

Vague Topological Modules and Vague Topological Vector Spaces

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Abstract:

In this paper, we introduce and develop the notions of vague topological ring, vague topological field, vague topological module, and vague topological vector space within the framework of vague set theory. By combining the algebraic structures of rings, fields, and modules with vague topology, we establish a generalised setting that extends classical topological algebra. We present fundamental definitions, illustrative examples, and examine important structural properties of these newly defined concepts. Furthermore, we investigate the relationships between vague topological structures and their corresponding classical counterparts, and analyse continuity conditions of algebraic operations under vague topology. The results obtained contribute to the advancement of vague algebraic topology and provide a foundation for further research in this direction.

Keywords: Vague topological space, vague topological ring, vague topological field, vague topological module, vague topological vector space.

1. Introduction

The theory of fuzzy topology was introduced by C.L. Chang [5] in 1967. Several research studies were conducted on the generalisations of the notions of fuzzy sets and fuzzy topology.

The theory of vague sets was proposed by Gaw and Buchere [7] as an extension of fuzzy set theory and vague sets are regarded as a special case of content – dependent fuzzy sets. The idea of vague sets is that the membership of every element can be divided into two aspects including supporting and opposing.

The theory of vague topology was introduced by Mariapresenti.L and Arockia Rani.I [11].

Now we introduce the concepts of vague topological rings, vague topological modules, vague topological fields and vague topological vector spaces.

2. Preliminaries

Definition 2.1:[7] A vague set A in the universal of discourse X is characterized by two membership functions given by:

A truth membership function $t_A : X \rightarrow [0,1]$ and

A false membership function $f_A : X \rightarrow [0,1]$,

Where $t_A(x)$ is a lower bound of the grade of membership of x derived from the “evidence for x ”, and $f_A(x)$ is a lower bound on the negation of x derived from the “evidence against x ” and $t_A(x) + f_A(x) \leq 1$.

Thus, the grade of membership of x in the vague set A is bounded by the subinterval $[t_A(x), 1 - f_A(x)]$ of $[0,1]$. This indicates that if the actual grade of membership of x is $\mu(x)$, then

$$t_A(x) \leq \mu(x) \leq 1 - f_A(x).$$

The vague set A is written as

$$A = \{ \langle x, [t_A(x), 1 - f_A(x)] \rangle / x \in X \}.$$

Where the interval $[t_A(x), 1 - f_A(x)]$ is called the value of x in the vague set A and denoted by $V_A(x)$.

Definition 2.2: [7] Let A and B be two vague sets of the form $A = \{ \langle x, [t_A(x), 1 - f_A(x)] \rangle \}$ and

$$B = \{ \langle x, [t_B(x), 1 - f_B(x)] \rangle \}$$
 then

(1). $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and

$$1 - f_A(x) \leq 1 - f_B(x) \text{ for all } x \in X.$$

(2). $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

$$(3). A^c = \{ \langle x, [f_A(x), 1 - t_A(x)] \rangle \}.$$

$$(4). A \cap B = \{ \min[t_A(x), t_B(x)], \min[1 - f_A(x), 1 - f_B(x)] \}.$$

$$(5). A \cup B = \{ \max[t_A(x), t_B(x)], \max[1 - f_A(x), 1 - f_B(x)] \}.$$

Definition 2.3: [11] A vague topology (VT) in short on X is a family T of vague sets (VS in short) in X satisfying the following axioms

$$(1). 0, 1 \in T.$$

$$(2). G_1 \cap G_2 \in T \text{ for any } G_1, G_2 \in T$$

$$(3). \cup G_i \in T \text{ for any family } \{ G_i / i \in \mathbb{N} \} \subseteq T$$

In this case the pair (X, T) is called a vague topological space (VTS in short) and any vague set in T is known as a vague open set (VOS) in X. The complement A^c of a VOS A in a VTS (X, T) is called a vague closed set (VCS in short) in

Definition 2.4: [3] An abelian group A is called a topological additive group if a topology is defined on the set A and following conditions are satisfied:

(1) The mapping $(a, b) \rightarrow a + b$ of the topological space $A \times A$ on to the topological space A is Continuous.

(2) Additive inversion continuity condition: The mapping $a \rightarrow (-a)$ of the topological space A onto itself is continuous.

Remark: Let A be an abelian group then A is a topological abelian group in the discrete or in the indiscrete topology.

Example 2.5: Let $G = \{0, 1\}$ and let $T = \{ \Phi, \{0\}, \{1\}, G \}$ is a discrete topology on G.

Let $A = \{0\}$, $B = \{1\}$ and define

$$A + B = (A - B) \cup (B - A)$$

+	Φ	A	B	G
Φ	Φ	A	B	G
A	A	Φ	G	B
B	B	G	Φ	A
G	G	B	A	Φ

Table 1

So $(G, T, +)$ is a topological additive group.

