Fixed Point Results in Extended Rectangular b-Metric Space

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Abstract

This research paper expands the concept of rectangular b-metric spaces by introducing extended rectangular b-metric spaces, which are used to establish a fixed point theorem. The main finding builds upon and refines numerous existing result

Keywords: extended Rectangular b-metric Space

1 Introduction

Fixed point theory has become an important field in mathematics due to its variety of applications in science, economics and game theory. In this field, one of the most effective tools is the Banach contraction theorem" suppose T be a mapping on a complete metric space (X,d) into itself satisfying d(Tx,Ty)≤ Kd(x,y) where $K \in [0,1,)$ ∀ x,y∈X,then T has a unique fixed point on X", which was introduced by Banach in 1922.

Many researchers proved Banach contraction principle in multitude of generalized metric space. In 1993 Czerwik introduced the concept of b-metric spaces, which relaxes the triangle in equality condition. This generalization has been useful in extending the Banach contraction mapping theorem .In 2009, the fixed point theory for several valued generalized contraction on a set with two b-metric was studied by Boriceanu et al. [10]

George et al.[6] introduced rectangular b-metric space in order to generalized rectangular metric spaces. inspired by the concept of extended b-metric space and rectangular b-metric space, we introduce extended rectangular b-metric space

2 Preliminaries

2.1 Defination

[5] let X be a non empty set. A mapping $\sigma : X \times X \to R^+$ is said to be a b-metric with coefficient $s \geq 1$, if σ satisfies the following (for all$u,v,w \in X$:)

1. $\sigma(u,v) = 0 \Longleftrightarrow u = v,$
2. $\sigma(u,v) = \sigma(v,u)$
3. $\sigma(u,v) \leq s[\sigma(u,w) + \sigma(w,v)]$

Then the pair (X,σ) is said to be a b-metric space.

2.2 Defination

[9] let X be a non empty set and $\phi : X \times X \to [1,\infty)$.A mapping $\sigma_\phi : X \times X \to R^+$ is said to be an extended b-metric space, if σ_ϕ satisfies the following for all $x,y,z \in X$:

1. $\sigma_\phi(x,y) = 0 \Longleftrightarrow x = y$
2. $\sigma_\phi(x,y) = \sigma_\phi(y,x)$
3. $\sigma_\phi(x,y) \leq \phi(x,y)[\sigma_\phi(x,z) + \sigma_\phi(z,y)]$

Then the pair(X,σ_ϕ) is said to be an extended b-metric space.

2.3 Defination

[2] let X be a non empty set. A mapping $p : X \times X \to R^+$ is said to be a rectangular metric on X,if ,p satisfies the following (for all u,v∈ X) and all distinct m,n∈ $X - x,y$: such that

1. p(u,v)=0 iff u=v
2. p(u,v)=p(v,u)
3. p(u,v)≤ p(u,m)+p(m,n)+p(n,v)

Then the pair (X,p) is said to be a rectangular metric space.

2.4 Defination

[6] let X be a non empty set with coefficient $s \geq 1$. A mapping $r_b : X{\times}X \to R^+$ is said to be a rectangular b-metric space on X if,r_b satisfies the following (for all x,y∈ X and all distinct m,n∈ $X - x,y$):

1. $r_b(x,y) = 0$ iff x=y
2. $r_b(x,y) = r_b(y,x)$
3. $r_b(x,y) \leq s[r_b(x,m) + r_b(m,n) + r_b(n,y)]$

Then the pair (X,r_b) is said to be a rectangular b-metric space.

3 Main Result

3.1 Defination

let X be a non empty set and $\phi : X{\times}X \to [1,\infty]$.A mapping $\alpha_\phi : X{\times}X \to R^+$ is said to be an extended rectangular b-metric on X if α_ϕ satisfies the following for all $x,y \in X$ and all distinct $u,v \in X - \{x,y\}$

1. $\alpha_\phi(x,y) = 0$ iff x=y
2. $\alpha_\phi(x,y) = \alpha_\phi(y,x)$
3. $\alpha_\phi(x,y) \leq \phi(x,y)[\alpha_\phi(x,u) + \alpha_\phi(u,v) + \alpha_\phi(v,y)]$

Then the pair (X,α_ϕ) is said to be an extended rectangular b-metric space.

