A Generalised Class of Analytic Functions Defined By Q-Analogue of Fractional Calculus Operator

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Abstract
The aim of this paper is to introduce a generalized class of analytic function defined by q-analogue by using fractional calculus operator. We obtain coefficient estimate, distortion theorems, radii of close to convexity, starlikeness and convexity for functions belonging to the class $TB_q^{\lambda}(\alpha,\beta)$ of analytic starlike and convex functions defined by q-analogue of fractional calculus operator. We further show that closure theorems,$N_{k,q,\delta}(e,g)$ neighborhood and partial sums for functions in this class.
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Introduction
Let A be the class of analytic and univalent functions of the form
$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in U = \{z: z \in \mathbb{C} : |z| < 1\}$ (1.1) Let S be the subclass of A, consisting of function f(z) of the form (1.1), which are univalent in U.
Also let $S^*(\alpha)$ and $C(\alpha)$ denote which are the subclasses of S which are, Starlike and Convex functions of order α $(0 \le \alpha < 1)$, studied earlier by Roberston and Silverman in 1997.

$$S^*(\alpha) = \left\{f: f \in S \text{ and } Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha\right\} \tag{1.2}$$

$$and\ C(\alpha) = \left\{f: f \in S \text{ and } Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha\right\} \tag{1.3}$$

From (1.2) and (1.3)
$f(z) \in C(\alpha) \leftrightarrow zf'(z) \in S^*(\alpha)$
for $0 < q < 1$ the Jackson's q-derivative of a function $f(z) \in S$ is given by [19] (see also [2,3,10,13,,21])

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & for\ z \neq 0 \\ f'(0) & for\ z = 0 \end{cases} \tag{1.4}$$

For f(z) of the form (1.1), we have –

$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}$ (1.5)

Where $[n]_q = \frac{1-q^n}{1-q}$ $(0 < q < 1; n \in N = \{1,2, \dots \dots\})$ (1.6)

The following definitions of fractional derivatives and fractional integrals are due to OWA [25,26] and Srivastava [34,35]

Definition:1 The fractional integral of order λ is defined for a function f(z) of the form (1.2) by

$D_z^{-\mu} \mathrm{f}(z) = \frac{1}{\Gamma\mu} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d(\xi)$

Where $\mu > 0$, f(z) is an analytic function in a simply connected region of the z plane containing the origin and the multiplicity of $(z-\xi)^{\mu-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition:2 The fractional derivative of order λ is defined for a function f(z) of the form (1.2) by

$D_z^{\lambda} \mathrm{f}(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi$

Where $0 \le \lambda \le 1$, f(z) is an analytic function in a simply connected region of the z plane containing the origin and the multiplicity of $(z - \xi)^{-\lambda}$ is removed as in definition 1 above.

Definition: 3 Under the hypothesis of definition 2 the fractional derivative of order $n + \lambda$ is defined for a function by

$D_z^{n+\lambda} \mathrm{f}(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z)$

Where $0 \le \lambda \le 1$ and n $\in N_0 = \{0,1,2, \dots \dots \dots\}$

In 2011, Dixit and Porwal [15,16] introduce a new fractional derivative operator for function of the form (1.2) as follows

$$\Omega^0 f(z) = f(z)$$

$$\Omega^1 f(z) = \Gamma 1 - \lambda\, z^{1+\lambda}\, D_z^{1+\lambda} f(z)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

....

$$\Omega^n f(z) = \Omega(\Omega^{n-1} f(z))$$

Thus we note that

$$\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\varphi(k,\lambda)]^n a_k z^k$$

$$where\ \varphi(k,\lambda) = \frac{\Gamma k + 1\, \Gamma 1 - \lambda}{\Gamma k - \lambda}$$

It is worthy to note that for λ = 0, $\Omega^n f(z)$ reduces to familiar Salagean operator introduced by Salagean in [6]

They define the above operator for the function f of the form (1.1) as follows

$$\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\varphi(k,\lambda)]^n a_k z^k$$

The application of q- calculus is a current and interesting topic of research in Geometric function theory very recently.

Srivastava gave definitions and properties of q-calculus and fractional q-calculus in detail and its applications in his survey cum expository review article.

