Defect Methods for Amenability and Ultrapower Stability in Banach Algebras

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Abstract:
This paper develops a defect-based approach to several themes appearing in the proposed doctoral study of non-standard Banach-algebra structures. The central idea is to replace an exact algebraic requirement by a family of quantitative defects measured on finite subsets. For amenability, the defect of a tensor records both its failure to commute with the algebra and its failure to act as an approximate identity after multiplication. This formulation is equivalent to the bounded approximate-diagonal criterion and gives a practical route for distinguishing amenable from non-amenable behavior. The same viewpoint is applied to approximately inner derivations, Arens regularity, Gelfand theory, group algebras, and ultrapowers. Particular attention is paid to the fact that tensor products do not automatically commute with ultrapower constructions, so uniform estimates must be separated from unjustified lifting arguments. Finite-dimensional and finite-group examples are included, together with a research programme for quotient algebras, semigroup algebras, operator algebras, and spectral applications. The paper therefore supplies a new title and a focused mathematical direction while remaining entirely within the scope of the original synopsis.

Keywords: Banach algebra; amenability defect; approximate diagonal; derivation; Arens regularity; ultrapower; Gelfand transform; group algebra.

1. Introduction

Banach algebras lie at the meeting point of algebra, topology, and analysis. Their multiplication is algebraic, but its interaction with the norm makes limiting arguments part of the structure. This is why apparently classical questions—existence of inverses, spectral mapping, representations, and ideal structure—acquire additional forms when one passes to biduals, tensor products, approximate identities, or ultrapowers. The doctoral synopsis motivating this paper collects precisely these themes: amenability, weak and approximate amenability, Arens products, Gelfand transforms, quasi-inverses, polynomial spectral mapping, group actions, and ultrapowers.

The expression "non-standard structure" is best understood here not as a new multiplication chosen arbitrarily, but as a controlled extension of the original Banach algebra. Four extensions are especially important. First, cohomology replaces equations inside the algebra by derivations into bimodules. Second, approximate amenability replaces exact innerness by convergence of inner derivations. Third, Arens multiplication extends the product from A to the bidual A** and exposes an order-of-limits problem.

Fourth, an ultrapower A_U encodes uniform asymptotic behavior of bounded families. Each extension preserves part of the original algebra while introducing new compactness or convergence questions.
The purpose of the present paper is to organize these questions around one measurable object: a defect. Given a finite set F in a Banach algebra and a candidate tensor m, the amenability defect measures how far m is from being central and how far its product is from acting as an identity on F. Small defects on every finite set characterize amenability when a uniform norm bound is available. This gives both a conceptual criterion and a possible computational method. Similar defects can be defined for derivations and for the discrepancy between the two Arens products.
The approach also clarifies the relation between local and global information. Finite-set estimates are local. A bounded approximate diagonal is global because it is a single net directed by all finite sets and tolerances. An ultrapower is again global, because it identifies bounded families modulo an ultrafilter. The main methodological warning is therefore that local witnesses can be transferred to an ultrapower only when their norms and tensor complexity are uniformly controlled. This point is essential in view of the corrigendum to the early ultrapower amenability results [9].
The paper is expository in foundation but research-oriented in method. It proves elementary equivalences and permanence statements, gives exact finite-group diagonals, and formulates a disciplined programme for studying new examples. No unproved characterization of ultra-amenability is asserted. Instead, the emphasis is on estimates that can be checked and on hypotheses that make passage between A, A^{**}, quotients, and A_U legitimate.

2. Banach-algebra preliminaries

Throughout, A denotes a complex Banach algebra. A Banach A-bimodule X is a Banach space with jointly continuous left and right actions satisfying $(ab)\cdot x=a\cdot(b\cdot x)$, $x\cdot(ab)=(x\cdot a)\cdot b$, and $a\cdot(x\cdot b)=(a\cdot x)\cdot b$. A bounded linear map $D:A\to X$ is a derivation when

$$D(ab)=a\cdot D(b)+D(a)\cdot b \quad (a,b\in A).$$

For $x\in X$, the map $\delta_x(a)=a\cdot x-x\cdot a$ is an inner derivation. If every bounded derivation $A\to X^*$ is inner for every Banach A-bimodule X, then A is amenable in the sense introduced by B. E. Johnson [3]. If one requires only derivations $A\to A^*$ to be inner, one obtains weak amenability. These definitions show immediately that amenability is a simultaneous vanishing statement over a large family of modules, while weak amenability is a single-module condition.

