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Eigenvalue and Singular Value Decomposition in Linear Algebra and Data Analysis2

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ABSTRACT

Singular value decompositions (SVD) have interesting and attractive algebraic properties and conveys important geometrical and theoretical insights about their linear transformation. The close connection between the singular value decomposition and the well-known theory of diagonalization for symmetric matrices makes the topic immediately accessible. In this paper, we have discussed Eigen Value Decomposition (EVD) and Singular Value Decomposition (SVD) and their applications.

INTRODUCTION

Singular Value Decomposition (SVD) is an important matrix factorization technique in linear algebra that applies to both real and complex matrices. Unlike eigenvalue decomposition, which is limited to square matrices, SVD can be used for any matrix of size m×n. It is closely connected to polar decomposition and provides a powerful way to analyze matrix properties.

SVD was first introduced in the field of differential geometry, where researchers aimed to understand whether two real bilinear forms could be transformed into each other using separate orthogonal transformations on the underlying vector spaces.

Eigenvalues and singular values play a vital role in many theoretical and practical applications. This project explores the concepts of eigenvalues, singular values, Eigenvalue Decomposition (EVD), and Singular Value Decomposition (SVD), along with their real-world applications. These techniques are widely used in areas such as machine learning, data analytics, image compression, and other computational disciplines.

1. EIGEN VALUE DECOMPOSITION (EVD)

If A is a real symmetric n×n matrix, it can be expressed as $A = VDV^T$ where V is an orthogonal matrix and D is diagonal. The columns of V are eigenvectors of A and together form an orthonormal basis for $\mathbb{R}^n$, while the diagonal elements of D are the corresponding eigenvalues. To highlight its relationship with the singular value decomposition, this factorization is called the eigenvalue decomposition (EVD) of A.

2. SINGULAR VALUE DECOMPOSITION (SVD)

In the singular value decomposition, we consider a general real m×n matrix A. The decomposition expresses A in terms of matrices U, Σ, and V, where U is an m×m orthogonal matrix and V is an n×n orthogonal matrix. As a result, Σ has the same dimensions as A and contains nonnegative entries σ_I along its main diagonal, arranged in decreasing order. These nonzero diagonal values are known as the singular values of A, while the columns of U and V are referred to as the left and right singular vectors, respectively.

The relationship between eigenvalue decomposition for symmetric matrices and singular value decomposition for general matrices becomes clearer when matrices are viewed as linear mappings. A symmetric matrix acts as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and the eigenvectors forming the columns of V provide a particularly convenient coordinate system. In this coordinate system, the transformation scales each component independently, either stretching or shrinking it based on the corresponding eigenvalue, while preserving orthogonality of the basis.

Consider the singular value decomposition of an m×n matrix A. In this setting, the associated linear mapping sends vectors from $\mathbb{R}^n$ into a different space, from $\mathbb{R}^m$, which naturally leads to choosing appropriate bases for both the input and output spaces. The columns of V and U serve as orthonormal bases for the domain and codomain, respectively. When vectors are expressed using these bases, the action of the transformation becomes easy to interpret: each component is scaled according to the corresponding singular value, with some components potentially eliminated or supplemented with zeros to reflect the change in dimensionality. Viewed this way, the SVD identifies orthonormal coordinate systems that reduce the transformation to its simplest representation—a diagonal matrix.

3. APPLICATIONS OF SVD

The singular value decomposition is most often used in problems that involve very large matrices, sometimes with dimensions in the thousands. Its practical importance comes from the availability of fast and reliable numerical algorithms that allow it to be computed efficiently on computers. The development of these algorithms relies on elegant mathematical ideas and is a subject of significant interest in its own right. Because of this combination of strong theory and computational efficiency, the SVD has become a fundamental tool in a wide range of applications.

3.1 PRINCIPAL COMPONENT ANALYSIS

Principal Component Analysis (PCA) is a major application of the singular value decomposition, offering a systematic, data-based way to represent high-dimensional, correlated data. It works by reducing the number of variables in a dataset, replacing them with a smaller set of new variables that preserve most of the original information. Although this reduction inevitably leads to some loss of precision, the goal is to balance accuracy with simplicity. By working with fewer dimensions, datasets become easier to visualize and interpret, and machine learning algorithms can operate more efficiently without being burdened by unnecessary variables.

3.2 LINEAR LEAST SQUARES

A linear least squares problem arises when we are given a collection of vectors and want to form their linear combination so that it best approximates a target vector. Suppose the vectors are $\{a_1, a_2, a_3, \dots, a_n\}$ and the vector to be approximated is b. The goal is to determine coefficients $x_1, x_2, x_3, \dots, x_n$ that minimize the error $|b - \sum_{i=1}^{n} x_i a_i|$

Such problems appear in many settings, including spaces of sequences, functions, or solutions to differential equations. When attention is restricted to linear combinations of a finite number of vectors, the problem can be expressed using matrices and vectors with finitely many components. Let A be the matrix whose columns are the vectors a_i, and let x be the vector of unknown coefficients. The task then becomes finding x that minimizes $|b - Ax|$. If A has dimensions m×n, each column vector a_i lies in $\mathbb{R}^m$. The least squares problem also has a clear geometric interpretation. The columns of A span a subspace S, and we seek the vector in S that lies closest to b. This vector is the orthogonal projection of b onto S. At

the solution, the error vector $b - Ax$ is orthogonal to the subspace S, which means it is orthogonal to each column of A. This condition leads to the equations

$a_i \,.(Ax - b) = 0 \quad for\ all\ i\ .$

In matrix form, this can be written as

$$A^T(Ax - b) = 0$$

or equivalently,

$$A^T Ax = A^T b.$$

These equations are known as the **normal equations** for the linear least squares problem. When the columns of A are linearly independent, the matrix $A^T A$ is invertible, and the solution is given by

$$x = (A^T A)^{-1} A^T b$$

3.3 DATA COMPRESSION USING REDUCED RANK APPROXIMATIONS

In this context, we start with an m×n matrix A that contains numerical data, and the objective is to represent A approximately using far fewer values than the original mn entries. The matrix is not viewed as a linear operator or an abstract algebraic structure; rather, it is treated simply as a collection of numbers. The aim is to find a reduced representation that preserves the most important patterns present in the data.

The rank of a matrix indicates how many of its rows or columns are linearly independent, making it a natural measure of redundancy. When a matrix has low rank, much of its information is repetitive, allowing it to be stored or described more compactly than by recording every entry. To illustrate this idea, consider a digital image created by scanning a photograph. The image can be represented as an m×n matrix in which each entry corresponds to a pixel intensity value between 0 and 1. Large-scale structures in the image lead to similarities among rows or columns of pixels, introducing redundancy. As a result, these dominant features can often be captured accurately by approximating the image with a matrix whose rank is significantly smaller than min (m,n).

CONCLUSION

In this paper, we examined the fundamental ideas behind Eigenvalue Decomposition and Singular Value Decomposition, highlighting both their theoretical foundations and practical significance. While EVD provides deep insight into the structure of symmetric matrices, SVD extends these ideas to general matrices and offers a unified geometric interpretation of linear transformations. By expressing matrices in terms of orthonormal bases and simple scaling operations, SVD reveals essential features such as rank, range, and redundancy in data. Its wide range of applications including principal component analysis, linear least squares problems, and data compression—demonstrates its importance in modern computational and data-driven fields. Together, EVD and SVD form powerful tools in linear algebra, bridging abstract mathematical theory with real-world applications and reinforcing their continued relevance in science and engineering.

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