Definition 2.6: A ring R is called a topological ring if a topology is defined on the set R and the additive group of the ring R is a topological group in this topology and the following condition is satisfied: The mapping $(a, b) \rightarrow a \cdot b$ of the topological space $R \times R$ to the topological space R is continuous.

Example 2.7: Let $X = \{0, 1, 2\} = Z_3$ and let $T = \{\Phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, X\}$ is a discrete topology on X . We denote $A = \{0\}, B = \{1\}, C = \{2\}, P = \{0, 1\}, Q = \{1, 2\}$ and $R = \{0, 2\}$ and define $A + B = (A - B) \cup (B - A)$ and $A \cdot B = A \cap B$

Addition Table:

+	Φ	A	B	C	P	Q	R	X
Φ	Φ	A	B	C	P	Q	R	X
A	A	Φ	P	R	B	X	C	Q
B	B	P	Φ	Q	A	C	X	R
C	C	R	Q	Φ	X	B	A	P
P	P	B	A	X	Φ	R	Q	C
Q	Q	X	C	B	R	Φ	P	A
R	R	C	X	A	Q	P	Φ	B
X	X	Q	R	P	C	A	B	Φ

Table 2

From the table 2 it is clear that additive identity is Φ and every set is its own inverse.

Multiplication Table

.	Φ	A	B	C	P	Q	R	X
Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ
A	Φ	A	Φ	Φ	A	Φ	A	A
B	Φ	Φ	B	Φ	B	B	Φ	B
C	Φ	Φ	Φ	C	Φ	C	C	C
P	Φ	A	B	Φ	P	B	A	P
Q	Φ	Φ	B	C	B	Q	C	Q
R	Φ	A	Φ	C	A	C	R	R
X	Φ	A	B	C	P	Q	R	X

Table 3

From the table 3 it is clear that additive identity is X.

Intersection satisfies associative law .

Let $A, B, C \in T$, then

$$\begin{aligned}
 A \cap (B + C) &= A \cap [(B - C) \cup (C - B)] \\
 &= [A \cap (B - C)] \cup [A \cap (C - B)] \\
 &= [A \cap B \cap (X - C)] \cup [A \cap C \cap (X - B)]. \quad (A \cap B) + (A \cap C) \\
 &= [(A \cap B) - (A \cap C)] \cup [(A \cap C) - (A \cap B)] \\
 &= [(A \cap B) - C] \cup [(A \cap C) - B] \\
 &= [A \cap B \cap (X - C)] \cup [A \cap C \cap (X - B)].
 \end{aligned}$$

So $A \cap (B + C) = (A \cap B) + (A \cap C)$.

Therefore $(X, T, +, \cdot)$ is a topological ring.

Definition 2.8: A topological module is a module over a topological ring such that scalar multiplication and addition are continuous.

Example 2.9: Let $X = \{0, 1, 2\} = Z_3$ be a ring then $M = T = \{\Phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, X\}$ is a discrete topology on X . Clearly M is a topological R – module.

Definition 2.10: Let X be an indiscrete space. The only continuous maps are constant maps from $X \rightarrow R$ and hence the ring R can be identified with the field R .

Example 2.11: Let X be an arbitrary set and $T = \{\Phi, X\}$ be an indiscrete topology on X .

It can be observed that from the following tables $(T, +, \cdot)$ is a topological field.

Define $A + B = (A - B) \cup (B - A)$ and $A \cdot B = A \cap B$

+	Φ	X
Φ	Φ	X
X	X	Φ

Table 4

\cdot	Φ	X
Φ	Φ	Φ
X	Φ	X

Table 5

Example 2.12: Let $X = Z_2 = \{0, 1\}$ be a Galoi’s field under addition modulo 2 and multiplication modulo 2 operations are as follows:

$+_2$	0	1
0	0	1
1	1	0

Table 6

\cdot_2	0	1
0	0	0
1	0	1

Table 7

Clearly $(X, +_2, \cdot_2)$ is a field.

If we take $T = \{\Phi, X\}$ be an indiscrete topology on X then $(T, +_2, \cdot_2)$ is a topological field.

Definition 2.13: A linear vector space X with a topology T on X is called a topological vector space if addition is a continuous function from $X \times X$ into X and multiplication by scalars is a continuous function from $F \times X$ into X .

Example 2.14: Let $X = Z_2 = \{0, 1\}$ be a vector space over the field $F = \{0, 1\} = Z_2$ and $T = \{\Phi, A, B, X\}$, where $A = \{0\}$, $B = \{1\}$, be the discrete topology on X . From the example 2.5, clearly $(T, +_2)$ is an abelian additive topological group.

