International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Zero Divisor Graph of a Lattice with two Atoms and Unique Sublattice

Dr. Pramod Namdeo Tayade

Associate Professor, Mathematics, Dr. Annasaheb G. D. Bendale Mahila Mahavidyalaya (Autonomous). Jalgaon (Ms)India

Abstract

Let L be a finite lattice with two atoms. In this paper we have shown existence of sublattice N such that N and L have two atoms. Studied zero-divisor graph of a lattice L and N. Further, explored relation between both zero divisor graphs and they are complete bipartite graph.

Keywords: Unique Sublattice, Zero Divisor Graph .

1. Introduction

In [3], I. Beck (1988), introduced the idea of zero divisor graph of commutative ring R with identity. He was interested in coloring of ring R. In [1], Anderson and Naseer continued this investigation. Anderson and Livingston in [2] associated a zero divisor graph to a commutative ring R with identity by different method. They defined zero divisor graph T(R) of a ring R, whose vertices are non-zero zero divisors of R and two distinct vertices x and y are adjacent if and only if xy=0.

In literature many authors have studied zero divisor graphs of rings, lattices and other graphs associated to other algebraic structures. Considering zero divisors and their annihilators, annihilating ideal graphs of different structures have studied in [9, 10]. For lattices see [6, 8].

Zero divisor graph of a finite lattice is studied in [5] by E.Estaji et al(2012). To expose relation between graph theory and lattice theory in [4], Deore et al (2019) studied the zero divisor graph of a lattice and introduced unique ideal. In [11], P. N. Tayade(December 2025) explored a class of finite, atomic lattices with two atoms. Further P. N. Tayade(January 2026) generalized the idea for atomic, finite lattice with finite number of atoms and introduced unique sublattice in[12]. In this paper P. N. Tayade (January 2026) studied zero divisor graph of a atomic, finite lattice with two atoms and its unique sublattice, both are bipartite graphs. If the lattice is Boolean lattice then both graphs are same.

2. Preliminaries Lattice Theory and Graph Theory

For self explanation, we recall some definitions and basic results.

A poset (L, ≤) is said to form a lattice if inf{a,b } and sup{a,b} exist in L, for any a, b∈L. In this case we write inf {a,b }=a∧b, sup{a,b}=a∨b.

A lattice with finite number of elements is finite lattice. A lattice is an algebra (L, ∧,∨) if the following conditions holds.

1. a∧a=a, a∨a=a.
2. a∧b= b∧ a, a∨b=b∨a.
3. a∧(b∧c)= (a∧b)∧c; a∨ (b∨c)= (a∨b) ∨c.

4. 4) $a\vee(a\wedge b)=a$, $a\wedge(a\vee b)=a$

For a lattice, L= (L,∧,∨), let ≤ be an order defined below.

For x, y in L, set x≤y if and only if x∧y=x, then (L, ≤) is an partially ordered set and any two elements have least upper bound (lub) and greatest lower bound (glb).

Conversely, let P be an partially ordered set with glb{x, y}, lub {x, y}∈P. Define

x∧y= glb{x, y} and x∨y= lub{x,y} then (P,∧, ∨) is lattice.

Let S be non empty subset of a lattice L. If x∧y, x∨y∈S for x, y∈S, then (S, ∧, ∨) is called sub lattice of a lattice L. For a, b ∈L, with a≤b be any two elements, the closed interval is denoted by [a,b] and [a, b] ={ x∈L: a≤ x ≤b}. The interval [a, b] is the sub lattice of L. For x∈[a,b] be any element, if their exist k∈L such that x∧k=a, x∨k=b then x is a complement of k relative to [a, b]. A lattice L is said be a bounded if there exist elements, say 0 and 1 in L such that 0≤x≤1, for all x∈L. Least element is 0 and 1 is greatest element of L. If there exist b∈L such that a∧b=0, a∨b=1, for a∈L, then b is said to be complement of a. If each element of L has complement the lattice is said to be complemented lattice. If each element of a bounded lattice has a unique complement, the lattice is said to be uniquely complemented. If b,c are elements in L such that b<x< c for no x in L then c is cover of b.

An element (0≠)a∈L is said to be atom of lattice L if a covers 0. Set of atoms of lattice L is denoted by A (L). A lattice L is said to be atomic, if for every nonzero element x∈L, their exist an atom a, such that a≤x . If a∧(b∨c)=(a∧b)∨(a∧c) for all a,b,c∈L, then the lattice L is to be distributive

For a≥ b, a∧(b∨c)=b∨(a∧c) ; a, b, c∈L, a lattice L is called modular

A non empty sub set I of L is said to be ideal of lattice L, if i∨j∈I and k∧i∈I, for i, j∈I, k∈L.

