Dynamic equations

Gas dynamics is the study of how gases, air, and things move through the atmosphere and how this affects physical systems. In several domains, including hydrodynamics, plasma physics, nonlinear optics, and others, more nonlinear equations that characterised the motion of isolated waves localised in a limited region of space were simulated in the current scenario. The ground breaking compacton equation [25], the time fractional Boussinesq equation, conservative nonlinear oscillators, microelectromechanical systems, etc. are examples of $K(n,n)$ equations. The physical laws of conservation, such as the mass, momentum, and energy of the air flow across the medium, serve as the foundation for mathematical models of gas dynamics. The study of shock fronts [18], rare fractions and contact discontinuities are three major nonlinear wave equations which describe ideal gas dynamics behaviour.

Gas dynamics is synonymous with aerodynamics, which is the discipline of the study of aerobic movements such as those found in aeroplanes, spacecraft, and other aircraft. This is a fundamental area of interest in the design of spacecraft and aeroplanes with their corresponding propulsion systems. This field coincided with the advancements in both transonic and supersonic flight as aircraft accelerated and air density changed, significantly raising air resistance as airspeed approached the speed of sound or light. These investigations were subsequently found to be an effect brought on by the creation of shock waves surrounding the aircraft in wind tunnel tests. The theories of gas dynamics were developed as a result of significant advancements made to demonstrate the behaviour of compressible and high-speed flows.

Consider the nonlinear gas dynamic one spatial dimensional partial differential equation is of the form as

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + u(1-u) = f(x,t);\;\; 0 \le x \le 1, t > 0$$

with the condition $u(x,t_0) = g(x)$, where $f(x,t)$ and $g(x)$ are differentiable functions and A is a physical constant in the gas dynamic system. For the case $f(x,t)$, this model becomes homogeneous or nonhomogeneous nonlinear gas dynamic. Many researchers have proposed many methods for solving the equation analytically and numerically. Some notable methods are Homotopy Analysis Method (HAM) [12], Homotopy Perturbation Method (HPM), Natural Decomposition Method (NDM), Transform Homotopy Perturbation Method (THPM), Finite Difference Schemes (FDS) in the case of system of equations of gas dynamics, Modified Homotopy Perturbation Method (MHPM), Laplace Variation Iteration Method (LVIM) [17] and Projected Differential Transform Method (PDTM) [9].

5.3.1 Nonlinear Homogeneous Gas dynamic equation

In this section, the present method applied to solve a nonlinear homogeneous gas dynamic equation [8, 15, 19]

$$u_t + \frac{1}{2}(u^2)_x - u(1-u) = 0;\;\; 0 \le x \le 1, \qquad t > 0 \tag{5.9}$$

with initial condition $u(x,0) = f_4(x)$. Evans and Bulut used the Adomian Decomposition Method (ADM) [10], Hossein Jafari et al. [15, 16] using the Variational Iteration Method (VIM) and the Homotopy Perturbation Method (HPM) [16] for the case $f_4(x) = e^{-x}$.

Applying Laplace Transform to (5.9) and substituting the Adomian polynomials of the nonlinear terms, $N(u) = u(1-u) = \sum_{n=0}^{\infty} A_n(x,t)$ and $M(u) = uu_x = \sum_{n=0}^{\infty} B_n(x,t)$ where

$$A_0 = u_0 - {u_0}^2$$
$$A_1 = u_1 - 2u_0u_1$$
$$A_2 = u_2 - 2u_0u_2 - {u_1}^2\ ;$$
$$B_0 = u_0u_{0x}$$
$$B_1 = u_0u_{1x} + u_1u_{0x}$$
$$B_2 = u_0u_{2x} + u_1u_{1x} + u_2u_{0x}$$

and so on. The iterative scheme of LADA yields the following equations.

