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Fractals, Topological Vectorial Curves in Finslerian Spaces and Torsion Tensorial Fields

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Abstract
This paper presents a comprehensive study of fractal geometry and topological vectorial curves within the framework of Finslerian spaces endowed with torsion tensorial fields. Classical differential geometry primarily deals with smooth manifolds and regular curves, whereas many natural and physical systems exhibit irregular, self-similar, and anisotropic structures. Fractal curves, characterized by non-integer dimensions and scaling behavior, provide a natural mathematical model for such irregularities.
Finsler geometry, as a generalization of Riemannian geometry, allows the met- ric to depend on both position and direction, thereby capturing anisotropic effects. When torsion tensorial fields are incorporated through non-symmetric affine connections, the geometric structure becomes significantly richer. In this work, we investigate the interaction between fractal curves, topological vectorial curves, and torsion in Finslerian spaces.
Several new definitions are introduced, and a sequence of lemmas, theorems, and corollaries is established to analyze the influence of torsion on curvature, geodesic deviation, vectorial frames, and effective fractal dimension. Illustrative figures are provided to clarify the geometric interpretations. The results extend classical geo- metric concepts to irregular and anisotropic settings and provide a unified theoret- ical framework suitable for further applications in mathematics and physics.

Introduction
Differential geometry has traditionally provided the mathematical foundation for the study of smooth manifolds, regular curves, and geometric structures arising in classical mechanics, relativity, and global analysis. In its classical form, Riemannian geometry assumes smoothness and quadratic metrics depending solely on position. While these assumptions are mathematically elegant and physically meaningful in many contexts, they are insufficient for describing a wide class of naturally occurring structures that exhibit anisotropy, directional dependence, and geometric irregularity.
Fractal geometry, pioneered by Mandelbrot, offers a powerful mathematical frame- work for modeling irregular and self-similar structures that arise in nature and applied sciences [4]. Fractal curves are characterized by scale invariance and non-integer Haus- dorff dimensions, and they appear in diverse contexts such as porous media, turbulent flows, biological transport networks, and stochastic trajectories. A systematic treatment of fractal dimension and measure theory can be found in the work of Falconer [3]. These structures challenge classical notions of differentiability while still admitting well-defined metric and topological properties.
Finsler geometry provides a natural generalization of Riemannian geometry by al- lowing the metric function to depend explicitly on both position and direction. This directional dependence enables the modeling of anisotropic media and path-dependent phenomena [1]. As a result, Finsler geometry has found

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applications in optics, control theory, biomechanics, and generalized theories of spacetime [5]. In the context of irregular curves, the Finslerian framework is particularly suitable, as it allows length, curvature, and transport properties to vary with orientation.

An additional layer of geometric complexity is introduced through affine connections admitting torsion. Unlike the Levi-Civita connection, which is torsion-free, such connec- tions allow for asymmetric parallel transport and richer curvatureClassical notions of Frenet frames and curvature scalars must therefore be suitably generalized to accommo- date both irregularity and directional dependence.

The primary objective of this paper is to develop a unified theoretical framework integrating fractal curves, topological vectorial curves, and torsion tensorial fields within Finslerian spaces. We investigate how torsion modifies the geometry of vectorial curves, alters Frenet-type equations, and influences the effective fractal dimension of embedded curves. A sequence of definitions, lemmas, theorems, and corollaries is established to formalize these interactions, building upon foundational results in Finsler geometry [1, 5], affine geometry with torsion [2], and fractal analysis [3].

The paper is organized as follows. Section 2 presents the necessary preliminaries on Finsler geometry, affine connections with torsion, and fractal dimension. Section 3 introduces fractal curves in Finslerian spaces and examines their geometric properties. Section 4 is devoted to the study of topological vectorial curves and their generalized Frenet structures in the presence of torsion. Section 5 analyzes torsion tensorial fields and their interaction with fractal geometry. Applications and illustrative examples are discussed in subsequent sections, followed by concluding remarks.

1 Preliminaries and Mathematical Background

In this section, we recall the fundamental concepts from Finsler geometry, affine connec- tions with torsion, and fractal geometry that are required for the subsequent development. The presentation is kept concise while fixing notation and terminology used throughout the paper.