Now we recall the concept of q-calculus which was first introduced by Jackson for k $\in N$, the q number is defined as follows:

$[k]_q = \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j$ and $[0]_q = 0$, 0< $q < 1$

Hence $[k]_q$ can be expressed as a geometric series $\sum_{i=0}^{\infty} q^i$, when k → ∞ , the series converges to $\frac{1}{1-q}$. As $q \to 1$, $[k]_q \to 1$ and this is the bookmark as a q-analogue the limit as q→ 1 recovers the classical object.

The q-derivative of a function f is defined by

$D_q(f(z)) = \frac{f(qz)-f(z)}{(q-1)z}$, q≠ $1, z \neq 0$

And $D_q(f(z)) = f'(0)$, provided $f'(0)$ exists.

For a function f(z)= z^k observe that

$$D_q f(z) = D_q z^k = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}$$

Then

$$lim_{q\to 1} D_q(h(z)) = lim_{q\to 1}[k]_q z^{k-1}$$
$$= kz^{k-1} = f'(z)$$

Where f' is the ordinary derivative.

The q- Jackson definite integral of the function f is defined by

$\int_0^z f(t)d_q t = (1-q)z \sum_{n=0}^{\infty} f(zq^n)\, q^n$, z ∈ C

Now, we let R (m, q, α, β) denote the subclass.

In the present paper, we study the coefficient bounds, distortion bounds, extreme points, convolution conditions, convex combinations and discuss a class preserving integral operator.

Coefficient estimates

Unless indicated, we assume that 0≤ $\alpha < 1, \beta \geq 0,\ \lambda \geq 0, 0 < q < 1$ and f(z) ∈ $\boldsymbol{\tau}$.

Theorem 2.1 A function f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty} [[k]_q(1+\beta) - (\alpha+\beta)] \frac{\ulcorner k+1 \ \ulcorner 1-\lambda}{\ulcorner k-\lambda} a_k \leq 1-\alpha \tag{2.1}$$

Proof : Assume that (2.1) holds. Then it suffices to show that

$\beta \left| \frac{zD_q(\Omega^m f(z))}{\Omega^m f(z)} - 1 \right| - Re\left\{ \frac{zD_q(\Omega^m f(z))}{\Omega^m f(z)} - 1 \right\} \leq 1-\alpha$

We have

$$\beta \left| \frac{zD_q(\Omega^m f(z))}{\Omega^m f(z)} - 1 \right| - Re\left\{ \frac{zD_q(\Omega^m f(z))}{\Omega^m f(z)} - 1 \right\}$$

$$\leq (1+\beta) \left| \frac{zD_q(\Omega^m f(z))}{\Omega^m f(z)} - 1 \right|$$

$$\leq (1+\beta) \left| \frac{\sum_{k=2}^{\infty} [\varphi(k,\lambda)]^m a_k ([k]_q - 1) a_k}{z - \sum_{k=0}^{\infty} [\varphi(k,\lambda)]^m a_k z^k} \right|$$

$$\leq \frac{(1+\beta) \sum_{k=2}^{\infty} \frac{\ulcorner k+1 \ \ulcorner 1-\lambda}{\ulcorner k-\lambda} ([k]_q - 1) a_k}{1 - \sum_{k=2}^{\infty} \frac{\ulcorner k+1 \ \ulcorner 1-\lambda}{\ulcorner k-\lambda} a_k}$$

This last expression is bounded above by $(1-\alpha)$ since (2.1) holds.

Conversely if f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$ and z is real, then

$$\operatorname{Re}\left\{\frac{1-\sum_{k=2}^{\infty}\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}[k]_q\, a_k\, z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}\, a_k\, z^{k-1}}-\alpha\right\} \geq \beta\left|\frac{\sum_{k=2}^{\infty}\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}([k]_q-1)\, a_k\, z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}\, a_k\, z^{k-1}}\right|$$

Letting z→ 1^- along the real axis, we obtain (2.1). Hence the proof is completed. □

3. Growth and distortion theorems

Theorem 3.1 For f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$ and $|z| = r < 1$, we have

$$|f(z)| \geq r - \frac{(1-\alpha)(2-\lambda)}{2[[2]_q(1+\beta)-(\alpha+\beta)]}\, r^2\,, \tag{3.1}$$

$$|f(z)| \leq r + \frac{(1-\alpha)(2-\lambda)}{2[[2]_q(1+\beta)-(\alpha+\beta)]}\, r^2 \tag{3.2}$$

Equalities hold for

$$f(z) \geq z - \frac{(1-\alpha)(2-\lambda)}{2[[2]_q(1+\beta)-(\alpha+\beta)]}\, z^2, \tag{3.3}$$

At z = r and z = r$e^{i(2k+1)\pi}(k \geq 2)$.