The projective tensor product $A\hat{\otimes}A$ is the completion of the algebraic tensor product for the norm

$$\|u\|_\pi = \inf \{ \Sigma_k \|a_k\| \|b_k\| : u=\Sigma_k a_k\otimes b_k \}.$$

It becomes an A-bimodule through $a\cdot(b\otimes c)=ab\otimes c$ and $(b\otimes c)\cdot a=b\otimes ca$. The multiplication map $\pi:A\hat{\otimes}A\to A$ is defined by $\pi(b\otimes c)=bc$ and extended continuously. A bounded approximate diagonal is a bounded net (m_α) in $A\hat{\otimes}A$ satisfying

$$a\cdot m_\alpha-m_\alpha\cdot a \to 0, \qquad \pi(m_\alpha)a \to a \qquad (a\in A).$$

For a non-unital algebra, these two conditions imply the existence of a bounded approximate identity. Johnson's criterion states that A is amenable if and only if it has a bounded approximate diagonal [3,4]. This tensor formulation is the foundation of the defect method used below.
For spectral questions, A# denotes the unitization when A has no identity. An element b is a quasi-inverse of a when $a+b-ab=0$ and $a+b-ba=0$; equivalently, $1-a$ is invertible in A#. For a unital commutative Banach algebra, $\Delta(A)$ is the character space and the Gelfand transform of a is $\hat{a}(\varphi)=\varphi(a)$. The spectrum is $\sigma_A(a)=\{\lambda\in C:a-\lambda 1$ is not invertible$\}$. The polynomial spectral-mapping theorem gives

$\sigma\_A(p(a))=p(\sigma\_A(a))$. These notions are later linked with defect estimates through quotient maps and characters.

3. A finite-set amenability defect

Let F be a non-empty finite subset of the closed unit ball of A. For $m\in A\hat{\otimes}A$ define

$$d\_F(m)=\max\_\{a\in F\} \{ \|a\cdot m-m\cdot a\|\_\pi + \|\pi(m)a-a\| \}.$$

The first term is a centrality defect and the second is an identity defect. For C>0 define the bounded defect number

$$\delta\_C(F)=\inf \{ d\_F(m) : m\in A\hat{\otimes}A \text{ and } \|m\|\_\pi\leq C \}.$$

The quantity $\delta\_C(F)$ decreases as C increases and increases when F is enlarged. It is therefore suitable for a directed construction indexed by triples (F,ε,C).

Proposition 3.1. A is amenable if and only if there is a constant C>0 such that $\delta\_C(F)=0$ for every finite subset F of the closed unit ball.

Proof. Suppose A is amenable. Choose a bounded approximate diagonal (m_α) with $\sup\_\alpha\|m\_\alpha\|\_\pi\leq C$. For fixed F and $\varepsilon>0$, both defining limits hold uniformly on the finite set F for sufficiently large α, so $d\_F(m\_\alpha)<\varepsilon$. Hence $\delta\_C(F)=0$. Conversely, assume such a C exists. Direct the pairs (F,ε), where F is finite and $\varepsilon>0$, by inclusion of F and decrease of ε. For each pair choose $m\_\{F,\varepsilon\}$ with $\|m\_\{F,\varepsilon\}\|\_\pi\leq C$ and $d\_F(m\_\{F,\varepsilon\})<\varepsilon$. For any fixed $a\in A$, rescale a into the unit ball and then choose indices whose finite set contains that rescaled element. The two defect terms converge to zero. Thus $(m\_\{F,\varepsilon\})$ is a bounded approximate diagonal, and A is amenable.

This proposition is not a replacement for Johnson's theorem; it is a finite-set reformulation of its tensor direction. Its value is methodological. To establish amenability one searches for tensors whose defects tend to zero with a uniform norm bound. To prove non-amenability one may try to show that for every C there is a finite set F and a positive lower bound η such that $\delta\_C(F)\geq\eta$. The latter form resembles a separation argument and is often more usable than attempting to construct a non-inner derivation directly.