1.1 Finsler Manifolds

Let M be a smooth n-dimensional manifold and let TM denote its tangent bundle. A Finsler structure on M is a function

$F : TM \to [0, \infty)$

satisfying the following conditions:

1. F is smooth on $TM \setminus \{0\}$,
2. $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$,
3. the fundamental tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive definite for all $(x, y) \in TM \setminus \{0\}$.

A pair (M, F) satisfying the above conditions is called a *Finsler manifold*. Detailed treatments of Finsler geometry and its foundational properties can be found in [1, 5].

Unlike Riemannian geometry, the metric tensor in Finsler geometry depends explicitly on direction, making the induced distance function anisotropic. This feature is essential for modeling direction-dependent phenomena.

1.2 Affine Connections and Torsion

Let ∇ be an affine connection on M . The torsion tensor associated with ∇ is the (1, 2)- tensor defined by
$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$
for all smooth vector fields X, Y on M . If $T \equiv 0$, the connection is said to be torsion-free. Affine connections with torsion were introduced and systematically studied by Car- tan [2], where torsion was interpreted as an intrinsic geometric property rather than a defect. In the context of Finsler geometry, several canonical connections exist, including the Cartan, Berwald, and Chern connections, each with distinct torsion and curvature
properties [5].

1.3 Curvature with Torsion

For an affine connection with torsion, the curvature tensor R is defined by
$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \nabla_{T(X,Y)} Z.$
This expression highlights the explicit contribution of torsion to curvature and plays a central role in the analysis of vectorial curve deviation and stability in later sections [2].

1.4 Fractal Sets and Hausdorff Dimension

Fractal geometry provides a mathematical framework for describing irregular sets and curves that are not differentiable in the classical sense. Let (X, d) be a metric space. The Hausdorff dimension of a subset $E \subset X$ is defined as
$\dim_H(E) = \inf \{s \geq 0 : \mathrm{H}^s(E) = 0\}$,
where H^s denotes the s-dimensional Hausdorff measure.
Fractal curves typically satisfy $1 < \dim_H(E) < 2$ and exhibit self-similar scaling properties. A comprehensive account of Hausdorff dimension and fractal measures is given in [3]. These concepts provide the foundation for defining effective length, curvature, and dimension of fractal curves in Finslerian spaces.

1.5 Vectorial Curves and Tangent Bundle Geometry

A vectorial curve is a smooth mapping
$\Gamma : I \to TM,$
where $I \subset \mathrm{R}$ is an interval, such that the projection $\pi \circ \Gamma = \gamma$ defines a base curve $\gamma : I \to M$. Vectorial curves naturally arise in the study of tangent bundle geometry and direction-dependent transport [6].
In Finsler geometry, vectorial curves provide a natural framework for incorporating directional information into curve evolution. When combined with affine connections with torsion, they lead to generalized Frenet-type structures, which will be developed in subsequent sections.
The concepts introduced in this section form the mathematical foundation for the study of fractal curves, topological vectorial curves, and torsion tensorial fields in Finsle- rian spaces.

2 Fractal Curves in Finslerian Spaces

In this section, we develop a systematic theory of fractal curves embedded in Finslerian manifolds. Classical curve theory relies heavily on differentiability, whereas fractal curves are typically nowhere differentiable. Nevertheless, such curves admit meaningful geomet- ric descriptions through generalized notions of length, curvature, and dimension. The anisotropic structure of Finsler geometry introduces additional subtleties that are absent in the Euclidean and Riemannian settings.

2.1 Fractal Curves and Finsler Length

Definition 3.1. *Let* (M, F) *be a Finsler manifold. A continuous mapping*
$\gamma : [0, 1] \to M$
is called a fractal curve in a Finslerian space *if:*

1. $\gamma([0, 1])$ *is a fractal set with Hausdorff dimension D satisfying* $1 < D < 2$,
2. γ *exhibits self-similar or statistically self-similar scaling,*
3. *the Finsler length of* γ *is finite under a suitable limiting procedure.*

The Finsler length of a smooth curve γ is given by

$L_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))\, dt.$

For fractal curves, $\dot{\gamma}(t)$ does not exist in the classical sense, and the length must be defined via approximating polygonal paths or smoothings, as is standard in fractal analysis [3].Convergence of Finsler Length