Suppose $0, 1 \in Z_2 = F$, then clearly

$$0(\Phi) = 0, 0(A) = 0, 0(B) = 0, 0(X) = 0,$$

$$1(\Phi) = \Phi, 1(A) = A, 1(B) = B, 1(X) = X.$$

Therefore T is a topological vector space.

Example 2.15: Let $X = Z_3 = \{0, 1, 2\}$ be a vector space over the field $F = \{0, 1, 2\} = Z_3$. Then the set T in the example 2.7 is a topological vector space.

3. Vague Topological Rings

Definition 3.1: A topological ring R is said to be a vague topological ring if it satisfies the following conditions:

1. $V_R(A \cup B) \leq \max \{ V_R(A), V_R(B) \}$
2. $V_R(A \cap B) \leq \min \{ V_R(A), V_R(B) \}$
3. $V_R(A + B) \leq \max \{ V_R(A), V_R(B) \}$
4. $V_R(AB) \leq \min \{ V_R(A), V_R(B) \}$.

Example 3.2: Let R be any ring and endow R with the indiscrete topology. Then $T = \{ \Phi, X \}$ is a topological ring.

Clearly the vague set $A = \{ \Phi \langle 0, 0 \rangle, X \langle 1, 1 \rangle \}$ is a vague topological ring.

Example 3.3: Let $X = \{0, 1\}$, $T = \{ \Phi, \{0\}, \{1\}, X \} = \{ \Phi, A, B, X \}$. Then $(T, +, \cdot)$ is a topological ring. The vague set R on $XR = \langle \Phi, [0, 0.2] \rangle \langle A, [0.1, 0.2] \rangle \langle B, [0.2, 0.2] \rangle \langle X, [0.2, 0.2] \rangle$ is a vague topological ring.

Example 3.4: Let $X = \{0, 1, 2\}$ and

$$T = \{ \Phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, X \}$$

$$= \{ \Phi, A, B, C, P, Q, R, X \}.$$

Then $(T, +_3, \cdot_3)$ is a topological ring.

The vague set $R = \langle x, [t_R(x), 1 - f_R(x)] \rangle$ on X defined by

$$t_R(x) = 0 \text{ if } x = \Phi$$

$$= 0.1 \text{ if } x = A$$

$$= 0.2 \text{ if } x = B$$

$$= 0.3 \text{ if } x = C, P, Q, R, X$$

$$1 - f_R(x) = 0 \text{ if } x = \Phi$$

$$= 0.2 \text{ if } x = A$$

$$= 0.3 \text{ if } x = B$$

$$= 0.4 \text{ if } x = C, P, Q, R, X$$

is a vague topological ring.

Lemma 3.5: If R is a vague topological ring of a ring X then for all $A \in X$, we have $V_R(-A) = V_R(A)$.

Theorem 3.6: Let R be a vague set of a ring X then R is a vague topological ring of a ring X if and only if R satisfies the following conditions:

1. $V_R(A - B) \leq \max \{ V_R(A), V_R(B) \}$
2. $V_R(A) \leq V_R(-A)$
3. $V_R(A \cap B) \leq \min \{ V_R(A), V_R(B) \}$.

Proof: Let R be a vague topological ring of a ring X. Then we have

$$t_R(A - B) \leq \max \{ t_R(A), t_R(-B) \}$$

$$= \max \{ t_R(A), t_R(B) \}$$

Similarly, we can prove that

$$1 - f_R(A - B) \leq \max \{ 1 - f_R(A), 1 - f_R(B) \}.$$

It follows that $V_R(A - B) \leq \max \{ V_R(A), V_R(B) \}$.

And by the definition, we get $V_R(A) \leq V_R(-A)$, $V_R(A \cap B) \leq \min \{ V_R(A), V_R(B) \}$.

Conversely suppose that R is a vague set of a ring X. Also we have $V_R(A) \leq V_R(-A)$.

Then clearly $t_R(A + B) = t_R(A - (-B))$

$$\leq \max \{ t_R(A), t_R(-B) \}$$

$$= \max \{ t_R(A), t_R(B) \}$$

Similarly we can prove that

$$1 - f_R(A + B) \leq \max \{ 1 - f_R(A), 1 - f_R(B) \}.$$

It follows that $V_R(A + B) \leq \max \{ V_R(A), V_R(B) \}$.

Hence R is a vague topological ring of a ring X.

Theorem 3.7 : Let R and S are vague topological rings of a ring X then $R \cap S$ is a vague topological ring of X.

Proof: Let R and S are vague topological rings of a ring X.

