Fixed Point Theorems for Generalized Weakly Contractive Mappings in m-Metric Spaces

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Abstract:
Motivated by the work of Samet et al., we introduce a novel class of generalized weakly contractive mappings in m-metric spaces. For this class, we establish existence and uniqueness results for fixed points and prove a coupled fixed-point theorem. These results properly extend and generalize several known fixed-point theorems in the literature, thereby unifying and strengthening earlier contributions. An example is provided to illustrate the applicability of the main results.

Keywords: Fixed Point, Coupled Fixed Point, Generalized Weak Contractive Mapping, m-Metric Space.

1. Introduction

Since the Banach contraction principle, fixed point theory has undergone extensive development through the relaxation of contractive conditions and the introduction of generalized distance structures. Early extensions by Alber et al. [1] and subsequent refinements by Rhoades [2] stimulated broad interest in alternative approaches to fixed point results in metric spaces. Further progress was achieved through the use of auxiliary tools such as altering distance functions, introduced by Khan et al. [15], which allowed the derivation of fixed point theorems under weaker and more flexible assumptions.

The study of fixed points in generalized metric frameworks has gained increasing importance, particularly in spaces where the classical notion of distance is inadequate. Partial metric spaces, investigated by Samet et al. [16], permit non-zero self-distances and have provided a useful setting for extending Banach-type results.

In recent years, fixed point theory in m-metric spaces has attracted growing attention. Several authors have demonstrated that many classical fixed point principles can be extended to this setting under appropriate assumptions. The flexibility of the m-metric structure allows the treatment of nonlinear problems that cannot be adequately addressed within standard metric spaces.The present work contributes to this line of research by establishing new fixed point results in m-metric spaces.

Definition:1.1. A function $f: X \to [0,\infty)$, where X is a metric space, is called lower semicontinuous if, for all $x \in X$ and $\{x_n\} \subset X$ with $x_n \to x$, we have

$$f(x) \leq \lim_{n\to\infty} \inf f(x_n)$$

Let us define $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ as follows:

$\boldsymbol{\psi} = \{\boldsymbol{\psi}: [\mathbf{0},\infty) \to [\mathbf{0},\infty) \mid \boldsymbol{\psi}$ is continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$.

Also, we denote
$\boldsymbol{\Phi} = \{\phi: [0,\infty) \to [0,\infty) \mid \phi \text{ is lower semicontinuous and } \phi(t) = 0 \Leftrightarrow t = 0\}$.

Lemma 1.2. ([17]) If a sequence $\{x_n\}$ in X is not Cauchy, then there exist $\epsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which

$$m(k) > n(k) > k,$$

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{1.2}$$

and

$$d(x_{m(k)-1}, x_{n(k)-1}) \leq \epsilon \tag{1.3}$$

Moreover, suppose that $d(x_n, x_{n+1}) = 0$
Then we have:

(1) $\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$
(2) $\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon$
(3) $\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \epsilon$
(4) $\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$

Definition 1.3. Let X be a non-empty set a function $m: X \times X \to [0,\infty)$ is called an m-metric if for all $x, y \in X$

1. $m(x,x) \leq m(x,y)$
2. $m(x,y) = m(y,x)$
3. $m(x,x) = m(y,y) = m(x,y) \Rightarrow x = y$
4. $m(x,z) \leq m(x,y) + m(y,z) - m(y,y)$

The pair (X, m) is called m-metric space.

2. Main Results

Let (X, m) be an m-metric space. Let $T: X \to X$ and $\psi: X \to [0,\infty)$ then T is an (ψ, ϕ) −generalized weakly contractive mapping if it satisfies the following condition:
$\psi(m(Tx,Ty)) + \psi(Tx) + \psi(Ty) \leq \psi(M_m(x,y,T,\psi)) - \phi(L_m(x,y,T,\psi))$, for all $x, y \in X$.
Where

$$M_m(x,y,T,\psi) = max\,\{m(x,y) + \psi(x) + \psi(y) + m(x,Tx) + \psi(x) + \psi(Tx),$$
$$m(y,Ty) + \psi(y) + \psi(Ty), \frac{1}{2}[m(x,Ty) + m(y,Tx)\} \tag{2.1}$$
$$L_m(x,y,T,\psi) = max\,\{m(x,y) + \psi(x) + \psi(y), m(y,T(y)) + \psi(y) + \psi(Ty)\}$$