Proof: Since for $k \geq 2$,

$$[[2]_q(1+\beta)-(\alpha+\beta)]\frac{2}{1-\lambda}\sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\ \frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}\, a_k$$
$$\leq 1-\alpha \qquad\qquad (3.4)$$

Then

$$\sum_{k=2}^{\infty} a_k \leq \frac{(1-\alpha)(1-\lambda)}{2[[2]_q(1+\beta)-(\alpha+\beta)]} \tag{3.5}$$

From (1.12) and (3.5), we have

$$|f(z)| \geq r - r^2\sum_{k=2}^{\infty} a_k \geq r - \frac{(1-\alpha)(1-\lambda)}{2[[2]_q(1+\beta)-(\alpha+\beta)]}\, r^2 \tag{3.6}$$

$$|f(z)| \leq r + r^2\sum_{k=2}^{\infty} a_k \leq r + \frac{(1-\alpha)(1-\lambda)}{2[[2]_q(1+\beta)-(\alpha+\beta)]}\, r^2 \tag{3.7}$$

This completes the proof. □

Corollary 3.1 For f(z) ∈ $S_p^{\lambda}(\alpha,\beta)$, then

$$|f(z)| \geq r - \frac{(1-\alpha)(1-\lambda)}{2(2+\beta-\alpha)}\, r^2, \tag{3.8}$$

$$|f(z)| \leq r + \frac{(1-\alpha)(1-\lambda)}{2(2+\beta-\alpha)}\, r^2. \tag{3.9}$$

Equalities hold for

$$|f(z)| = z - \frac{(1-\alpha)(1-\lambda)}{2(2+\beta-\alpha)}\, z^2, \tag{3.10}$$

Proof: Letting q → 1^- in theorem 3.1, we can show (3.8) and (3.9). □

Theorem 3.2 Let f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$. Then for $|z| = r < 1$,

$$|f'(z)| \geq 1 - \frac{(1-\alpha)(1-\lambda)}{[[2]_q(1+\beta)-(\alpha+\beta)]}\, r, \tag{3.11}$$

$$|f'(z)| \leq 1 + \frac{(1-\alpha)(1-\lambda)}{[[2]_q(1+\beta)-(\alpha+\beta)]}\, r, \tag{3.12}$$

The sharpness are attained for f(z) given by (3.3).

Proof: For k ≥ 2 , we have

$$|f'(z)| \leq 1 + r\sum_{k=2}^{\infty} k a_k$$

We find from (2.1) and (3.5) that

$$[2]_q(1+\beta)(1-\lambda)^{-1}\sum_{k=2}^{\infty} ka_k \leq (1-\alpha)+2(\alpha+\beta)(1-\lambda)^{-1}\sum_{k=2}^{\infty} a_k$$

$\leq (1-\alpha)+\frac{(1-\alpha)(\alpha+\beta)}{[[2]_q(1+\beta)-(\alpha+\beta)]}$

$\leq \frac{[2]_q(1+\alpha)-(1+\beta)}{[2]_q(1+\beta)-(\alpha+\beta)}$

That is, that

$\sum_{k=2}^{\infty} ka_k \leq \frac{(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}$ (3.13)

From (3.11) and (3.12) that

$|f'(z)| \geq 1-\frac{(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}\, r$ (3.14)

$|f'(z)| \leq 1+\frac{(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}\, r$ (3.15)

This completes the proof. □

Theorem 3.3 for f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$ and $|z| = r < 1$,

$|D_q f(z)| \geq 1-\frac{|2|_q(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}\, r,$ (3.16)

$|D_q f(z)| \leq 1+\frac{|2|_q(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}\, r$ (3.17)

The sharpness are attained for f(z) given by (3.3).