Proposition 3.2. Let $q:A\to B$ be a continuous surjective homomorphism. If A has a bounded approximate diagonal of bound C, then B is amenable. More quantitatively, for each finite $F\subset B$ and each choice of lifts $E\subset A$ with $q(E)=F$, the tensors $(q\otimes q)(m\_\alpha)$ have norms at most $\|q\|^2C$ and their B-defects on F tend to zero.

Proof. The projective tensor product is functorial for bounded maps, so $q\otimes q:A\hat{\otimes}A\to B\hat{\otimes}B$ is bounded. It intertwines the module actions and the multiplication maps. Applying $q\otimes q$ to the two approximate-diagonal identities therefore gives the corresponding identities in B. The norm estimate follows from $\|q\otimes q\|\leq\|q\|^2$.

This quotient principle suggests a practical non-amenability test: if one can exhibit a non-amenable quotient B, then A cannot be amenable. It also explains why characters and finite-dimensional representations matter. Every character $\varphi:A\to\mathbb{C}$ is a one-dimensional quotient map onto its range, and a family of representations can convert tensor defects into matrix defects that are easier to estimate.

4. Derivation defects and approximate amenability

The cohomological definition can be expressed by an analogous finite-set defect. Let X be a Banach A-bimodule, $D:A\to X^*$ a bounded derivation, F a finite subset of the unit ball, and $x\in X^*$. Put

$$e\_F(D,x)=\max\_\{a\in F\} \|D(a)-(a\cdot x-x\cdot a)\|.$$

Then D is approximately inner exactly when $\inf\_x\ e\_F(D,x)=0$ for every finite F. Approximate amenability requires this condition for every X and every bounded derivation $D:A\to X^*$. Approximate contractibility uses derivations into X rather than X*. These notions were developed systematically by Ghahramani and Loy [5] and subsequently refined in several directions [6].

The defect formulation prevents a common logical error. Pointwise approximation on each a is equivalent to finite-set approximation because finite sets allow a common implementer x. By contrast, choosing a different implementer for every single a without compatibility does not produce an approximately inner derivation. The finite-set maximum records the required compatibility.

Uniform boundedness of the implementers is a stronger condition. If for a given derivation there is M such that for every finite F and $\varepsilon>0$ one can choose x with $\|x\|\leq M$ and $e\_F(D,x)<\varepsilon$, then a weak-star cluster point argument may produce an exact inner implementer in an appropriate dual setting. This observation explains why bounded approximate amenability is closer to amenability than unrestricted approximate amenability, although the precise implication depends on the modules and approximation hypotheses involved.

Proposition 4.1. If A is amenable, then every bounded derivation $D:A\to X^*$ has zero derivation defect on finite sets; in fact D is inner.

This is immediate from the definition of amenability, but it identifies the hierarchy clearly: amenability gives one exact implementer, bounded approximate amenability gives a uniformly bounded net of implementers, and approximate amenability gives a possibly unbounded net. The separation between these levels is one of the principal "non-standard" phenomena proposed in the synopsis.

For weak amenability one takes X=A. A useful strategy is therefore to search for continuous point derivations at characters. If $\varphi\in\Delta(A)$, a continuous functional d satisfying $d(ab)=\varphi(a)d(b)+d(a)\varphi(b)$ is a derivation into the one-dimensional φ-module. A non-zero point derivation obstructs certain amenability properties and can often be detected through the local behavior of the Gelfand transform. Thus the spectral and cohomological parts of the synopsis are not independent: characters provide concrete modules on which derivations can be tested.

5. Arens products and regularity defects

Arens extended the multiplication of A to its bidual A** in two natural ways [1]. For $\Phi,\Psi\in A^{**}$ and $f\in A^*$, the first and second Arens products are obtained by extending the two variables in opposite orders. Writing $\square$ and $\diamond$ for these products, A is Arens regular when $\Phi\square\Psi=\Phi\diamond\Psi$ for all $\Phi,\Psi\in A^{**}$.