Then we have ,

$$\begin{aligned} t_{R \cap S}(A - B) &= \min \{ t_R(A - B), t_S(A - B) \} \\ &\leq \min \{ \max \{ t_R(A), t_R(B) \}, \max \{ t_S(A), t_S(B) \} \} \\ &= \max \{ \min \{ t_R(A), t_S(A) \}, \min \{ t_R(B), t_S(B) \} \} \\ &= \max \{ t_{R \cap S}(A), t_{R \cap S}(B) \} \end{aligned}$$

Similarly, we can prove that

$$1 - f_{R \cap S}(A - B) \leq \max \{ 1 - f_{R \cap S}(A), 1 - f_{R \cap S}(B) \}.$$

It follows that

$$V_{R \cap S}(A - B) \leq \max \{ V_{R \cap S}(A), V_{R \cap S}(B) \}.$$

Clearly $V_R(A) \leq V_R(-A)$.

$$\begin{aligned} t_{R \cap S}(AB) &= \min \{ t_R(AB), t_S(AB) \} \\ &\leq \min \{ \max \{ t_R(A), t_R(B) \}, \max \{ t_S(A), t_S(B) \} \} \\ &= \max \{ \min \{ t_R(A), t_S(A) \}, \min \{ t_R(B), t_S(B) \} \} \\ &= \max \{ t_{R \cap S}(A), t_{R \cap S}(B) \} \end{aligned}$$

Similarly, we can prove that

$$1 - f_{R \cap S}(AB) \leq \max \{ 1 - f_{R \cap S}(A), 1 - f_{R \cap S}(B) \}.$$

It follows that

$$V_{R \cap S}(AB) \leq \max \{ V_{R \cap S}(A), V_{R \cap S}(B) \}.$$

Therefore $R \cap S$ is a vague topological ring.

Definition 3.8: Let f be a mapping from a set X into a set Y. Let B be a vague set in Y. Then the inverse image of B, $f^{-1}(B)$ is the vague set in X by $V_{f^{-1}(B)}(x) = V_B(f(x))$ for all $x \in X$.

Definition 3.9: Let f be a mapping from a set X into set Y. Let A be a vague set in X. Then the image of A, $f(A)$ is the vague set in Y by

$$\begin{aligned} V_{f(A)}(y) &= \text{isup} \{ V_A(z) / z \in f^{-1}(y), \text{ if } f^{-1}(y) \neq \emptyset \} \\ &= [0, 0] \quad \text{otherwise.} \end{aligned}$$

Theorem3.10: Let R and S are vague topological rings and f be a homomorphism from R into S. Let B be a vague topological ring of S, then the inverse image $f^{-1}(B)$ is a vague topological ring of R.

Proof: Let R and S are vague topological rings.

Then for all $C, D \in R$, we have

$$\begin{aligned} V_{f^{-1}(B)}(C - D) &= V_B(f(C - D)) \\ &= V_B(f(C) - f(D)) \\ &\leq \max \{V_A(f(C)), V_A(f(D))\} \end{aligned}$$

$$= \max \{V_{f^{-1}(B)}(C), V_{f^{-1}(B)}(D)\}$$

Similarly, we can prove that

$$V_{f^{-1}(B)}(CD) \leq \max \{V_{f^{-1}(B)}(C), V_{f^{-1}(B)}(D)\}.$$

Hence $f^{-1}(B)$ is a vague topological ring of R.

Theorem3.11: Let R and S are vague topological rings and f be a homomorphism from R into S. Let A be a vague topological ring of R, then the image $f(A)$ is a vague topological ring of S.

Proof: Let R and S are vague topological rings. Let $U, V \in S$ such that

$$V_{f(R)}(U) = \text{isup} \{ V_R(U) / C \in f^{-1}(U) \} \text{ and}$$

$$V_{f(R)}(V) = \text{isup} \{ V_R(V) / D \in f^{-1}(V) \}.$$

$$\text{Then } V_{f(R)}(U - V) = \text{sup} \{ V_R(W) / W \in f^{-1}(U - V) \}$$

$$\leq \max \{ V_R(C), V_R(D) \}$$

$$= \text{imax} \{ V_{f(A)}(u), V_{f(A)}(v) \}.$$

Thus the image $f(A)$ of A is a vague topological ring of S.

4.Vague Topological Modules

Definition4.1: Let R be a ring and M be a topological R- module. Then the set S of M is a vague topological module if it satisfies the following conditions:

- 1) $V_S(A + B) \leq \max \{V_S(A), V_S(B)\}$ for all $A, B \in M$
- 2) $V_S(rA) \leq V_S(A)$ for all $A \in M$ and for all $r \in R$
- 3) $V_S(\Phi) = (0, 0)$.

Example 4.2: Let $(T, M, +)$ be a topological module in the example 2.7.