Proof: for k ≥ 2, we have

$$|D_q f(z)| \leq 1 - r\sum_{k=2}^{\infty}[k]_q\, a_k .$$

We find from (2.1) and (3.5) that

$$2(1+\beta)(1-\lambda)^{-1}\sum_{k=2}^{\infty}[k]_q\, a_k \leq 2(1-\alpha)+(\alpha+\beta)(1-\lambda)^{-1}\sum_{k=2}^{\infty} a_k$$

$$\leq (1-\alpha)+\frac{(\alpha+\beta)(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)]}$$

$\leq \frac{[2]_q(1+\beta)(1-\alpha)}{[[2]_q(1+\beta)-(\alpha+\beta)]}$,

That is, that

$\sum_{k=2}^{\infty}[k]_q\, a_k \leq \frac{[2]_q(1-\alpha)}{2[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}$ (3.18)

From (3.16) and (3.17) that

$|D_q f(z)| \geq 1 - r\sum_{k=2}^{\infty}[k]_q\, a_k \geq 1-\frac{[2]_q(1-\alpha)}{2[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}\, r$ (3.19)

And

$|D_q f(z)| \leq 1 + r\sum_{k=2}^{\infty}[k]_q\, a_k \geq 1+\frac{[2]_q(1-\alpha)}{2[[2]_q(1+\beta)-(\alpha+\beta)](1-\lambda)^{-1}}\, r$ (3.20)

This completes the proof. □

Corollary 3.3 For f(z) ∈ $S_p^{\lambda}(\alpha,\beta)$, then

$|f'(z)| \geq 1 - \frac{(1-\alpha)(1-\lambda)}{2(2+\beta-\alpha)}\ r,$ (3.21)

$|f'(z)| \leq r + \frac{(1-\alpha)(1-\lambda)}{2(2+\beta-\alpha)}\ r.$ (3.22)

The sharpness is attained for f(z) given by (3.10).

Proof: letting q→ 1^- in theorem 3.4, we can show (3.21) and (3.22). Then corollary 3.5 corresponds to theorem 3.3 when **q**→ 1^- **.**

Closure theorem

Let $f_j(z)$ be defined, for j = 1, 2,……m , by

$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j}\, z^k \qquad (a_{k,j} \geq 0, z \in U).$ (4.1)

Theorem 4.1 Let $f_j(z) \in TB_q^{\lambda}(\alpha,\beta)$ for j = $1,2, \dots \dots \dots m$. then

g(z) = $\sum_{j=1}^{m} c_j f_j(z)$, (4.2)

is also in the same class, where $c_j \geq 0, \sum_{j=1}^{m} c_j = 1$.

Proof: According to (4.2), we can write

g(z) = $z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j a_{k,j}\right) z^k$. (4.3)

Further, since $f_j(z) \in TB_q^{\lambda}(\alpha,\beta)$, we get

$\sum_{k=2}^{\infty}\left[[k]_q(1+\beta)-(\alpha+\beta)\right] \frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}\ a_{k,j} \leq 1-\alpha$ (4.4)

Hence

$$\sum_{k=2}^{\infty}\left[[k]_q(1+\beta)-(\alpha+\beta)\right] \frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}\ a_k\left(\sum_{j=1}^{m} c_j a_{k,j}\right)$$
$$= \sum_{j=1}^{m} c_j\left[\sum_{k=2}^{\infty}\left[[k]_q(1+\beta)-(\alpha+\beta)\right] \frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}\ a_{k,j}\right]$$

$\leq \left(\sum_{j=1}^{m} c_j\right)(1-\alpha) = 1-\alpha$, (4.5)

Which implies that g(z) $\in TB_q^{\lambda}(\alpha,\beta)$. Thus, we have the theorem. □

Corollary 4.1: the class $TB_q^{\lambda}(\alpha,\beta)$ is closed under convex linear combination.