We have different types of generalized metric spaces, certain relationships between them are expected but the inverse implications need not be true :

↓ metric space → b-metric space ↓→ extended b-metric space↓

Rectangular metric space→ rectangular b-metric space → extended rectangular b-metric space

3.2 Defination

A sequence $\{x_n\}$ in (X,$\phi\xi$) is said to be cauchy if $\lim_{m,n\to\infty} \phi_\xi(x_n,x_m) = 0$

3.3 Defination

A sequence $\{x_n\}$ in (X,ϕ_ξ) is said to be convergent to $x \in X$ if $\lim_{n\to\infty} \phi_\xi(x_n,x) = 0$

3.4 Defination

An extended rectangular b-metric space (x,α_ϕ) is said to be complete if every cauchy in X is convergent to some point in X.

3.5 lemma

3.1 let (X,α_ϕ) be an extended rectangular b-metric space and $\{x_n\}$ a cauchy sequence in X such that $x_n \neq x_m$ whenever $m \neq n$. Then sequence $\{x_n\}$ converges at most one point.

In 1974, Ciric considered the concept of orbit and proved some fixed point results

3.6 Defination

[3] let (X,α_ϕ) be an extended rectangular b-metric space. for a self mapping $f : X \to X$, we define (for $x \in X$ and $n \in N$)

O(x;n)=x,fx,...,$f^n x$ and O(x,∞)=x,fx,...,$f^n x$,...

The set O(x,∞) or simply O(x) is called an orbit of f.

Theorem 3.1 let (X,α_ϕ) be an extended rectangular b-metric space and $f : X \to X$.suppose that the following conditions hold:

1. for all $x,y \in X$,we have α_ϕ (fx,fy)≤$\lambda\alpha_\phi$(x,y) where $\lambda \in [0,\frac{1}{2})$
2. $\lim_{n,m\to\infty} \phi(x_n,x_m) < \frac{1}{\lambda}$
3. (X,α_ϕ) is f-orbitally complete iv. f is orbitally continuous

Then f has a unique fixed point.

Proof: with initial point $x_0 \in X$ construct an iterative sequence $\{x_n\}$ by:

x_1=fx_0, $x_2 = f^2x_0$, $x_3 = f^3x_0$,..., $x_n = f^nx_0$,... now we assert that $\lim_{n\to\infty} \alpha_\phi(x_n,x_{n+1})$=0 on setting x=$x_n$ and y=x_{n+1} in condition (1) we get $\alpha\phi(fnx0),fn+1x0=\alpha\phi(xn,xn+1)$

$\leq \lambda\alpha\phi(xn,xn+1) \leq \lambda^n\alpha_\phi(x_0,x_1)$ which on making $n \to\infty$,gives rise $\lim n\to\infty\ \alpha\phi(fnx0,fn+1x0)$ =0 and $\lim_n\to\infty$ $\alpha_\phi(f^nx_0,f^{n+2}x_0)$ =0 now we show that $\{x_n\}$ is a cauchy sequence in (X,α_ϕ). In doing so, we distinguish two cases as under:

case 1: Firstly let p is odd , that is p=2m+1 for any $m \geq 1$. Now using ($3\alpha_\phi$) for any $n \in N$, we have $\alpha\phi(xn,xn+2m+1) \leq \phi(xn,xn+2m+1)[\alpha\phi(xn,xn+1) + \alpha\phi(xn+1,xn+2) + \alpha\phi(xn+2,xn+2m+1)]$

$\leq \phi(xn,xn+2m+1)[\lambda n\alpha\phi(x0,x1) + \lambda n+1\alpha\phi(x0,x1)] + \phi(xn,xn+2m+1) \times \alpha\phi(xn+2,xn+2m+1)$

$=\phi(x_n,x_n+2m+1)(\lambda^n + \lambda^{n+1})\alpha_\phi(x_0,x_1) + \phi(x_n,x_n+2m+1) \times \alpha_\phi(x_n+2,x_n+2m+1)$

$\leq \phi(x_n,x_n+2m+1)(\lambda^n + \lambda^{n+1})\alpha_\phi(x_0,x_1) + \phi(x_n,x_n+2m+1) \times \phi(x_n+2,x_n+2m+1)$

$(\lambda^{n+2}+\lambda^{n+3})\alpha_\phi(x_0,x_1)+\ldots+\phi(x_n,x_n+2m+1)\ldots\phi(x_n+2m-2,x_n+2m+1)(\lambda^{n+2m-2}+\lambda^{n+2m-1})\times\alpha_\phi(x_0,x_1)+$

$\phi(x_n,x_n+2m+1)\ldots\phi(x_n+2m-2,x_n+2m+1)\lambda^{n+2m}\alpha_\phi(x_0,x_1)$