$$\mathcal{L}\big(u_0(x,t)\big) = \frac{1}{s}e^{-x} \tag{5.10}$$
$$\mathcal{L}\big(u_1(x,t)\big) = \frac{1}{s}\mathcal{L}\big(A_0(x,t)\big) - \frac{1}{s}\mathcal{L}\big(B_0(x,t)\big) \tag{5.11}$$
$$\mathcal{L}\big(u_2(x,t)\big) = \frac{1}{s}\mathcal{L}\big(A_1(x,t)\big) - \frac{1}{s}\mathcal{L}\big(B_1(x,t)\big) \tag{5.12}$$
$$\mathcal{L}\big(u_3(x,t)\big) = \frac{1}{s}\mathcal{L}\big(A_2(x,t)\big) - \frac{1}{s}\mathcal{L}\big(B_2(x,t)\big) \tag{5.13}$$

In general,

$$\mathcal{L}\big(u_{n+1}(x,t)\big) = \frac{1}{s}\mathcal{L}\big(A_n(x,t)\big) - \frac{1}{s}\mathcal{L}\big(B_n(x,t)\big) \tag{5.14}$$

From (5.10), using the inverse Laplace transforms the initial term as $u_0(x,t) = e^{-x}$. Putting $n = 0, 1, 2, \ldots.$ into (5.14), applying the initial term and Adomian polynomials, we get the successive terms of the solution as

$$u_1(x,t) = te^{-x}$$
$$u_2(x,t) = \frac{t^2}{2!}e^{-x}$$
$$u_3(x,t) = \frac{t^3}{3!}e^{-x}$$
$$u_4(x,t) = \frac{t^4}{4!}e^{-x} \text{ and so on.}$$

Hence the Analytic solution is is $u(x,t) = te^{-x} + \frac{t^2}{2!}e^{-x} + \frac{t^3}{3!}e^{-x} + \frac{t^4}{4!}e^{-x} + \cdots \approx e^{t-x}$

which is the exact solution. This problem was solved by Ramesh Rao [19] using Pade Approximants. The numerical solution obtained by Modified New Iterative Method (MNIA)by Falade Kazeem Iyanda [11].

For the special case,

$$u_t + uu_x - u(u-1)\ln(a) = 0; \;\; 0 \le x \le 1, \qquad a > 0 \tag{5.15}$$

satisfies the condition $u(x,t) = a^{-x}$. The iterative scheme of LADA, the initial term of (5.15) is $u_0(x,t) = a^{t-x}$ and the sum of the successive iteration yields the analytic solution $u(x,t) = a^{t-x}$, which is the exact solution.

5.3.2 Nonlinear nonhomogeneous Gas dynamic equation

Another nonlinear nonhomogeneous gas dynamic equation [11, 19]

$$u_t + \frac{1}{2}u_x + u(u-1) = -e^{t-x} \tag{5.16}$$

with $0 \le x \le 1,\; t > 0$ satisfies the condition $u(x,0) = -e^{-x}$.

Using Laplace iterative scheme in (5.15) and (5.16), can be transformed as

$$\mathcal{L}\big(u_0(x,t)\big) = -\frac{e^{-x}}{(s-1)^2} + \frac{1-e^{-x}}{s-1}$$

$$\mathcal{L}\big(u_1(x,t)\big) = -\frac{1}{s-1}\mathcal{L}({u_0}^2) - \frac{1}{s-1}\mathcal{L}(u_0u_{0x})$$

$$\mathcal{L}\big(u_2(x,t)\big) = -\frac{1}{s-1}\mathcal{L}(2u_0u_1) - \frac{1}{s-1}\mathcal{L}(u_0u_{1x} + u_1u_{0x})$$

Using inverse Laplace transform, these equations yields the following terms in the iterative procedure.

$$u_0(x,t) = e^t - e^{t-x}(1+t) \tag{5.17}$$

$$u_1(x,t) = te^{2t} - e^{2t} + e^t \tag{5.18}$$

$$u_2(x,t) = e^t - 2e^{2t} + e^{3t} + (e^{2t-x} - e^{3t-x})t \tag{5.19}$$

Hence the successive approximations of the analytic solution are as follows.