Theorem:2.1. Let (X, m) be a complete m-metric space. If T is a generalized weakly contractive mapping $\psi \in \Psi$, $\varphi \in \phi$ and ψ *and* η are lower semi continuous, then there exists $z \in X$ such that $Tz = z$, $\psi(z) = 0, \eta(z,z) = 0$.

Proof: Let $x_0 \in X$ be a fixed point and define a sequence $\{x_n\}$ by $x_{n+1} = T(x_n)$ for all $n = 0, 1, 2 \ldots$
If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = T(x_n)$
So, x_n is a fixed point of T. (Hence proved)
Assume, $x_n \neq x_{n+1}$ for all $n = 0, 1, 2 \ldots$
From (1), we have
$x_n = x_{n-1}$ *and* $y = x_n$, we have

$$M_m(x_{n-1},x_n,T,\psi\,) = max\,\{m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n) + m(x_{n-1},Tx) + \psi(x_{n-1}) \\ +\psi(Tx_{n-1})\,, m(x_n,Tx_n) + \psi(x_n) \\ +\psi(Tx_n), \frac{1}{2}[m(x_{n-1},Tx_n) + m(x_n,Tx_{n-1})\,\}$$

Since, $Tx_{n-1} = x_n$ and $Tx_n = x_{n+1}$, this reduces to

$$M_m(x_{n-1},x_n) = max\,\{m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n), m(x_n,x_{n+1}) + \psi(x_n) + \psi(x_{n+1})\}$$

And

$$L_m(x_{n-1},x_n,m,\psi\,) \\ = max\,\{m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n), m\big(x_n,T(x_n)\big) + \psi(x_n) + \psi(Tx_n)\}$$

Using contraction condition with $x = x_{n-1}, y = x_n$, we get

$$\psi\big(m(x_n,x_{n+1}) + \psi(x_n) + \psi(x_{n+1})\big) \le \psi(m(Tx_{n-1},Tx_n) + \psi T(x_{n-1}) + \psi T(x_n) \\ \le \psi\big(M_m(x_{n-1},x_n\,)\big) - \phi\big(L_m(x_{n-1},x_n)\big)$$

If $m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n) < m(x_n,x_{n+1}) + \psi(x_n) + \psi(x_{n+1})$

Then from inequality above, we obtain

$$\psi(t) \le \psi(t) - \phi(t), t = m(x_n,x_{n+1}) + \psi(x_n) + \psi(x_{n+1}),$$

which implies that

$\phi(t) = 0$, hence $t = 0$

This contradicts $x_n \neq x_{n+1}$

Therefore

$$m(x_n,x_{n+1}) + \psi(x_n) + \psi(x_{n+1}) \le m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n),$$

Now, we have

$$M_m(x_{n-1},x_n) = m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n)$$

Similarly, we have

$$L_m(x_{n-1},x_n\,) = m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n)$$

Thus, the inequality becomes

$$\psi\big(m(x_n,x_{n+1}) + \psi(x_n) + \psi(x_{n+1})\big) \\ \le \psi(m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n)) - \phi(m(x_{n-1},x_n) + \psi(x_{n-1}) + \psi(x_n))$$

It follows that $a_n = m(x_n,x_{n+1}) + \psi(x_n) + \psi(x_{n+1})$ is non-increasing and bounded below by 0.

Hence, $a_n \to r \ge 0$.

Letting $n \to \infty$, using continuity of ψ and semi continuity of ϕ, we obtain

$$\psi(r) \le \psi(r) - \phi(r)$$

Which implies $\phi(r) = 0$, here $r = 0$.