The vague set $S = \langle x, [t_S(x), 1 - f_S(x)] \rangle$ on X defined by

$$t_S(x) = 0 \text{ if } x = \Phi$$

$$= 0.2 \text{ if } x = A$$

$$= 0.3 \text{ if } x = B$$

$$= 0.4 \text{ if } x = C, P, Q, R, X$$

$$1 - f_S(x) = 0 \text{ if } x = \Phi$$

$$= 0.3 \text{ if } x = A$$

$$= 0.4 \text{ if } x = B$$

$$= 0.5 \text{ if } x = C, P, Q, R, X$$

is a vague topological module

Theorem 4.3: The intersection of a family of vague topological modules is also a vague topological module.

Proof: Let $\{M_i / i \in I\}$ be a family of vague topological modules and $M = \cap M_i$.

$$\begin{aligned} \text{Then } V_M(A + B) &= \inf_{i \in I} \{ V_{M_i}(A + B) \} \\ &\leq \inf_{i \in I} \{ \max (V_{M_i}(A), V_{M_i}(B)) \} \\ &= \max \{ V_M(A), V_M(B) \}. \end{aligned}$$

and

$$\begin{aligned} V_M(rA) &= \sup_{i \in I} \{ V_{M_i}(rA) \} \\ &\leq \sup_{i \in I} \{ \max V_{M_i}(A) \} \\ &= \max \{ \sup_{i \in I} V_{M_i}(A) \} \\ &= V_M(A). \end{aligned}$$

Hence the intersection of a family of vague topological modules is also a vague topological module.

Theorem 4.4: Let M_1 and M_2 are vague topological ring modules over the ring R and f be a linear transform of M_1 and M_2 . Let M be a vague topological module of M_2 , then the inverse image $f^{-1}(M)$ is a vague topological module of M_1 .

Proof: Let $A, B \in M_1$ and $a, b \in R$, then

$$\begin{aligned} V_{f^{-1}(M)}(aA + bB) &= V_M(f(aA + bB)) \\ &= V_M(af(A) + bf(B)) \\ &\leq \max \{ V_M(f(A)), V_M(f(B)) \} \\ &= \max \{ V_{f^{-1}(M)}(A), V_{f^{-1}(M)}(B) \}. \end{aligned}$$

Hence $f^{-1}(M)$ is a vague topological ring of M_1 .

Theorem 4.5: Let M_1 and M_2 are vague topological ring modules over the ring R and f be a linear transform of M_1 and M_2 . Let M be a vague topological module of M_1 , then the inverse image $f(M)$ is a vague topological module of M_2 .

Proof: Let $\alpha, \beta \in M_2$.

If either $f^{-1}(\alpha)$ or $f^{-1}(\beta)$ is empty then the inequality theorem

$$V_{f(M)}(aA + bB) \leq \max \{ V_{f(M)}(A), V_{f(M)}(B) \}$$

is satisfied for all $A, B \in M$ and for all $a, b \in R$.

Suppose that neither $f^{-1}(\alpha)$ nor $f^{-1}(\beta)$ is empty.

Let $A_0 \in f^{-1}(\alpha)$, $B_0 \in f^{-1}(\beta)$ then

$$V_M(A_0) = \sup_{A \in f^{-1}(\alpha)} V_M(A),$$

$$V_M(B_0) = \sup_{B \in f^{-1}(\beta)} V_M(B).$$

Then $V_{f(M)}(a\alpha + b\beta)$

$$= \sup_{r \in f^{-1}(a\alpha + b\beta)} V_M(r) \text{ where } r \in f^{-1}(A + B)$$

$$\begin{aligned} &\leq \max \{ V_M(A_0), V_M(B_0) \} \\ &= \max \{ V_{f(M)}(\alpha), V_{f(M)}(\beta) \} \end{aligned}$$

Thus the image $f(M)$ is a vague topological module of M_2 .

5. Vague Topological Fields

Definition 5.1: A topological ring F is said to be a vague topological field if it satisfies the following conditions:

- $V_F(A \cup B) \leq \max \{ V_F(A), V_F(B) \}$

2. $V_F(A \cap B) \leq \min \{ V_F(A), V_F(B) \}$
3. $V_F(A + B) \leq \max \{ V_F(A), V_F(B) \}$
4. $V_F(AB) \leq \min \{ V_F(A), V_F(B) \}$.

Example 5.2: Let X be any arbitrary set and $T = \{ \Phi, X \}$ be an indecrete topological space
Then $F = (T, +, \cdot)$ is a topological field.

Clearly the vague set $A = \{ \Phi <0, 0>, X <1, 1> \}$ is a vague topological field.

Example 5.3: Let $X = \{0, 1\}$ is a field, $T = \{ \Phi, \{0\}, \{1\}, X \} = \{ \Phi, A, B, X \}$.