For x∈L, (x] ={y∈L:y≤x} is principal ideal generated by element x. An element a ∈L is said to be zero divisor of lattice L if their exist (0≠) b∈L such that a∧b=0. The set of all zero divisors in lattice L is denoted by Z (L). The set of non-zero zero divisors is Z* (L)={0≠k ∈L : there exist 0≠b∈L such that k∧b=0}.

In this paper all lattices are atomic, finite, with two atoms and some conditions, if not specified.In the next section we prove that there exist a finite , atomic, distributive and uniquely complemented sub lattice.

We recall some definitions. A graph **G**=(V(G), E(G)), consist of two sets. V(G), the vertex set of graph, sometimes denoted by V, is the set of elements called vertices. E(G) set of edges e={u,v}.

Edge is the arc or line segment joining two points u and v in set V. For two distinct vertices u and v if there exist finite number of distinct vertices $u=u_1, u_2, u_3 \ldots\ldots\ldots u_n=v$ such that each pair of vertices (u_k, u_{k+1}), $1\le k\le n-1$ is an edge then such finite sequence of vertices is called path from vertex u to v. The number of edges in the path is the length of the path. Let the length of the shortest path from u to v is d(u, v). If there is no such path then d(u, v)=0. The length of the largest path between two vertices in G is diameter . Cycle is the closed path and the number of edges in the cycle is its length. The length of smallest cycle in G is girth of G, denote gr(G). If G not contains cycle then gr(G)=0. A graph is complete if each pair of vertices joined by an edge, denoted by $K_{n.}$ where n is the number of vertices in graph G. For an integer r≥2, the vertex set V of graph G, is partitioned into r classes such that every edge has its end vertex in different classes such graph is r –partite graph is denoted by $K_{n1, n2,\ldots nr.}$ where n_i is the number of vertices in i^{th} class. An undirected graph without loop and parallel edges is simple graph. For graph theory, see[7].

Let L= (L, ∧, ∨) be a lattice with least element 0. An element x ∈L is said to be zero divisor if their exist a non zero element y∈L such that x∧y=0. The set of all zero divisors in lattice L is denoted by Z(L). The set of non zero zero- divisors is Z* (L)= Z(L)\{0}. The zero divisor graph of lattice L, denoted G(L) is a undirected graph whose vertex set is Z* (L) and two distinct vertices x and y are adjacent if x∧y=0.

All lattices under consideration are finite, atomic with two atoms.

3. Existence of Unique Sublattice

Let L be bounded lattice with least bound say 0 and least upper bound say 1. An element a∈L is called co atom if a is covered by 1. A lattice L is said to be diamond lattice if each element a∈L, a∉{0,1} is an atom and co atom. A diamond lattice with n number of atoms is denoted by D_n.

Theorem .3.1.([4], Theorem 2) Let L be a non diamond lattice with the least element 0 and having a finite set of atoms. Then there exists a finite, distributive and uniquely complemented ideal B of L such that A(L)=A(B).

Corollary .3.2. ([4], Corollary2) Let L be a lattice with least element 0 and has two atoms and B be unique ideal of L. Then,

(a) Z (B) =B −{0, 1_J}, 1_J is the greatest element of ideal B.

(b) Z (B)⊆Z(L).

Definition 3.3. Let L be a lattice and a, b ∈L, with a≤b be any two elements. The closed interval is denoted by [a,b] and defined as [a,b] ={x∈L:a≤x≤b}. For an element i∈[a,b], if there exist k∈L such that i∧k=a and i∧k=b then k is complement of i relative to interval [a,b].

Theorem .3.4. Let L be a non diamond(excluding D_2) finite lattice and having two atoms. Then there exist a finite, distributive and uniquely complemented sublattice N of L such that A(N)=A(L).

Proof: Let A(L)={a,b} and N ={0,a,b, a∨b} then obviously , N is sublattice of L with A(N)= A(L). N is finite, each element has one complement.

Uniqueness, Suppose their exist another atomic sublattice P with two atoms such that N≠P. Implies ∃ x∉N and x∈P then x= 0 or a or b or a∨b it gives, x∈N contradicts.

