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Dynamical System Structure of a Modified Equation

Esen Hanaç Duruk

Department of Mathematics, Adıyaman University, Altınsehir, Adıyaman, 02040, Turkey,

Abstract

We study the dynamical–system structure underlying the viscous conservation law $u_t + \sqrt{u}u_x = u_{xx}$which arises in models of nonlinear transport with diffusion. By introducing a traveling–wave ansatz, the equation is reduced to a planar autonomous system whose phase–plane geometry completely characterizes steady wave profiles. The reduced system exhibits a continuous line of equilibria corresponding to uniform states, with stability determined by the relative magnitude of the wave speed and the characteristic speed $\sqrt{u}$. A critical manifold separates stable and unstable regimes and plays a role analogous to a sonic boundary. We demonstrate through phase-plane analysis that heteroclinic relations between usually unstable and normally stable equilibria correlate to viscous shock profiles, whereas rarefaction behavior results from the absence of such connections. The geometric framework makes the role of degeneracy on the nonlinear flux near vanishing states clear and offers a transparent interpretation of entropy admissibility. Both the general characteristics of scalar viscous conservation laws and the unique asymmetries generated by the square-root nonlinearity are emphasized by a comparison with Burgers' equation. These results enable the development and verification of numerical approximations and provide insight into the qualitative dynamics of propagating waves.

Keywords: Phase plane analysis, travelling wave, Burgers equation

1. Introduction

Over the years, numerous researchers have examined nonlinear partial differential equations, which are now widely used in a variety of fields, including biological and physical contexts [1, 2, 4, 6–9, 14]. It falls into two categories: integrable and nonintegrable. It is simple to integrate the integrable equations up to their initial conditions. because the initial circumstances of the integrable equations dictate their structure. However, after a symbolic transformation is applied, some nonlinear differential equations become integrable. There are many exact solutions to these equations, and there isn't just one optimum way to find them. There are many prominent methods, including the Jacobi elliptic approach and the tanh function method [11]. The goal of using phase plane analysis to research dynamical systems is to get relevant information about any physical system that is represented by linear or nonlinear equations. Due to the stability of the differential equation's equilibrium points, the system's long-term structure is examined in this work. In particular, we look at the stability or instability that is influenced by the differential equation parameter. Here, we use phase plane trajectories that establish whether or not the associated ordinary differential equation system has a heteroclinic orbit to define the form of differential equation solutions. This leads to the identification of traveling wave solutions for partial differential equ-

ations. We look at the modified Burgers equation in this paper, specifically,

$u_t + \sqrt{u}u_x = u_{xx}$ *(1)*

with inital data

$$u(x,t) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

where *t* and *x* stand for time and distance, respectively. Using the phase plane analysis method, we attempt to get the exact solutions to the given equation.

2. The equation's dynamical system

In this part, we use the modification for travelling wave solutions of the altered Burgers equation to look at the dynamical system of Equation *(1),* namely,

$u(x,t) = U(\rho), \qquad \rho = x - s^*t$

$$U' = V$$

$V' = (\sqrt{U} - s^*)V$ *(2)*

Several authors, notably Murray [3], have studied dynamic system *(2).* The Jacobian of dynamic system *(2)* is

$$DF(U,V) = \begin{vmatrix} 0 & 1 \\ \frac{V}{2\sqrt{U}} & \sqrt{U} - s^* \end{vmatrix}, \qquad U>0.$$

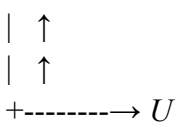
The dynamical system's equilibrium points are $(U,V)=(U_o, 0)$, $U_0 \geq 0$. Consequently, a continuous line (the *U*-axis) is formed by
the equilibria. Eigenvalues must be used to categorize each point as equilibria are not isolated. Eigenvalues are obtained by linearization at $(U_0, 0)$:

$\lambda_1 = 0, \qquad \lambda_2 = \sqrt{U_0} - s^*.$

The sign of $\lambda_2 = \sqrt{U_0} - s^*$ determines entirety of the phase portrait. The *U*-axis is divided into three distinct regions as a result.

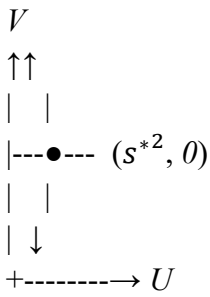
- Region $\sqrt{U_0} < s^*$, eigenvalues are $\lambda_1 = 0$, $\lambda_2 < 0$ which means in phase portrait behavior, motion along the *U*-direction is neutral and trajectories descend vertically toward *V*=0. This indicates that the phase portrait has stable equilibrium line. This is where viscous shocks are in the right state.

V
↑↓
| ↓
|---●--- $(U_0,0)$



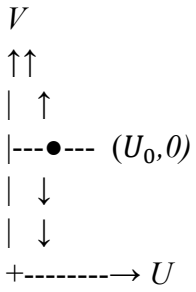
Trajectories terminate when they fall vertically onto the equilibrium line.