The distinction can be seen through iterated limits. If bounded nets (a_i) and (b_j) in A converge weak-star to Φ and Ψ, then, whenever both scalar iterated limits exist,

$$\langle\Phi\square\Psi,f\rangle = \lim_i \lim_j f(a_i b_j),$$
$$\langle\Phi\diamond\Psi,f\rangle = \lim_j \lim_i f(a_i b_j).$$

Arens regularity therefore says that the two orders agree for all admissible data. For finite sets $E\subset A^{**}$, $H\subset A^*$, one may introduce the regularity defect

$$r_{E,H}=\sup \{ |\langle\Phi\square\Psi-\Phi\diamond\Psi,f\rangle| : \Phi,\Psi\in E,\ f\in H \}.$$

Vanishing of all such defects is exactly Arens regularity. Although this definition is formally simple, computing the products can be difficult because A** is large. In applications one therefore uses weakly almost periodic functionals, compactness criteria, or representations on reflexive spaces.

Proposition 5.1. Every reflexive Banach algebra is Arens regular.

Proof. If A is reflexive, the canonical embedding identifies A** with A. Both Arens products extend the original multiplication and hence coincide on all of A**.

Important non-reflexive algebras may also be Arens regular. In particular, for a C*-algebra the bidual is its enveloping von Neumann algebra and the two Arens products agree with the operator-algebra product. On the other hand, many convolution algebras are strongly Arens irregular, showing that amenability and Arens regularity should not be conflated. Amenability concerns derivations and approximate diagonals; Arens regularity concerns uniqueness of bidual multiplication.

The two topics nevertheless interact through compactness. A bounded approximate diagonal lives in $A\hat{\otimes}A$, while a virtual diagonal lives in $(A\hat{\otimes}A)^{**}$. Arens multiplication controls module actions on biduals. A research programme combining these objects should therefore track exactly which product is used and whether the relevant module actions are weak-star continuous. Statements that ignore this distinction can fail even when all formulas look formally correct.

6. Ultrapowers and uniform stability

Let U be an ultrafilter on an index set I. The ultrapower A_U is the quotient of $\ell\infty(I,A)$ by the closed ideal of families (a_i) with $\lim\_U\|a\_i\|=0$. Coordinatewise multiplication makes A_U a Banach algebra. The diagonal map $a\mapsto[(a)\_i]$ is an isometric homomorphism.

Ultrapowers convert uniform local information into exact objects. For example, a family of approximate solutions whose errors converge to zero along U defines an exact solution in the quotient. This principle explains their relevance to the defect method. If tensors m_i have uniformly bounded projective norms and their finite-set defects tend to zero in a sufficiently uniform manner, one expects an induced diagonal-type object in an ultrapower.

The difficulty is that $(A\hat{\otimes}A)\_U$ and $A\_U\hat{\otimes}A\_U$ need not be naturally identical. A coordinatewise family of tensors certainly determines an element of $(A\hat{\otimes}A)\_U$, but it may not determine an element of $A\_U\hat{\otimes}A\_U$ with the required control. This tensor-lifting problem is the main reason that an apparently obvious ultrapower version of the approximate-diagonal criterion requires additional approximation hypotheses. Daws' original paper developed the theory of amenability in ultrapowers [8], and the later corrigendum explicitly corrected the unrestricted diagonal characterization and added an approximation condition [9].

Definition 6.1. A Banach algebra A is ultra-amenable when every ultrapower A_U is amenable.

This is substantially stronger than amenability. The property requires amenability to survive all uniform asymptotic enlargements, not merely quotients or bidual constructions. A safe way to study it is to separate three questions: whether finite-set defects in A admit bounds independent of the size of F; whether the witnesses have uniformly controlled tensor complexity; and whether the relevant tensor map into $A\_U\hat{\otimes}A\_U$ is sufficiently surjective.

Proposition 6.2. If A is finite-dimensional and amenable, then A is ultra-amenable.

Proof. In a finite-dimensional Banach space, every bounded family has a unique ultralimit in the norm-compact closure of its range. The map $[(a\_i)]\mapsto\lim\_U a\_i$ is an isometric linear isomorphism from A_U onto A and respects coordinatewise multiplication. Hence every ultrapower is isomorphic to A and is amenable.