Proof: Let $f_j(z) \in TB_q^{\lambda}(\alpha,\beta)$ (j= 1,2) and

g(z) = $\mu f_1(z) + (1-\mu) f_2(z)$ (0≤ $\mu \leq 1$), (4.6)

Then by, taking m= 2, $c_1 = \mu$ and $c_2 = 1-\mu$ in theorem 5, we have g(z) $\in TB_q^{\lambda}(\alpha,\beta)$.

Theorem 4.2 Let $f_1(z) = z$ and

$f_k(z) = z - \frac{1-\alpha}{[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}\ z^k$ (k≥ 2). (4.7)

Then f(z) $\in TB_q^{\lambda}(\alpha,\beta)$ if and only if

f(z) = $\sum_{k=1}^{\infty} \mu_k f_k(z)$, (4.8)

where $\mu_k \geq 0\ (k \geq 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof : Suppose that

f(z) = $\sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}\ \mu_k z^k$. (4.9)

then it follows that

$\sum_{k=2}^{\infty} \frac{[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}{1-\alpha} \cdot \frac{1-\alpha}{[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}\ \mu_k = \sum_{k=2}^{\infty} \mu_k = 1-\mu_1 \leq 1$. (4.10)

So by Theorem 2.1, f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$. Conversely, assume that f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$. then

$$a_k \leq \frac{1-\alpha}{[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}} \quad (k\geq 2)\,. \tag{4.11}$$

Setting

$$\mu_k = \frac{[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}{1-\alpha} a_k \quad (k\geq 2) \tag{4.12}$$

And

$$\mu_1 = 1-\sum_{k=2}^{\infty}\mu_k\,, \tag{4.13}$$

We see that f(z) can be expressed in the form (4.8). This completes the proof. □

Corollary 4.2 The extreme points of $TB_q^{\lambda}(\alpha,\beta)$ are $f_k(z)$ (k ≥ 1) given by theorem 4.3.

Some radii of the class $TB_q^{\lambda}(\alpha,\beta)$

Theorem 5.1 Let f(z) ∈ $TB_q^{\lambda}(\alpha,\beta)$. Then for 0≤ $\rho < 1$, $k \geq 2$, f(z) is

(i) Close-to-convex of order ρ in $|z| < r_1$, were

(ii) $$r_1 = r_1(q,\alpha,\beta,\lambda,\rho) := inf_k\left[\frac{(1-\rho)[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \tag{5.1}$$

(iii) Starlike of order ρ in $|z| < r_2$, where

$$r_2 = r_2(q,\alpha,\beta,\lambda,\rho) := inf_k\left[\frac{(1-\rho)[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \tag{5.2}$$

(iv) Convex of order ρ in $|z| < r_3$, where $r_3 = r_3(q,\alpha,\beta,\lambda,\rho) :=$

$$inf_k\left[\frac{(1-\rho)[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \tag{5.3}$$

The result is sharp for f(z) is given by (2.3).

Proof: To prove (i) we mut show that

$|f'(z)-1| \leq 1-\rho$ for $|z| < r_1 = r_1(q,\alpha,\beta,\lambda,\rho)$.

From (1.12), we have

$$|f'(z)-1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}$$

$$|f'(z)-1| \leq 1-\rho,$$

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\rho}\right) a_k\, |z|^{k-1} \leq 1 \tag{5.4}$$

But, by theorem 2.1, (5.4) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{(1-\rho)[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}{1-\alpha}$$

That is, if

$$|z| \leq \left[\frac{(1-\rho)[[k]_q(1+\beta)-(\alpha+\beta)]\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda}}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k\geq 2)\,, \tag{5.5}$$

Which gives (5.1)

To prove (ii) and (iii) it suffices to show

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho \quad \text{for } |z| \leq r_2\,, \tag{5.6}$$

$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\rho$ for $|z| \leq r_3$, (5.7)

Respectively, by using arguments as in proving (i), we have the results. □

Inclusion relations involving $N_{k,q,\delta}(e)$

In this section following the works of Goodman [21] and Ruscheweyh [33] (see also [5], [6], [9], [16], [26], and [28]) defined the k, δ neighborhood of function f(z) $\in T$ by

$N_{k,\delta}(f;g) = \{g \in T: g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta\}$ (6.1)

In particular, for the identity function e(z) = z, we have

$N_{k,\delta}(e;g) = \{g \in T: g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|b_k| \leq \delta\}$ (6.2)