Therefore, we have

$$m(x_n,x_{n+1}) = 0, \quad \lim_{n\to\infty} \psi(x_n) = 0.$$

Now, we prove that the sequence $\{x_n\}$ is Cauchy. If $\{x_n\}$ is not Cauchy, then there exists $\epsilon > 0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(x_{m(k)},x_{n(k)}) \ge \epsilon$

And $m\big(x_{m(k)-1},x_{n(k)}\big) < \epsilon$

Where $m(k)$ is the smallest index with $m(k) > n(k)$

From definition of $M_{m,}$ we have

$$M_m\big(x_{n(k)},x_{m(k)}\big) = max\,\{m\big(x_{n(k)},x_{m(k)}\big) + \psi\big(x_{n(k)}\big) + \psi\big(x_{m(k)}\big) + m\big(x_{n(k)},x_{n(k)+1}\,\big) \\ +\psi\big(x_{n(k)}\big) + \psi\big(Tx_{n(k)+1}\big)\,, m\big(x_{m(k)},Tx_{m(k)+1}\big) + \psi\big(x_{m(k)+1}\big)$$

$$+\psi\left(x_{m(k)+1}\right), \frac{1}{2}\left[m\left(x_{n(k)}, x_{m(k)+1}\right) + m\left(x_{m(k)}, x_{n(k)}\right)\right\}$$

Letting $k \to \infty$, we have

$$M_m\left(x_{n(k)}, x_{m(k)}\right) = \epsilon$$

Also,

$$L_m\left(x_{n(k)}, x_{m(k)}\right)$$
$$= max\left\{m\left(x_{n(k)}, x_{m(k)}\right) + \psi\left(x_{n(k)}\right) + \psi\left(x_{m(k)}\right), m\left(x_{m(k)}, x_{m(k)+1}\right) + \psi(x_n) + \psi(Tx_n)\right\}$$

Letting $k \to \infty$, we have

$$\lim_{n\to\infty} L_m\left(x_{n(k)}, x_{m(k)}\right) = \epsilon$$

From contractive inequality, we have

$$\psi\left(m\left(x_{n(k)+1}, x_{m(k)+1}\right) + \psi\left(x_{n(k)+1}\right) + \psi\left(x_{m(k)+1}\right)\right) \leq \psi(M_m\left(x_{n(k)}, x_{m(k)}\right)) - \phi(L_m\left(x_{n(k)}, x_{m(k)}\right))$$

Letting $n \to \infty$, using continuity of ψ and lower semi continuity of ϕ, we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$$

Since $\epsilon > 0$ we have

$$\phi(\epsilon) > 0$$

which is a contradiction. Therefore, the assumption that the sequence $\{x_n\}$ is not Cauchy, is wrong. Hence the sequence $\{x_n\}$ is Cauchy sequence.

$\Rightarrow \lim_{n\to\infty} x_n = z \in X$ exists.

As X is complete and ψ lower semi continuous, we have

$$\psi(z) \leq \lim_{n\to\infty} inf\ inf\ \psi(x_n) \leq \lim_{n\to\infty} \psi(x_n) = 0$$

Hence

$$\psi(z) = 0$$

It follows that

$$M_m(x_n, z) = max\{m(x_n, z) + \psi(x_n) + \psi(z) + m(x_n, Tx_n) + \psi(x_n) + \psi(Tx_n),$$
$$m(z, Tz) + \psi(z) + \psi(Tz), \frac{1}{2}[m(x_n, Tz) + m(z, Tx_n)\}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} M_m(x_n, z) = m(z, Tz) + \psi(Tz)$$

Also, we have

$$L_m(x_n, z) = max\left\{m(x_n, z) + \psi(x_n) + \psi(z), m\left(z, T(z)\right) + \psi(z) + \psi(Tz)\right\}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} L_m(x_n, z) = m(z, Tz) + \psi(Tz)$$

Using the contractive condition with $x = x_n$ and $y = z$, we have

$$\psi\left(m(x_{n+1}, Tz)\right) + \psi(x_{n+1}) + \psi(Tz) = \psi\left(m(Tx_n, Tz)\right) + \psi(Tx_n) + \psi(Tz)$$
$$\leq \psi\left(M_m(x_n, z)\right) - \phi\left(L_m(x_n, z)\right)$$