Then $(T, +, \cdot)$ is a topological field. The vague set R on X

$R = \langle \Phi, [0, 0.2] \rangle \langle A, [0.1, 0.2] \rangle \langle B, [0.2, 0.2] \rangle \langle X, [0.2, 0.2] \rangle$
is a vague topological field.

Lemma 5.4: If F is a vague topological field of a field X then

- 1). $A \in X$, we have $V_F(-A) = V_F(A)$,
- 2). $A \in X$, we have $V_F(A^{-1}) = V_F(A)$.

Proof: By definition, we have $V_F(-A) \leq V_F(A)$ for all $A \in X$(1)

Also $V_F(A) = V_F(-(-A)) \leq V_F(-A)$ (2)

From (1) and (2), $V_F(-A) = V_F(A)$

Similarly, $V_F(A^{-1}) \leq V_F(A)$ for all $A \in X$(3)

Also $V_F(A) = V_F((A^{-1})^{-1}) \leq V_F(A^{-1})$ (4)

From (3) and (4), $V_F(-A) = V_F(A)$.

Theorem 5.5: Let F be a vague topological subset of a field X . Then F is a vague topological field of X if and if F satisfies the following conditions:

- 1) $V_F(A - B) \leq \max \{ V_F(A), V_F(B) \}$ for all $A, B \in F$;
- 2) $V_F(A \cap B^{-1}) \leq \min \{ V_F(A), V_F(B) \}$ for all $A, B \in F$.

Proof: Let F be a vague topological subset of a field X . Then we have

$$V_F(A - B) \leq \max \{ V_F(A), V_F(-B) \}$$

$$= \max \{ V_F(A), V_F(B) \}.$$

Similarly we can prove that

$$V_F(A \cap B^{-1}) \leq \min \{ V_F(A), V_F(B) \}$$

for all $A, B \in F$.

Theorem 5.6: The intersection of a family of vague topological fields is a vague topological field.

Proof: Let $\{F_i / i \in I\}$ be a family of vague topological fields and $F = \cap F_i$.

$$\text{Then } V_F(A - B) = \min_{i \in I} [V_{F_i}(A - B)]$$

$$\leq \min_{i \in I} \{ \max \{ V_{F_i}(A), V_{F_i}(B) \} \}$$

$$= \max \{ V_F(A), V_F(B) \}.$$

Similarly we can prove that

$$V_F(A \cap B^{-1}) \leq \min \{ V_F(A), V_F(B) \}.$$

Hence the intersection of a family of vague topological fields is also vague topological field.

6.Vague Topological Vector Space

Definition 6.1: Let S be a topological vector space over a field F. Then the set S is a vague topological vector space if it satisfies the following conditions:

1. $V_S(A + B) \leq \max \{ V_S(A), V_S(B) \}$;
2. $V_S(aA) \leq V_S(A)$ for all $a \in F$;
3. $V_S(0) = (0, 0)$.

Example 6.2: Let (X, T) be a topological vector space in the example 2.15.

The vague set $S = \langle x, [t_S(x), 1 - f_S(x)] \rangle$ on X defined by

$$\begin{aligned}
 t_S(x) &= 0 \text{ if } x = \Phi \\
 &= 0.3 \text{ if } x = A \\
 &= 0.4 \text{ if } x = B \\
 &= 0.5 \text{ if } x = C, P, Q, R, X \\
 1 - f_S(x) &= 0 \text{ if } x = \Phi \\
 &= 0.4 \text{ if } x = A \\
 &= 0.5 \text{ if } x = B \\
 &= 0.6 \text{ if } x = C, P, Q, R, X
 \end{aligned}$$

is a vague topological vector space.

Example 6.3: For a Vague Topological Vector space over the field Z2:

$$X = Z_2 \times Z_2 = \{0,1\} \times \{0,1\}$$

$$= \{(0,0), (0,1), (1,0), (1,1)\}$$

$$X = \{e, a, b, c\} \text{ where } e = (0,0)$$

$$a = (0,1) \quad b = (1,0) \quad c = (1,1)$$

$$\mathcal{T} = \{ \Phi, \{e\}, \{a\}, \{b\}, \{c\}, \{e,a\}, \{e,b\}, \{e,c\}, \{a,b\},$$

$$\{b,c\}, \{c,a\}, \{e,a,b\}, \{e,b,c\}, \{e,c,a\}, \{a,b,c\},$$

$$\{e,a,b,c\} \}$$

For convenience we will take $A = \{e\}$ $B = \{a\}$ $C = \{b\}$ $D = \{c\}$ $E = \{e,a\}$ $F = \{e,b\}$ $G = \{e,c\}$ $P = \{a,b\}$ $Q = \{b,c\}$