Aouf et al. [12] defined the k, q,δ neighborhood of function f(z)$\in T$ by

$N_{k,q,\delta}(f;g) = \left\{g \in T: g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty}[k]_q|a_k - b_k| \leq \delta_q\right\}$ (6.3)

In particular, for the identity function e(z) = z, we have

$N_{k,q,\delta}(e;g) = \left\{g \in T: g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty}[k]_q|b_k| \leq \delta_q\right\}$ (6.4)

Theorem 6.1 Let $\delta_q = \frac{(1-\alpha)}{2[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}}$ (6.5)

Then $TB_q^{\lambda}(\alpha,\beta) \in N_{k,q,\delta}(e)$.

Proof: For f $\in TB_q^{\lambda}(\alpha,\beta)$, Theorem 2.1, (3.5) and (3.18), and in view of the (6.4), Theorem 6.1 follows.

A function f $\in T$ is in the class $TB_q^{\lambda}(\alpha,\beta,\xi)$ if there exists a function g $\in TB_q^{\lambda}(\alpha,\beta)$ such that

$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \xi_q$ $(z \in U, 0 \leq \xi_q \leq 1)$ (6.6)

Now we determine the neighborhood for the class $TB_q^{\lambda}(\alpha,\beta,\xi)$.

Theorem 6.2 If g $\in TB_q^{\lambda}(\alpha,\beta)$ and

$\xi_q = 1 - \frac{\delta_q 2[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}}{2\{[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}-(1-\alpha)\}}$ (6.7)

Where

$$\delta_q \leq \frac{2\{[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}-(1-\alpha)\}}{\delta_q 2[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}}$$

Proof : Suppose that f $\in N_{k,q,\delta}(g)$ then

$$\sum_{k=2}^{\infty}[k]_q|a_k - b_k| \leq \delta_{q,}$$

Where δ_q is given by (6.5), which implies that the coefficient inequality

$$\sum_{k=2}^{\infty}|a_k - b_k| \leq \frac{\delta_q}{[2]_q}$$

Next, since g $\in TB_q^{\lambda}(\alpha,\beta)$, we have

$\sum_{k=2}^{\infty} b_k \leq \frac{(1-\alpha)}{2[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}}$

So that

$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=2}^{\infty}|a_k-b_k|}{1-\sum_{k=2}^{\infty} b_k} \leq \frac{\delta_q}{[2]_q} \times \frac{2[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}}{2[[2]_q(1+\beta)-(\alpha+\beta)][1-\lambda]^{-1}-(1-\alpha)} \leq 1 - \xi_q$

Provided that ξ_q is given precisely by (6.7). Thus, by definition, g $\in TB_q^{\lambda}(\alpha,\beta)$,which complete the proof.

Partial sums

For f(z) of the form (1.1), the sequence of partial sums is given by

$f_z(z) = z + \sum_{k=2}^{\infty} a_k\, z^k$ (m ∈ $N\backslash\{1\}$)

Now following the work of [38] and also the works cited in [11,15,19,25,27,31,37] on partial sums of analytic functions, to obtain our results. Let

$$\Phi_{q,k}^{\lambda} = \Phi_q^{\lambda}(k,\alpha,\beta) = \sum_{k=2}^{\infty}[[k]_q(1+\beta)-(\alpha+\beta)]\,\frac{\Gamma k+1\ \Gamma 1-\lambda}{\Gamma k-\lambda} \quad (7.1)$$

Theorem 7.1 If f ∈ S , satisfies the condition (2.1), then

$$\mathrm{Re}\left(\frac{f(z)}{f_m(z)}\right) \geq \frac{\Phi_{q,m+1}^{\lambda}-1+\alpha}{\Phi_{q,m+1}^{\lambda}}, \quad (7.2)$$

Where

$$\Phi_{q,k}^{\lambda} \geq \begin{cases} 1-\alpha, if\ k = 2,3 \ldots, m \\ \Phi_{q,m+1}^{\lambda}, \quad if\ k = m+1, m+2, \ldots \end{cases} \quad (7.3)$$