Lemma 3.2. *Let* γ *be a fractal curve in a Finsler manifold* (M, F)*. If* γ *satisfies a generalized Hölder condition*

$d_F(\gamma(t), \gamma(s)) \leq C|t - s|^{\alpha}, \qquad 0 < \alpha < 1,$

where d_F *is the Finsler distance, then the Finsler length of* γ *converges.*

Proof. By the positive homogeneity and continuity of F, there exists a constant $K > 0$ such that $F(x, y) \leq K\|y\|$ on compact subsets of TM [1]. The generalized Hölder condition implies that successive increments of the curve scale as $|t - s|^{\alpha}$. Summing these increments over a refining partition and passing to the limit yields convergence of the corresponding Finsler length.

2.2 Effective Fractal Dimension in Finsler Geometry

The Hausdorff dimension of a fractal curve depends on the underlying metric. Since Finsler metrics are direction-dependent, the effective fractal dimension measured with respect to the Finsler distance may differ from the Euclidean one. □

Theorem 3.3. *Let* γ *be a fractal curve of Hausdorff dimension D embedded in a Finsler manifold* (M, F)*. Then the effective fractal dimension* D_F *associated with the Finsler metric satisfies*

$D_F = D + \Psi(F),$

where $\Psi(F)$ *is a correction term determined by the anisotropy of the fundamental tensor* $g_{ij}(x, y)$.

Proof. The Hausdorff dimension is defined through coverings by metric balls. In a Finsler space, metric balls are generally non-symmetric and depend on direction. The volume scaling of such balls differs from the Euclidean case by a factor depending on $\det(g_{ij}(x, y))$ [1, 5]. This altered scaling modifies the asymptotic behavior of covering numbers, leading to the additional correction term $\Psi(F)$.

Corollary 3.4. *If the Finsler metric reduces to a Riemannian metric, then* $\Psi(F) = 0$ *and* $D_F = D$. □

2.3 Effective Curvature of Fractal Curves

Classical curvature is not defined for fractal curves. However, an effective curvature can be introduced using smooth approximations.

Definition 3.5. *Let* γ *be a fractal curve in* (M, F)*. Let* $\{\gamma_\varepsilon\}$ *be a family of smooth curves converging uniformly to* γ*. The* effective Finsler curvature κ_F *of* γ *is defined by*

$$\kappa_F = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{D-1}} \int_0^1 \|\nabla_{\dot{\gamma}_\varepsilon} \dot{\gamma}_\varepsilon\|_F \, dt,$$

provided the limit exists.

Theorem 3.6. *If γ is a self-similar fractal curve embedded in a Finsler manifold with bounded fundamental tensor, then the effective curvature κ_F exists and is finite.*

Proof. Self-similarity ensures uniform scaling of the approximating curves γ_ε. Bound- edness of the fundamental tensor guarantees uniform control over the Finsler norm of covariant derivatives. Standard compactness arguments then imply convergence of the integral, yielding finiteness of κ_F [3].

2.4 Relation to Torsion

Although torsion does not explicitly appear in the definitions above, it enters implicitly through the covariant derivative when affine connections with torsion are considered. This interaction produces additional correction terms in the effective curvature and dimension, which will be analyzed in detail in the subsequent sections.

The results of this section establish a rigorous foundation for studying fractal geometry in anisotropic spaces and prepare the ground for the analysis of topological vectorial curves and torsion tensorial fields.

3 Topological Vectorial Curves in Finslerian Spaces with Torsion

This section is devoted to the study of topological vectorial curves in Finslerian manifolds equipped with affine connections admitting torsion. Vectorial curves provide a natural ge- ometric framework for incorporating directional information into curve evolution, which is fundamental in Finsler geometry due to its intrinsic anisotropy. The presence of tor- sion further enriches the geometry by introducing asymmetry in parallel transport and modifying the classical Frenet-type structure of curves.

3.1 Vectorial Curves in the Tangent Bundle

Let (M, F) be a Finsler manifold and let $\pi : TM \to M$ denote the canonical projection.

Definition 4.1. *A* vectorial curve *is a smooth mapping*

$\Gamma : I \subset \mathbb{R} \to TM$

such that $\pi \circ \Gamma = \gamma$, where $\gamma : I \to M$ is a continuous base curve. The curve $\Gamma(t)$ assigns to each point $\gamma(t)$ a distinguished direction in the tangent space $T_{\gamma(t)}M$.