Corollary .3.5. Let L be a lattice and N be unique sublattice of L. Then,

(a) Z^* (N) =N −{0, 1_J}, 1_J is the greatest element of sublattice N.

(b) Z (N)⊆Z(L).

Theorem.3.5. Let L be lattice with A(L)= {a,b} then |Z*(N)|=2.

Proof, Let set of atoms A(L)= {a,b}. Then, a∧b=0.

Theorem 3.6. Let (L, ≤) be a lattice with A(L)={a, b} and N be sub lattice then G(N) ⊆ G(L).

Proof: The set of nonzero, zero divisors of sub lattice N ; Z^*(N)⊆Z*(L).

G(N) has only one edge joining two atoms only, hence, it lies in G(L).

Theorem 3.7. Let L be a lattice and A(L) = { a,b } be its set of atoms. Then no element of subset $[a\vee b]^u$ is element of Z(L), where $[a\vee b]^u$={x∈L| x≥[a∨b }.

Proof: For x ∈[a∨b]^u , atomicity implies there exist a ≤x, , follows x∧ a = a ≠0.

Theorem 3.8. Let L be a Boolean lattice L= P(1,2), ∩, ∪) then L = N, is unique sublattice of L.

Corollary 3.9. . Let L be a Boolean lattice L= P(1,2), ∩, ∪) then G(N) =G(L).

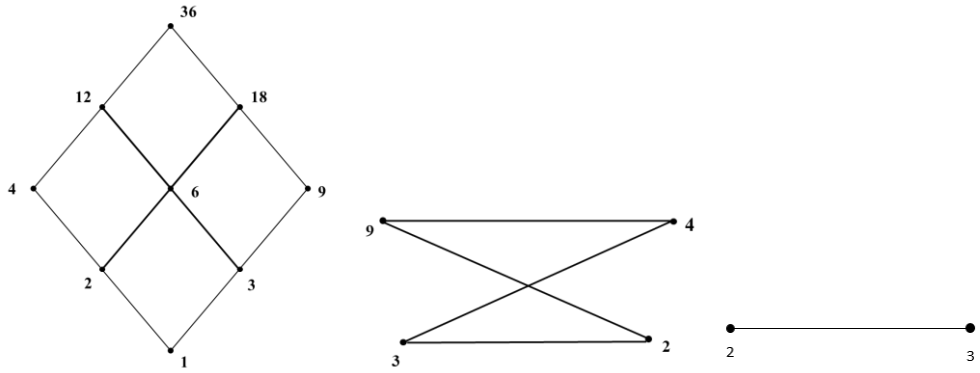
4. Examples: Zero Divisor Graph of lattice and Its Unique Sublattice

In this section, the diagram of a lattice, its zero divisor graph, and zero divisor graph of unique sub lattice have been drawn.

Example.4.1. Figure 1, L=(D(36), |) lattice of divisors of 36 under divisibility order.

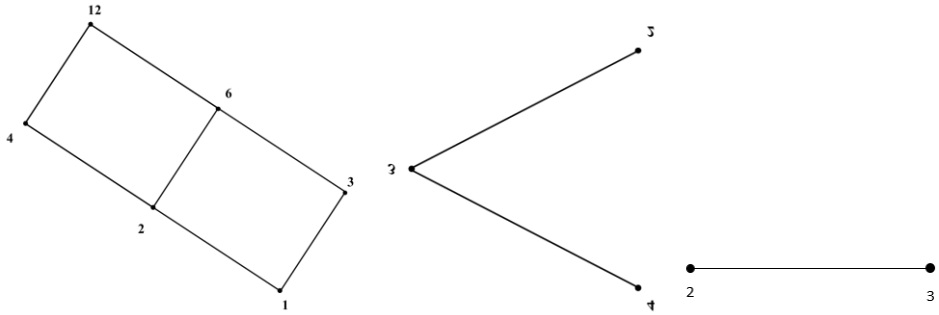
Figur2.G(L), zero divisor graph of L, Figure3.G(N) zero divisor graph of N