This proposition gives a model case in which all uniformity problems disappear. It also suggests a finite-dimensional approximation programme: compress an infinite-dimensional algebra to finite-dimensional subspaces or quotients, compute defect bounds there, and determine whether the bounds remain stable. Stability of the bounds is the essential issue; convergence of each finite-dimensional approximation separately is not enough.

Ultrapowers also interact with local reflexivity. Daws showed that module and algebra versions of local reflexivity can be used to recover Arens multiplication from a suitable ultrapower representation [7]. Consequently, one promising direction is to compare the Arens-regularity defect with the failure of an ultrapower representation to be multiplicative. This makes local reflexivity a bridge between the bidual and the ultrapower parts of the synopsis.

7. Gelfand transforms, spectra, and quotient tests

Suppose A is commutative and unital. The Gelfand transform $\Gamma:A\to C(\Delta(A))$ is a contractive homomorphism, $\Gamma(a)=\hat{a}$. If A is semisimple, Γ is injective. For every polynomial p,

$$\Gamma(p(a)) = p(\Gamma(a)), \quad \sigma_A(p(a))=p(\sigma_A(a)).$$

These identities allow algebraic defects to be observed pointwise on the character space. If $m=\Sigma_k b_k\otimes c_k$ is an algebraic tensor, then applying a character φ to both tensor factors gives the scalar

$$(\varphi\otimes\varphi)(m)=\Sigma_k \varphi(b_k)\varphi(c_k).$$

Similarly, applying φ to $\pi(m)a-a$ produces $(\varphi(\pi(m))-1)\varphi(a)$. Thus a small identity defect forces $\varphi(\pi(m))$ to be close to one whenever $\varphi(a)$ is not small. Although scalar characters cannot detect the full non-commutative centrality defect, they provide necessary tests and can reveal obstructions caused by the absence of a bounded approximate identity.

Quasi-inverses offer another connection. If b is a quasi-inverse of a, then $1-a$ is invertible in A# and the spectral value 1 is excluded from $\sigma_{A\#}(a)$. Under a homomorphism $T:A\to B$, quasi-inverse relations are preserved. Consequently, spectral obstructions found in a quotient or representation can be pulled back to A. This is compatible with Proposition 3.2: amenability descends to quotients, while non-amenability or pathological spectral behavior in a quotient can obstruct the desired structure in A.

For function algebras, continuous point derivations connect the local geometry of $\Delta(A)$ with cohomology. The proposed research can therefore use three layers of tests. The tensor layer studies $d_F(m)$. The representation layer applies finite-dimensional or scalar homomorphisms to those tensors. The spectral layer studies characters, quasi-inverses, and polynomial images. Agreement among the three layers gives stronger evidence than any single calculation.

A useful computational model is a finite-dimensional commutative algebra represented by matrices. The projective tensor norm can be replaced by a computable upper bound, the centrality equations become linear constraints, and the identity equations become linear approximation constraints. One may then minimize a convex surrogate of d_F. The resulting numerical value is not, by itself, a proof of amenability, but stable lower bounds can suggest an obstruction and stable upper bounds can guide an analytic construction.

8. Group algebras and group actions

Group algebras provide the clearest application of the theory. For a locally compact group G, the convolution algebra $L^1(G)$ is amenable exactly when G is amenable, a fundamental theorem of Johnson [3,4]. Thus an invariant-mean property of the group is equivalent to a cohomological property of the Banach algebra.

For a finite group G, the diagonal is exact. In $\ell^1(G)$ let δ_g denote the point mass at g and define

$$M_G = (1/|G|)\, \Sigma_{g\in G}\, \delta_g\otimes\delta_{g^{-1}}.$$

Then $\pi(M_G)=\delta_e$, the identity of $\ell^1(G)$, and $\delta_h\cdot M_G=M_G\cdot\delta_h$ for every $h\in G$. Hence $d_F(M_G)=0$ for every finite F. The proof is a direct reindexing of the sum. This example supplies a concrete bridge from finite-group representation theory to Banach-algebra amenability.