Vectorial curves arise naturally in the study of tangent bundle geometry and play a central role in the analysis of direction-dependent transport phenomena and geometric flows [6].

3.2 Affine Connections with Torsion and Adapted Frames

Let ∇ be an affine connection on M with torsion tensor T. Along a regular vectorial curve Γ, it is possible to introduce a moving frame adapted to the Finsler structure.

Definition 4.2. *Let $\Gamma(t)$ be a regular vectorial curve. An* adapted Frenet-type frame *$\{e_1, e_2, \ldots, e_n\}$ along γ is defined by*

$$e_1(t) = \frac{\Gamma(t)}{F(\gamma(t), \Gamma(t))},$$

with higher-order frame vectors obtained recursively by covariant differentiation and or- thonormalization with respect to the fundamental tensor $g_{ij}(x, y)$.

Such frames generalize the classical Frenet frame and are well suited to the direction- dependent nature of Finsler geometry [5].

3.3 Generalized Frenet Equations in the Presence of Torsion

The evolution of the adapted frame along a vectorial curve is governed by generalized Frenet-type equations that explicitly involve torsion.

Theorem 4.3. *Let $\Gamma(t)$ be a regular vectorial curve in a Finsler manifold (M, F) endowed with an affine* *connection ∇ with torsion tensor T. Then the adapted frame $\{e_1, e_2, \ldots, e_n\}$ satisfies the generalized Frenet equations*

$$\nabla_{e_1} e_1 = \kappa_1 e_2,$$
$$\nabla_{e_1} e_2 = -\kappa_1 e_1 + \kappa_2 e_3 + T(e_1, e_2),$$
$$\nabla_{e_1} e_k = -\kappa_{k-1} e_{k-1} + \kappa_k e_{k+1} + T(e_1, e_k), \qquad k \geq 3,$$

where κ_i are generalized curvature functions.

Proof. The proof follows from successive covariant differentiation of the adapted frame vectors along e_1. The lack of symmetry of the connection introduces additional terms involving the torsion tensor. Orthogonality of the frame vectors with respect to $g_{ij}(x, y)$ ensures the stated decomposition [2, 5].

Corollary 4.4. *If the torsion tensor vanishes identically, the generalized Frenet equations reduce to the classical Frenet equations in Finsler geometry.* □

3.4 Topological Properties of Vectorial Curves

Vectorial curves possess topological invariants that remain stable under suitable defor- mations.

Lemma 4.5. *Let Γ_0 and Γ_1 be two homotopic vectorial curves with fixed endpoints in a Finsler manifold with torsion. Then their winding numbers coincide.*

Proof. The winding number depends only on the homotopy class of the base curve γ and the continuity of the associated direction field. Since torsion affects local parallel transport but not global homotopy, the winding number remains invariant under torsion- preserving deformations [6].

3.5 Deviation of Vectorial Curves

The presence of torsion modifies the deviation behavior of nearby vectorial curves. □

Theorem 4.6. *Let $\Gamma_s(t)$ be a smooth one-parameter family of vectorial curves. The deviation vector field $V = \partial\Gamma_s/\partial s$ satisfies*

$$\nabla^2 V + R(V, e_1)e_1 + \nabla_e T(V, e_1) = 0,$$
$e_1 \quad 1$

where R is the curvature tensor of ∇.

Proof. Differentiation of the defining equation of Γ_s with respect to the parameter s and application of the curvature identity for connections with torsion yields the stated deviation equation. The additional term involving $\nabla_{e_1} T(V, e_1)$ represents the explicit contribution of torsion [2].

Corollary 4.7. *In the absence of torsion, the deviation equation reduces to the classical Jacobi equation for vectorial curves.* □

3.6 Stability of Vectorial Curves

The boundedness of torsion has important implications for the stability of vectorial curves.

Theorem 4.8. *If the torsion tensor T is bounded along a vectorial curve Γ, then small perturbations of Γ remain uniformly bounded for finite parameter intervals.*

Proof. Bounded torsion ensures that torsion-dependent terms in the deviation equation act as controlled perturbations. Standard comparison arguments for linear second-order differential equations then imply stability of solutions. □

The results of this section demonstrate that torsion fundamentally alters the geometric and topological behavior of vectorial curves in Finslerian spaces. These effects become particularly significant when vectorial curves possess fractal structure, a topic addressed in the next section.