The results (7.2) is sharp for

$$f(z) = z + \frac{1-\alpha}{\Phi_{q,m+1}^{\lambda}} z^{m+1} \quad (7.4)$$

Proof: Define g(z) by

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\left[\frac{f(z)}{f_m(z)} - \frac{\Phi_{q,m+1}^{\lambda}-1+\alpha}{\Phi_{q,m+1}^{\lambda}}\right] = \frac{1+\sum_{k=2}^{m} a_k\, z^{k-1}+\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} a_k\, z^{k-1}}{1+\sum_{k=2}^{m} a_k\, z^{k-1}} \quad (7.5)$$

It suffices to show that $|g(z)| \leq 1$. Now from (7.5) we have

$$g(z) = \frac{\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} a_k\, z^{k-1}}{2+2\sum_{k=2}^{m} a_k\, z^{k-1}+\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} a_k\, z^{k-1}}$$

hence, we obtain

$$|g(z)| \leq \frac{\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum_{k=m+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{m} a_k z^{k-1}+\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} a_k z^{k-1}}$$

Now $|g(z)| \leq 1$ if and only if

$$2\left(\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)\sum_{k=m+1}^{\infty} |a_k| \leq 2-2\sum_{k=2}^{m} |a_k|,$$

Or, equivalently,

$$\sum_{k=2}^{m}|a_k| + \sum_{k=m+1}^{\infty}\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}|a_k| \leq 1$$

From (2.1) , it is sufficient to show that

$$\sum_{k=2}^{m}|a_k| + \sum_{k=m+1}^{\infty}\frac{\Phi_{q,m+1}^{\lambda}}{1-\alpha}|a_k| \leq \sum_{k=2}^{\infty}\frac{\Phi_{q,k}^{\lambda}}{1-\alpha}|a_k|,$$

Which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{\Phi_{q,k}^{\lambda}-1+\alpha}{1-\alpha}\right)|a_k| + \sum_{k=m+1}^{\infty}\left(\frac{\Phi_{q,k}^{\lambda}-\Phi_{q,m+1}^{\lambda}}{1-\alpha}\right)|a_k| \geq 0 \quad (7.6)$$

For z = r $e^{\frac{i\pi}{m}}$ we have

$\frac{f(z)}{f_m(z)} = 1 + \frac{1-\alpha}{\Phi^{\lambda}_{q,m+1}} z^k \to 1 - \frac{1-\alpha}{\Phi^{\lambda}_{q,m+1}} = \frac{\Phi^{\lambda}_{q,m+1}-1+\alpha}{\Phi^{\lambda}_{q,m+1}}$ where r→ 1^- ,

Which shows that f(z) is given by (7.4) gives the sharpness. □

Remark 7.1 (i) Putting λ= 0 and (ii) λ= 1 in Theorem 7.1, we obtain the following results, respectively.

Corollary 7.2 If f ∈ S, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0 \ (0 < |z| < 1), then$

$$\mathrm{Re}\left(\frac{f(z)}{f_m(z)}\right) \geq \frac{[[m+1]_q(1+\beta)-(\alpha+\beta)]-1+\alpha}{[[m+1]_q(1+\beta)-(\alpha+\beta)]} \tag{7.7}$$

The results is sharp for

$$f(z) = z + \frac{1-\alpha}{[[m+1]_q(1+\beta)-(\alpha+\beta)]} z^{m+1} \tag{7.8}$$

corollary 7.3 If f ∈ S, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0$ (0< $|z|$ < 1), then

$$\mathrm{Re}\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{1-\alpha}{[m+1]_q[[m+1]_q(1+\beta)-(\alpha+\beta)]} \tag{7.9}$$

The result is sharp for

$$f(z) = z + \frac{1-\alpha}{[m+1]_q[[m+1]_q(1+\beta)-(\alpha+\beta)]} z^{m+1} \tag{7.10}$$

theorem 7.4 If f ∈ S, satisfies the condition (2.1), then

$$\mathrm{Re}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{\Phi^{\lambda}_{q,m+1}}{\Phi^{\lambda}_{q,m+1}+1-\alpha}, \tag{7.11}$$

Where $\Phi^{\lambda}_{q,m+1}$ is defined by (7.1) and satisfies (7.3) and f(z) given by (7.4) gives the sharpness.