$R = \{C,A\}$ $S = \{e,a,b\}$ $T = \{e,b,c\}$ $U = \{e,c,a\}$ $V = \{a,b,c\}$ $X = \{e,a,b,c\}$

The Addition Table is Given Below

$$A+B = (A - B) \cup (B - A)$$

+	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X
Φ	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X
A	A	Φ	E	F	G	B	C	D	S	T	U	P	Q	R	X	V
B	B	E	Φ	P	R	A	S	U	C	V	D	F	X	G	Q	T
C	C	F	P	Φ	Q	S	A	T	B	D	V	E	G	X	R	U
D	D	G	R	Q	Φ	U	T	A	V	C	B	X	F	E	P	S
E	E	B	A	S	U	Φ	P	R	F	X	G	C	V	D	T	Q
F	F	C	S	A	T	P	Φ	Q	E	G	X	B	D	V	U	R
G	G	D	U	T	A	R	Q	Φ	X	F	E	V	C	B	S	P
P	P	S	C	B	V	F	E	X	Φ	R	Q	A	U	T	D	G
Q	Q	T	V	D	C	X	G	F	R	Φ	P	U	A	S	B	E

R	R	U	D	V	B	G	X	E	Q	P	Φ	T	S	A	C	F
S	S	P	F	E	X	C	B	V	A	U	T	Φ	R	Q	G	D
T	T	Q	X	G	F	V	D	C	U	A	S	R	Φ	P	E	B
U	U	R	G	X	E	D	V	B	T	S	A	Q	P	Φ	F	C
V	V	X	Q	R	P	T	U	S	D	B	C	G	E	F	Φ	A
X	X	V	T	U	S	Q	R	P	G	E	F	D	B	C	A	Φ

Additive identity elements is Φ

Every set has its own inverse.

Inverse of A is A only. Similarly for the other elements also.

Multiplication Table: $A \cdot B = A \cap B$

.	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X
Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ	Φ
A	Φ	A	Φ	Φ	Φ	A	A	A	Φ	Φ	Φ	A	A	A	Φ	A
B	Φ	Φ	B	Φ	Φ	B	Φ	Φ	B	Φ	B	B	Φ	B	B	B
C	Φ	Φ	Φ	C	Φ	Φ	C	C	C	C	Φ	Φ	C	Φ	C	C
D	Φ	Φ	Φ	Φ	D	Φ	Φ	D	Φ	D	D	Φ	D	D	D	D
E	Φ	A	B	Φ	Φ	A	A	A	B	Φ	B	E	B	E	B	E
F	Φ	A	Φ	C	Φ	A	F	A	C	C	Φ	F	F	A	C	F
G	Φ	A	Φ	Φ	Φ	A	A	G	Φ	D	D	A	A	G	D	G
P	Φ	Φ	B	C	Φ	B	C	Φ	P	E	B	P	C	B	P	P
Q	Φ	Φ	Φ	C	D	Φ	C	D	C	Q	D	C	Q	D	Q	Q
R	Φ	Φ	B	Φ	D	B	Φ	D	B	D	R	B	Φ	R	R	R
S	Φ	A	B	C	Φ	E	F	A	P	Q	B	S	F	E	P	S
T	Φ	A	Φ	C	D	A	F	G	C	Q	D	F	T	G	Q	T
U	Φ	A	B	Φ	D	E	A	G	B	D	R	E	G	U	Q	U
V	Φ	Φ	B	C	D	B	C	D	P	Q	R	P	Q	R	V	V
X	Φ	A	B	C	D	E	F	G	P	Q	R	S	T	U	V	X

From the above two tables (\mathcal{T}, \cdot) is a Topological vector space. It is represented by V

Now the vague set $S = (x, [ts(x), fs(x)])$ on V defined by

$$ts(x) = 0 \text{ if } x = \Phi$$

$$= 0.6 \text{ otherwise} \quad 1 - fs(x) = 0 \text{ if } x = \Phi$$

$$= 0.7 \text{ otherwise}$$

Then S is a Vague Topological Vector space.

Example 6.4:

Let $X = \frac{\mathbb{Z}_2}{\langle 1+x+x^2 \rangle} = \{0, 1, \alpha, 1 + \alpha\}$ be a set. Define addition modulo 2 and multiplication modulo 2 on X as follows:

Addition Table:

+2	0	1	α	$1 + \alpha$
0	0	1	α	$1 + \alpha$
1	1	0	$1 + \alpha$	α
α	α	$1 + \alpha$	0	1
$1 + \alpha$	$1 + \alpha$	α	1	0

Multiplication Table:

\times_2	0	1	α	$1 + \alpha$
0	0	0	0	0
1	0	1	α	$1 + \alpha$
α	0	1	$1 + \alpha$	α
$1 + \alpha$	0	$1 + \alpha$	1	α

Then $(X, +_2, \times_2)$ is a Field.