By letting $n \to \infty$, and by applying continuity of ψ and lower semi continuity of ϕ, we have

$$\psi\left(m(z, Tz)\right) + \psi(Tz) \leq \psi\left(m(z, Tz)\right) + \psi(Tz)) - \phi\left(m(z, Tz) + \psi(Tz)\right)$$

This implies that

$$\phi\left(m(z, Tz) + \psi(Tz)\right) = 0$$

Since, $\phi(t) = 0$ implies $t = 0$, we conclude that

$m(z,Tz) = 0$ and $\psi(Tz) = 0$
Hence, $z = Tz$
This implies z is the fixed point of T.
Let u be another fixed point of T.
Then $u = Tu$ and $\phi(u) = 0$
Then, by applying $x = z, and\ y = u$, we have

$$\psi\big(m(z,u)\big) = \psi\big(m(Tz,Tu)\big) = \psi\big(m(Tz,Tu)\big) + \psi(Tz) + \psi(Tu) \leq \psi\big(m(z,u)\big) - \phi\big(m(z,u)\big)$$

This implies that

$$z = u$$

This completes the proof of the theorem.
Example:2.2. Let $X = [0,\infty)$ and $m: X \times X \to [0,\infty)$, for all $x, y \in X$.
Every m-metric index a usual metric $d_m(x,y) = m(x,y) - \{m(x,x), m(y,y\}$
Let $m(x,y) = \{x,y\}$, $x, y \in X$

$$d_m(x,y) = |x - y|$$

Define $\psi(t) = \frac{3}{2}t \qquad t \geq 0$

$$\psi(t) = \{\frac{t}{2} \qquad 0 \leq t \leq 1, \quad \frac{t}{2} + \frac{1}{2} \quad 1 \leq t \leq 2\ t \qquad t > 2$$

$$\varphi(t) = \frac{3t}{4 + 2t}$$

Then, $\psi \in \Psi$, ψ is lower semicontinuous and $\frac{1}{2}t \leq \psi(t) \leq$t ,t≥0
Define map $T: X \to X$ by

$$T(x) = \frac{x^2}{2(1 + x)}$$

Without loss of generality, suppose that $x \geq y$,
In m-metric space,

$$\psi\big(m(Tx,Ty) + \psi(Tx) + \psi(Ty)\big) \leq \psi\big(M_m(x,y)\big) - \phi(L_m(x,y))$$

Where

$$M_m(x,y) = max\,\{m(x,y) + \psi(x) + \psi(y), m(x,Tx) + \psi(x) + \psi(Tx), m(y,Ty) + \psi(y) + \psi(Ty), \frac{1}{2}[m(x,Ty) + \psi(x) + \psi(Ty) + m(y,Tx) + \psi(y) + \psi(Tx)]\}$$

$$= max\,\{x + \frac{x}{2} + \frac{Tx}{2}, y + \frac{y}{2} + \frac{Ty}{2}, \frac{1}{2}(x + \frac{x}{2} + \frac{y}{2} + \frac{y}{2} + \frac{y}{2} + \frac{Tx}{2})$$

$\geq x$.

$$L_m(x,y) = max\,\{m(x,y) + \psi(x) + \psi(y), m(x,Ty) + \psi(x) + \psi(Ty), m(y,Tx) + \psi(y) + \psi(T(x))\}$$

Then

$$L_m(x,y) = max\,\{x + \psi(x) + \psi(x), y + \psi(y) + \psi(y)\}$$
$$= max\,\{x + 2\psi(x), y + 2\psi(y)\}$$

Using monotonicity of $\psi, we\ obtain$
$\psi\big(m(Tx,Ty)\big) \leq \psi\big(L_m(x,y)\big) - \phi(L_m(x,y))$ for all $x, y \in X$
Hence T is generalized (ψ, ϕ) contractive condition in an m- Metric space.
To prove fixed point :
Let assume $x = Tx$ implies

$$x = \frac{x^2}{2(1+x)}$$

This implies $x = 0$.
Hence T has a unique fixed point in X.

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