Proof The proof follows by defining

$$\frac{1+g(z)}{1-g(z)} = \frac{\Phi^{\lambda}_{q,m+1}+1-\alpha}{1-\alpha}\left[\frac{f_m(z)}{f(z)} - \frac{\Phi^{\lambda}_{q,m+1}}{\Phi^{\lambda}_{q,m+1}+1-\alpha}\right]$$

And much akin are to similar arguments in Theorem 7.1 so, we omit it. □

Remark 7.2 (i) Putting $\lambda = 0$ and (ii) $\lambda = 1$ in Theorem 7.4, we obtain the following sharp results, respectively.

Corollary 7.5 If f ∈ S , satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, then

$$\mathrm{Re}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{[[m+1]_q(1+\beta)-(\alpha+\beta)]}{[m+1]_q(1+\beta)-(\alpha+\beta)+1-\alpha} \tag{7.12}$$

Corollary 7.6 If f ∈ S, satisfies the condition (2.1) and $\frac{f(z)}{z} \neq 0$ $(0 < |z| < 1)$, then

$$\mathrm{Re}\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{[m+1]_q[[m+1]_q(1+\beta)-(\alpha+\beta)]}{[m+1]_q[[m+1]_q(1+\beta)-(\alpha+\beta)]+1-\alpha} \tag{7.13}$$

Theorem 7.7 If f ∈ S , satisfies the condition (2.1), then

$$\mathrm{Re}\left(\frac{f'(z)}{f'_m(z)}\right) \geq \frac{\Phi^{\lambda}_{q,m+1}-(m+1)(1-\alpha)}{\Phi^{\lambda}_{q,m+1}} \tag{7.14}$$

$$\mathrm{Re}\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{\Phi^{\lambda}_{q,m+1}}{\Phi^{\lambda}_{q,m+1}+(m+1)(1-\alpha)} \tag{7.15}$$

Where $\Phi^{\lambda}_{q,m+1} \geq (m+1)(1-\alpha)$ and

$$\Phi^{\lambda}_{q,k} \geq \begin{cases} k(1-\alpha), if\ k = 2,3,\ldots\ldots,m \\ k\left(\frac{\Phi^{\lambda}_{q,m+1}}{(m+1)}\right), if\ k = m+1, m+2, \ldots \end{cases} \tag{7.16}$$

f(z) is given by (7.4) gives the sharpness.

Proof We write

$$\frac{1+g(z)}{1-g(z)}=\frac{\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)}\left[\frac{f'(z)}{f_m'(z)}-\frac{\Phi_{q,m+1}^{\lambda}-(m+1)(1-\alpha)}{\Phi_{q,m+1}^{\lambda}}\right],$$

Where

$$g(z)=\frac{\left(\frac{\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty}ka_kz^{k-1}}{2+2\sum_{k=2}^{m}ka_kz^{k-1}+\left(\frac{\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty}ka_kz^{k-1}}$$

Now $|g(z)|\leq 1$ if and only if

$$\sum_{k=2}^{m}k|a_k|+\left(\frac{\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty}k|a_k|\leq 1$$

From (2.1), it is sufficient to show that

$$\sum_{k=2}^{m}k|a_k|+\left(\frac{\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)}\right)\sum_{k=m+1}^{\infty}k|a_k|\leq\sum_{k=2}^{\infty}\frac{\Phi_{q,k}^{\lambda}}{(1-\alpha)}|a_k|,$$

Which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{\Phi_{q,k}^{\lambda}-k(1-\alpha)}{(1-\alpha)}\right)|a_k|+\sum_{k=m+1}^{\infty}\left(\frac{(m+1)\Phi_{q,k}^{\lambda}-k\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)}\right)|a_k|\geq 0$$

To prove the result (7.15), define the function g(z) by

$$\frac{1+g(z)}{1-g(z)}=\frac{(m+1)(1-\alpha)+\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)}\left[\frac{f_m'(z)}{f'(z)}-\frac{\Phi_{q,m+1}^{\lambda}}{(m+1)(1-\alpha)+\Phi_{q,m+1}^{\lambda}}\right],$$

And by similar arguments in first part, we get desired result. □

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