Figure1. L **Figure 2. G(L)** **Figure 3. G(N)**



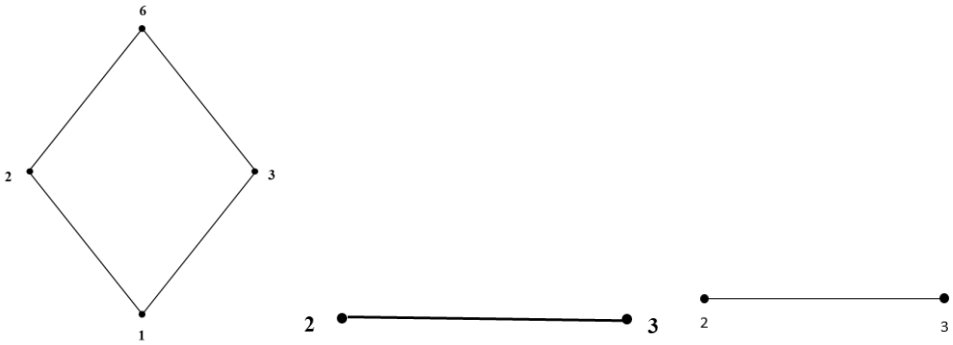
Example 4.2. Figure 4: L = (D(12), |) is lattice of divisors of 12 under divisibility order |, with D(12), ={1, 2, 3, 4, 6, 12}. Figure 5 : G(L) is zero divisor graph of L. Fig6:G(N) is zero divisor graph of unique sub lattice N.

Figure 4: L **Figure 5: G(L)** **Figure 6: G(N)**



Example 4.3 Figure 7: L=(D(6),|) is a lattice of divisors of 6 under divisibility order. Figure 8 G(L) is the zero divisor graph of lattice L. Figure 9:G(N) is zero divisor graph of unique sub lattice N=L.

Figure 7: L **Figure 8: G (L)** **Figure 9: G(N)**



5. Conclusions

A finite lattice containing two atoms have a finite, distributive and atomic unique sub lattice. Unique sub lattice contains two zero divisors. For lattice L with unique sub lattice N explored that $G(N) \subseteq G(L)$ and former is induced sub graph of latter. Sub lattices are isomorphic but their lattices are not. Further shown that, zero divisor graphs of unique lattices are isomorphic but, the lattices are not isomorphic. In a lattice containing two atoms the zero divisor graph of a unique sub lattice is complete bipartite graph $K_{1,1.}$ If, the lattice is Boolean then it is self unique sub lattice.

6. Acknowledgment:

I, extend my heartiest gratitude to the respected Prof. Rajendra P. Deore for his valuable suggestions and constructive cooperation for this research work.

I am thankful towards my wife, Jyotsna, sons Vishvesh, Khushvant, neace Shubhangi and brother Rajesh for their patience and support.

8. References

1. D. D. Anderson and M. Naseer, Beck's Colouring of Commutative Ring, J. Algebra. (1993) 159 (2), 500- 514.
2. D. F Anderson and P. S Livingston, The Zero Divisor Graph of Commutative Ring J. Algebra. (1999) 217(2), 434-447.
3. I. Beck, Coloring of Commutative Rings, J.Algebra, (1998) Vol. 116(1), 208-226.
4. R. P.Deore and P. N. Tayade, Zero Divisor Graph of a Lattice and its Unique Ideal, J. Math. (2019)9825495.
5. [5] E.Estaji and K.Khashyarmanesh, The Zero Divisor Graph of Lattice, Results in Mathematics (2012), Vol 61, 1-11.
6. G. Gratzer, General Lattice Theory, Springer, Germany, 1978.
7. John Clark Derek Allan Holton,A first look at graph theory, World Scientific Publishing Co. Pe.Ltd, 5 toh Tuck Link, Singapore596224.
8. Vijay K. Khanna, Lattices and Boolean Algebras, Vikas publishing house PVT LTD, New Delhi (2008).
9. P. N. Tayade, K. Surwade and R. P. Deore, Some Results on Annihilating Ideal Graph of Commutative Rings, JP Journal of Algebra, Number Theory and Applications, (2018) Vol.40 no 4..
10. Pramod Tayade, Kamalakar. Surwade, Rajendra Deore, Characterization of Annihilating Ideal Graph, JP Journal of Algebra, Number Theory and Applications, (2019) Vol.42 no 2. .
11. P. N. Tayade, Class of Lattices With Two Atoms, IJSDR-International Journal of Scientific Development and Research(www.IJSDR.org), ISSN:2455-2631, December)2025 Vol.10,Issue12, b741-b744,.
    DOI: https://doi.org/10.56975/ijsdr.v10i12.306704
12. P.N. Tayade, Unique Sublattice of a Lattice With Finite Number of Atoms,"IJSDR-International Journal of Scientific Development and Research (www.IJSDR.org), ISSN: 2455-2631,Vol.11, (January)2026 Vol.11,Issue1,.a657-a661.