- Region $U_0 = s^{*2}$, eigenvalues are $\lambda_1 = 0,\ \lambda_2 = 0$. This indicates that the phase portrait has a bifurcation point.



Behavior that attracts and repels is distinguished at this point.

- Region $\sqrt{U_0} > s^*$, eigenvalues are $\lambda_1 = 0,\ \lambda_2 > 0$. This indicates that the phase portrait has typically unstable equilibrium line. Thus, solutions decrease exponentially in ρ. This is where viscous shocks are in the left state.



The equilibrium line is moved away from the trajectories.

The phase-plane geometry of the dynamical system *(2)* is therefore entirely controlled by the signs of V and $\sqrt{U_0} - s^*$ away from the equilibrium line $V=0$. Trajectories are bent toward the equilibrium line in the first region because the vertical component V' has the opposite sign of V. Points with $V>0$ travel downstairs and rightward, whereas points with $V<0$ move upward and leftward, creating a damping effect toward $V=0$. In contrast, trajectories are pushed away from the equilibrium line in the last area where V' has the same sign as V. Points with $V>0$ travel upward and rightward, whereas points with $V<0$ go lower and leftward, resulting in vertical magnitude expansion. The flow is entirely horizontal along the vertical line $U_0 = s^{*2}$, where the vertical velocity disappears. Consequently, $U_0 = s^{*2}$divides the phase plane away from equilibria into two regions: one where trajectories are repelled from the equilibrium line and the other where they are attracted to it.

3. Physical meaning of the Dynamic System

The process by which spatial gradients change in a frame traveling at speed s^* is described by the dynamical system. Whether the local convective speed is slower or faster than the wave determines the stability of the line of equilibria formed by uniform states. Viscous shock profiles are only possible when input enters the wave from upstream and diffuses downstream because diffusion stabilizes states behind the wave and destabilizes those ahead of it. The system generates time-dependent rarefaction waves rather than constant traveling fronts when this equilibrium is disrupted.

4. Numerical Significance

The phase-plane structure affects numerical simulations in a number of significant ways. First, the convective term's degeneracy as $u \to 0$ suggests that conventional high-resolution schemes created for Burgers' equation might show excessive numerical diffusion close to low-density states. Therefore, it is important to maintain accuracy in areas where diffusion predominates and convection is weak.

Additionally, the presence of a stability boundary at $U_0 = s^{*2}$, implies that the proper direction of information propagation should be respected by numerical approaches. Inaccurate shock speeds or nonphysical oscillations could arise from failing to capture the proper upstream-downstream stability. In order to ensure convergence to the physically permissible viscous profile, entropy-consistent discretizations are necessary.

In conclusion for verifying numerical results, the dynamical-system viewpoint offers an invaluable diagnostic tool. In the phase plane, computed traveling waves can be considered discrete approximations of heteroclinic orbits, providing a precise standard for determining whether a numerical method accurately captures the underlying continuous dynamics.

5. Conclusion

In the present research we examined the viscous conservation law,

$$u_t + \sqrt{u}u_x = u_{xx}$$

with a focus on traveling wave solutions' stability and structure. The problem was simplified to a planar dynamical system by introducing the traveling-wave ansatz $u(x,t)=U(x - s^*t)$. This formulation enabled it possible to apply phase-plane approaches to fully characterize wave profiles geometrically.

The translational invariance of the underlying partial differential equation is reflected in the simplified system's continuous line of equilibria, which corresponds to uniform states. Stability is solely determined by the relative magnitude of the wave speed and the characteristic speed $\sqrt{U_0}$, as demonstrated by linearization about these equilibria. States that satisfy $\sqrt{U_0} > s^*$ are typically unstable, whereas those that satisfy $\sqrt{U_0} < s^*$ are typically stable. The critical condition $\sqrt{U_0} = s^*$, which functions similarly to a sonic or characteristic boundary, indicates a shift in stability. The vertical stability boundary $U_0 = s^{*2}$ and the equilibrium line are two invariant sets that regulate the global phase-plane geometry. Heteroclinic linkages between unstable and stable equilibria provide rise to viscous shock profiles, whose existence is equal to the Rankine–Hugoniot and entropy conditions being satisfied. In the event that these geometric requirements are violated, the solution evolves as a time-dependent rarefaction rather than a constant propagating wave. The validity of shock and

rarefaction structures is thus explained in a straightforward and coherent manner by the dynamical-system theory.

A comparison with Burgers' equation demonstrates the current model's specificity as well as its universality. The nonlinear flux $f'(u) = \sqrt{u}u'$ introduces degeneracy at $u=0$ and confines the dynamics to the physically significant region $u\geq 0$, even though the entropy mechanism and qualitative phase–plane structure are shared. These characteristics set the current equation aside from the uniformly convex Burgers flux by generating asymmetric traveling waves and increased diffusive effects at low states.

Overall, the analysis illustrates that the geometric principles applicable to viscous traveling waves in this system are the same as those regarding standard scalar conservation laws, but with modifications that take into account the underlying physical constraints relevant to applications like traffic flow and porous-media transport.

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