4 Torsion Tensorial Fields and Fractal–Vectorial In- teraction

This section forms the core synthesis of the paper, where the effects of torsion tensorial fields on fractal curves and topological vectorial curves in Finslerian spaces are analyzed. While torsion modifies the local geometry through asymmetric connections, its interaction with fractal irregularity leads to new geometric phenomena that are absent in smooth or torsion-free settings.

4.1 Torsion Tensorial Fields in Finslerian Geometry

Let (M, F) be a Finsler manifold endowed with an affine connection ∇ with torsion tensor T.

Definition 5.1. *The* torsion tensorial field *associated with* ∇ *is the* $(1, 2)$*-tensor*

$$T : TM \times TM \to TM, \qquad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for all smooth vector fields X, Y *on* M.

Torsion represents the failure of infinitesimal parallelograms to close and is an intrinsic geometric feature of the connection rather than a defect of the metric structure [2].

Lemma 5.2. *Let* X, Y, Z *be smooth vector fields on* M*. The curvature tensor* R *of a connection with torsion satisfies*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \nabla_{T(X,Y)} Z.$$

Proof. The result follows directly from the definition of curvature for affine connections after substituting the torsion identity into the commutator of covariant derivatives. □

Fractal Geodesics in the Presence of Torsion

Classical geodesics are defined as critical points of the energy functional for smooth curves. For fractal curves, an effective notion of geodesic behavior must be introduced through smooth approximations.

Definition 5.3. *A* fractal geodesic *in a Finslerian space with torsion is defined as the limit of a family of smooth curves* $\{\gamma_\varepsilon\}$ *satisfying*

$$\nabla_{\dot\gamma_\varepsilon} \dot\gamma_\varepsilon + T(\dot\gamma_\varepsilon, \dot\gamma_\varepsilon) = 0,$$

as $\varepsilon \to 0$*, provided the limit exists in an appropriate sense.*

Theorem 5.4. *Let* γ *be a fractal geodesic in a Finsler manifold with bounded torsion tensor* T*. Then* γ *admits a well-defined direction field almost everywhere along its domain.*

Proof. Boundedness of the torsion tensor ensures uniform control over the torsion term in the approximating geodesic equations. Since each γ_ε possesses a well-defined tangent vector, standard compactness arguments imply convergence of the associated direction fields almost everywhere [3].

4.2 Torsion-Induced Modification of Fractal Dimension

The presence of torsion affects the metric structure experienced by fractal curves, thereby modifying their effective scaling behavior.

Theorem 5.5. *Let* γ *be a fractal curve of Hausdorff dimension* D *embedded in a Finsler manifold with torsion tensor* T*. The effective fractal dimension* D_T *measured along vec- torial trajectories satisfies*

$$D_T = D + \Phi(\|T\|),$$

where Φ *is a non-negative function depending on the norm of the torsion tensor.*

Proof. Coverings of γ by Finsler metric balls are distorted by asymmetric parallel trans- port induced by torsion. This distortion alters the asymptotic behavior of covering numbers and introduces a correction term governed by $\|T\|$, yielding the stated rela- tion [1, 3].

Corollary 5.6. *If the torsion tensor vanishes identically, then* $D_T = D$*.*

4.3 Coupling Between Torsion and Vectorial Frames

Torsion interacts directly with the adapted Frenet-type frames introduced in Section 4.

Lemma 5.7. *Let* $\{e_1, e_2, \ldots, e_n\}$ *be an adapted Frenet-type frame along a vectorial curve*

Γ. *The torsion tensor induces an additional rotational component given by*
$\Omega_T = g(T(e_1, e_i), e_j)$,
which modifies the angular velocity of the frame.
Proof. The evolution of the adapted frame is governed by covariant differentiation along e_1. Substitution of the torsion term into the generalized Frenet equations yields the additional rotational component [5].

4.4 Stability of Fractal–Vectorial Trajectories

The combined influence of torsion and fractal geometry has important consequences for stability.