If we take $X = \{0, 1, \alpha, 1 + \alpha\}$

From the Example of Vague Topological Vector space, we can consider

$$\mathcal{T} = \{\Phi, \{0\}, \{1\}, \{\alpha\}, \{1 + \alpha\}, \{0, 1\}, \{0, \alpha\}, \{0, 1 + \alpha\}, \{1, \alpha\}, \{1, 1 + \alpha\}, \{\alpha, 1 + \alpha\}, \{0, 1, \alpha\}, \{0, 1, 1 + \alpha\}, \{0, \alpha, 1 + \alpha\}, \{1, \alpha, 1 + \alpha\}, \{0, 1, \alpha, 1 + \alpha\}\}$$

For convenience we will take

$$\mathcal{T} = \{\Phi, A, B, C, D, E, F, G, P, Q, R, S, T, U, V, X\}$$

$$\text{Where } A = \{0\} \quad B = \{1\} \quad C = \{\alpha\} \quad D = \{1 + \alpha\} \quad E = \{0, 1\} \quad F = \{0, \alpha\} \quad G = \{0, 1 + \alpha\} \quad P = \{1, \alpha\}$$

$$Q = \{1, 1 + \alpha\} \quad R = \{\alpha, 1 + \alpha\} \quad S = \{0, 1, \alpha\}$$

$$T = \{0, 1, 1 + \alpha\} \quad U = \{0, \alpha, 1 + \alpha\}$$

$$V = \{1, \alpha, 1 + \alpha\} \quad X = \{0, 1, \alpha, 1 + \alpha\}$$

We can write the tables for addition and multiplication tables as in the 6.3 example of Topological vector space.

Hence $(\mathcal{T}, +, \cdot)$ is a Topological Vector space.

We can define Vague Topological vector space as follows now the vague set $S = (x, [ts(x), fs(x)])$ on \mathcal{T} defined by

$$ts(x) = 0 \text{ if } x = \Phi$$

$$= 0.5 \text{ otherwise} \quad 1 - fs(x) = 0 \text{ if } x = \Phi$$

$$= 0.6 \text{ otherwise}$$

Theorem 6.5: If S is a vague topological vector space of a vector space X over a field F , then $V_S(\lambda A) = V_S(A)$ for all $0 \neq \lambda \in F$.

Proof: For all $A \in S$, we have $V_S(\lambda A) \leq V_S(A)$ for some $0 \neq \lambda \in F$.

Also we have $V_S(A) = V_S(\lambda A \lambda^{-1}) \leq V_S(\lambda A)$.

That is $V_S(\lambda A) = V_S(A)$.

Theorem 6.6: Let X be a topological vector space over a field F and S is a vague set of X . Then S vague S is a vague topological vector space of X if and only if

$$V_S(aA + bB) \leq \max \{ V_S(A), V_S(B) \} \text{ for all } A, B \in S, a, b \in F.$$

Proof: Suppose that S is a vague topological vector space of X .

We have $V_S(aA) = V_S(A)$ and $V_S(bB) = V_S(B)$.

Then we have

$$V_S(aA + bB) \leq \max \{ V_S(A), V_S(B) \}.$$

Conversely suppose that if $a = b = 1$, we have $V_S(A + B) \leq \max \{ V_S(A), V_S(B) \}$

and if $b = 0$ then $V_S(aA) \leq V_S(A)$.

So S is a vague topological vector space of X .

Theorem 6.7: Let R and S are vague topological vector spaces of vector space X over a field F . Then we have $R \cap S$ is a vague topological vector space of X .

Proof: Let R and S are vague topological vector spaces of vector space X over a field F .

Then we have ,

$$\begin{aligned} V_{R \cap S}(A + B) &= \min \{ V_R(A + B), V_S(A + B) \} \leq \min \{ \max \{ V_R(A), V_R(B) \}, \max \{ V_S(A), V_S(B) \} \} \\ &= \max \{ \min \{ V_R(A), V_S(A) \}, \min \{ V_R(B), V_S(B) \} \} \\ &= \max \{ V_{R \cap S}(A), V_{R \cap S}(B) \} \end{aligned}$$

and

$$\begin{aligned} V_{R \cap S}(\lambda A) &= \min \{ V_R(\lambda A), V_S(\lambda A) \} \\ &= \min \{ \min [V_R(A), V_S(A)] \} \end{aligned}$$

So $R \cap S$ is a vague topological vector space.

Conclusion:

We investigated vague topological rings, vague topological fields, vague topological modules and vague topological vector spaces. It is hoped that these concepts will rise to the notations like vague normed linear spaces, vague Hilbert spaces and vague Banach spaces etc.

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