Representations can refine the test. If $\pi:G\to GL(V)$ is a finite-dimensional representation, its integrated form maps $\ell^1(G)$ into $B(V)$. Applying $\pi\otimes\pi$ to M_G produces an averaging operator. In unitary representations this is closely related to projection onto invariant vectors. Therefore the tensor diagonal, invariant means, and averaging in representation spaces are three manifestations of the same symmetry principle.

For infinite groups one replaces the exact average by Følner-type or invariant-mean approximations when they exist. The defect method suggests examining the rate at which averaging tensors centralize a prescribed finite set. Quantitative estimates may distinguish classes of amenable groups even when ordinary amenability gives only a yes-or-no answer. Conversely, expander-type phenomena or uniformly non-amenable actions may produce positive lower bounds for centrality defects in finite-dimensional quotients.

The group-action language in the synopsis should therefore be tied to explicit modules. If G acts on a compact space K, then the induced action on C(K) and its dual measures produces Banach modules. Cocycles for the action give derivations, while invariant probability measures correspond to fixed points in dual modules. This provides a natural route from measure theory and probability to Banach-algebra cohomology without treating those subjects as unrelated applications.

9. Proposed research programme and applications

The defect framework leads to a concrete work plan.

(i) Quotient and representation stage. Begin with algebras admitting many finite-dimensional representations, such as group algebras of finite or residually finite groups, finite semigroup algebras, and matrix-function algebras. Compute images of candidate diagonals under these representations. A persistent positive defect in a quotient gives a rigorous obstruction to amenability of the original algebra.

(ii) Approximate-amenability stage. For selected Banach sequence algebras and semigroup algebras, describe derivations into natural dual modules and estimate e_F(D,x). The aim is to identify when implementers must become unbounded. This directly addresses the synopsis statement concerning bounded approximate contractibility and the work of Loy and collaborators.

(iii) Bidual stage. Determine whether the natural functionals are weakly almost periodic and estimate the difference between the two iterated limits defining the Arens products. The goal is not to infer amenability from Arens regularity, but to locate precisely where bidual multiplication is stable.

(iv) Ultrapower stage. Formulate uniform versions of the finite-set defect and record the additional approximation property needed to transfer tensor witnesses. Every claim should be checked against the tensor-map obstruction highlighted by Daws' corrigendum. Finite-dimensional amenable algebras and finite group algebras provide benchmark cases.

(v) Spectral stage. Use the Gelfand transform and polynomial spectral mapping to translate approximate identities and quasi-inverse relations into pointwise information. Continuous point derivations should be tested as possible obstructions to weak or approximate amenability.

Applications arise in harmonic analysis, operator theory, and probability. In harmonic analysis, the method studies convolution algebras through averaging. In operator theory, Arens products and ultrapowers compare local operator models with bidual multiplication. In probability, invariant means and stationary measures appear as fixed points of dual actions. In numerical work, finite-dimensional defect minimization can be used as an exploratory tool before an exact proof is attempted.

The expected contribution of this programme is not a single universal criterion beyond Johnson's theorem. Its contribution is a compatible set of tests that can be applied across several extensions of Banach algebras. The tests preserve the distinction between exact, approximate, boundedly approximate, bidual, and ultrapower properties. This distinction is mathematically necessary and gives the phrase “non-standard structure” a precise operational meaning.

10. Conclusion

This paper has presented a new title and a focused development of the original doctoral synopsis. Amenability was reformulated through a bounded finite-set defect, approximate amenability through derivation defects, and Arens regularity through the discrepancy of the two bidual products. Quotient maps, characters, finite-dimensional representations, and finite-group diagonals were used to connect the abstract theory with computable models.

The ultrapower discussion identified uniformity as the central issue. Small defects in A do not automatically yield a diagonal in A_U because projective tensor products and ultrapowers need not commute. Any proposed theorem in this direction must state the required approximation or lifting hypothesis. This methodological caution is as important as the positive finite-dimensional result.

The resulting research direction—defect methods for amenability and ultrapower stability—is sufficiently distinct from the earlier title while remaining fully related to the synopsis. It provides a coherent basis for further work on semigroup algebras, group actions, weak amenability, Gelfand theory, local reflexivity, and operator-algebra ultrapowers.

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