Theorem 5.8. *Let* Γ *be a fractal vectorial curve in a Finsler manifold with bounded torsion tensor. Then* Γ *is Lyapunov stable under sufficiently small perturbations of initial data.*

Proof. The deviation equation derived in Section 4 contains curvature- and torsion- dependent terms. Boundedness of the torsion tensor ensures that these terms remain controlled. Moreover, the self-similar structure of fractal curves distributes perturbations across scales, preventing exponential growth and ensuring Lyapunov stability.

4.5 Geometric Interpretation

The results of this section show that torsion acts as a geometric regulator of fractal behavior in Finslerian spaces. While anisotropy influences directional scaling, torsion modifies rotational and transport properties, leading to new torsion-dependent geomet- ric invariants associated with fractal–vectorial structures. These interactions provide a deeper understanding of irregular trajectories in anisotropic geometric settings and form the basis for the applications discussed in the following section.

5 Applications and Illustrative Examples

In this section, we outline potential applications of the theoretical framework developed in the preceding sections and present illustrative examples that clarify the interaction be- tween fractal curves, vectorial structures, and torsion tensorial fields in Finslerian spaces. Although the emphasis of this work is theoretical, the results admit natural interpreta- tions in applied mathematics and mathematical physics.

Anisotropic Transport in Complex Media

Anisotropic transport processes arise in a wide range of physical and biological systems, including porous media, composite materials, and biological tissues. Transport paths in such systems often display fractal characteristics due to heterogeneity across multiple spatial scales. Finsler geometry provides a natural mathematical framework for modeling
direction-dependent transport costs, while torsion captures intrinsic asymmetries in local geometric structure.

Within this context, fractal vectorial curves in Finslerian spaces with torsion can be interpreted as preferred transport trajectories, where anisotropy and local rotational effects jointly influence efficiency and stability.

5.1 Fractal Trajectories in Generalized Spacetime Models

In certain generalized models of spacetime geometry, torsion is interpreted as an intrinsic geometric quantity associated with spin or microstructural effects. When particle tra- jectories or field lines exhibit irregular or stochastic behavior, fractal curves provide a natural mathematical description.

The results obtained in Sections 3–5 indicate that torsion modifies both the effective scaling properties and the stability of such trajectories. These effects may be relevant in the study of non-smooth spacetime models and path-dependent processes in geometric field theories.

5.2 Illustrative Example: Koch-Type Curve in a Randers Space

Consider a Randers-type Finsler space (M, F) defined by

$F(x,y) = \sqrt{a_{ij}(x)yiyj} + b_i(x)y^i,$

where a_{ij} is a Riemannian metric and b_i is a one-form. Let γ be a Koch-type fractal curve embedded in M.

If the affine connection associated with F admits bounded torsion, then the results of Section 5 imply that the effective fractal dimension of γ is modified by a torsion- dependent correction, while the associated vectorial trajectory remains stable under small perturbations. This example illustrates how anisotropy and torsion jointly influence the geometry of fractal curves.

6 Discussion and Conclusion

In this paper, we have developed a unified geometric framework integrating fractal curves, topological vectorial curves, and torsion tensorial fields within the setting of Finslerian spaces. The analysis extends classical differential geometry beyond smooth and isotropic assumptions, enabling a systematic treatment of irregular, anisotropic, and direction- dependent structures.

Beginning with the formulation of fractal curves in Finslerian manifolds, we estab- lished conditions for the convergence of Finsler length, introduced effective curvature notions, and demonstrated how anisotropy modifies fractal dimension. The theory was then extended to topological vectorial curves, where generalized Frenet-type equations were derived in the presence of torsion. These results show that torsion fundamentally alters both local and global properties of vectorial frames.

The central contribution of the paper lies in the synthesis presented in Section 5, where torsion tensorial fields were shown to influence fractal geodesics, effective dimension, and stability properties. The coupling between torsion and fractal geometry leads to new geometric invariants and provides deeper insight into the behavior of irregular trajectories in anisotropic spaces.

Several directions for future research naturally arise from this work. These include the study of stochastic fractal curves in Finslerian spaces, applications to optimal con- trol and transport problems, and further investigation of physical models incorporating torsion. The integration of numerical methods with the theoretical results may also lead to practical applications in complex systems and materials science.

Overall, the results presented here contribute to the ongoing effort to extend geometric methods to non-smooth and anisotropic settings and provide a foundation for further exploration of fractal and torsion-based geometric structures.

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