$$S_1(x,t) = 2e^t - e^{t-x}(1+t) + te^{2t-x} - e^{2t} \tag{5.20}$$

$$S_2(x,t) = 3e^t - e^{t-x}(1+t) + te^{2t-x} - 3e^{2t} + e^{3t} + (e^{2t-x} - e^{3t-x})t \tag{5.21}$$

The numerical solution obtained in [11] by Modified New Iterative Method (MNIA) is compared with the present method and exact solution as shown in Table 2. The graphical 3-D plots of the solution up to $S_1(x,t)$ and $S_2(x,t)$ can be seen in Fig.5, Fig.6 which can be compared with the exact solution in Fig.7.

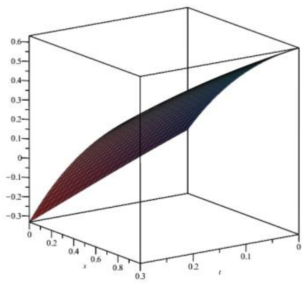
Fig.5: 3-D plot for gas flow, up to $S_1(x,t)$

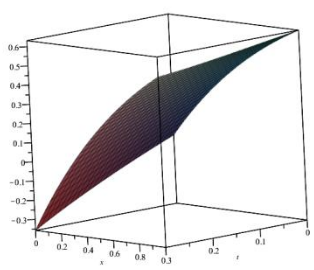
Fig.6: 3-D plot for gas flow, up to $S_2(x,t)$

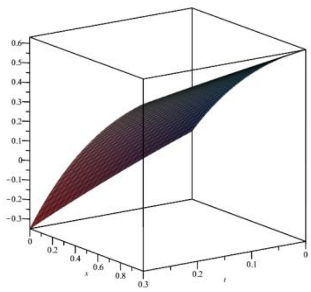
Fig.7: 3-D plot for gas flow - exact solution

t	S_1	S_2	**Exact**	**Absolute Error**
0.00	0.000999500	0.000999500	0.0009995002	0.00000000
0.05	-0.050156780	-0.0502236115	-0.050220351	6.4519E-05
0.10	-0.103515655	-0.1041242107	-0.104066300	5.5617E-05
0.15	-0.158687816	-0.1609942591	-0.160672989	1.9955E-04
0.20	-0.215170817	-0.2212914496	-0.220181966	5.0138E-03
0.25	-0.272329635	-0.2856994172	-0.282742033	1.0352E-02
0.30	-0.329374416	-0.3552042405	-0.348509623	1.8847E-02

Table 2: Convergence, comparison with exact and error estimates of the solution up to S_2

The authors have already utilised LADA to solve nonlinear integral/ integro-differential equations, nonlinear ordinary differential equations with initial/boundary conditions, and systems of these kinds of equations. One can see from these applications, LADA developed by the first author [4, 5, 6, 7, 8, 9, 20, 21] provides an exact solution with simple computations. Furthermore, we want to avoid making any assumptions about the first term. However, an initial guess of the first term that meets initial/boundary requirements is required for the other methods (HPM, VIM, and HAM). Compared to LADA, there is less convergence of this algorithm to the exact.

6 Conclusion

The Laplace-Adomian Decomposition Algorithm for nonlinear partial differential equations occurring in gas flow, advection, and $K(2,2)$, the special case of $K(n,n)$ is described in the paper along with a full discussion and graphic explanation of its physical aspects. The outcomes showed that LADA is highly effective, accurate, user-friendly, and quickly convergent, and it offers precise/analytic answers for a variety of problems. Even when there are more nonlinear components, this method is very simple to implement. This work also demonstrates the validity and great promise of the LADA for solving nonlinear problems in science and engineering. All the computations are done using Maple.

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