$=\lambda n(1+\lambda)\alpha\phi(x0,x1)\mathrm{P}mi=0-1\ \lambda 2i\ \mathrm{Q}ij=0\ \phi(xn+2j,xn+2m+1)+\lambda n+2m\ \mathrm{Q}mj=0-1\ \phi(xn+2j,xn+2m+1\alpha\phi(x0,x1))$

yielding thereby

$\sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \phi(x_{n+2j},x_{n+2m+1}) \leq \sum_{i=0}^{m-1} \lambda^{2i} \prod_{j=0}^{i} \phi(x_{2j},x_{n+2m+1})$

As in view of condition (ii) we have $\lim_{n,m\to\infty} \phi(x_n,x_m)\lambda < 1$

therefore by the ratio test, we conclude that the series $\sum_{i=0}^{\infty} \lambda^{2i} \prod_{j=0}^{i} \phi(x_{2j},x_{n+2m+1}$) is convergent for each $m \in N$. Assume that

$^{\mathrm{P}}\infty i=0\ \lambda^{2i}\ {}^{\mathrm{Q}}ij_{=0}\ \phi(x_2j,x_n+2m+1)\ sn = \mathrm{P}ni=0\ \lambda 2i\ \mathrm{Q}ij=0\ \phi(x2j,xn+2m+1)$ therefore from the above in equality we have $\alpha_\phi(x_n,x_n+2m+1) \leq \lambda^n(1 + \lambda)\alpha_\phi(x_0,x_1)[s_m-1 - s_n-1] + \lambda^{n+2m\ \mathrm{Q}m}j_{=0}{}^{-1}\ \phi(x_n+2j,x_n+2m+1)\alpha_\phi(x_0,x_1)$

(3.1)

letting $n \to\infty$ in equation (3.1), we can conclude that $\alpha_\phi(x_n,x_n+2m+1) \to 0$ Case 2:

Assume that p is even that is p=2m for any $m \geq 1$ then $\alpha_\phi(x_n,x_n+2m) \leq \phi(x_n,x_n+2m)[\alpha_\phi(x_n,x_n+1) + \alpha_\phi(x_n+1,x_n+2) + \alpha_\phi(x_n+2,x_n+2m)] \leq \phi(xn,xn+2m[\lambda n\alpha\phi(x0,x1) + \lambda n+1\alpha\phi(x0,x1) + \phi(xn,xn+2m) \times \alpha\phi(xn+2,xn+2m)]$

$=\phi(xn,xn+2m)(\lambda n + \lambda 6n + 1)\alpha\phi(x0,x1) + \phi(xn,xn+2m)\alpha\phi(xn+2,xn+2m)$

$\leq \phi(xn,xn+2m)(\lambda n +\lambda 6n + 1)\alpha\phi(x0,x1)+\phi(xn,xn+2m)\times\phi(xn+2,xn+2m)(\lambda n+2+\lambda n+3)\alpha\phi(x0,x1)+ \ldots+$ $\phi(x_n,x_n+2m)\ldots\phi(x_n+2m,x_n+2m+1)\lambda^{n+2m}\alpha_\phi(x_0,x_1)$

$=\lambda n(1+\lambda)\alpha\phi(x0,x1)\mathrm{P}mi=0-1\ \lambda 2i\ \mathrm{Q}ij=0\ \phi(xn+2j,xn+2m+1)+\lambda n+2m-2\mathrm{Q}mj=0-1\ \phi(xn+2j,xn+2m)\alpha\phi(x0,x2)$

(2)

Taking limit as $n \to\infty$ in (2) we get $\alpha\phi(xn,xn+2m) \to 0$

Therefore in both the cases we have $\lim n \to \infty\ \alpha\phi(xn,xn+p) = 0$
which shows that the sequence $\{x_n\}$ is cauchy sequence in X. Since X is f-orbitally complete then there exists $x \in X$ such that $x_n \to x$. Since f is orbitally continuous so we have $\alpha_\phi(fx,x) \leq \phi(fx,x)[\alpha_b(fx,x_n) + \alpha_\phi(x_n,x_{n+1}) + \alpha_\phi(x_{n+1,x1})]$ $\alpha_\phi(fx,x) \leq \phi(fx,x)[\alpha_b(fx,x_n) + \alpha_\phi(fx_{n-1},fx_n) + \alpha_\phi(x_{n+1},x)]$
$=\phi(fx,x)[\alpha_b(fx,x_n) + \lambda\alpha_\phi(x_{n-1},x_n) + \alpha_\phi(x_{n+1},x)]$ $n \to \infty$ gives arise $\alpha_\phi(fx,x) \to 0$
So that $\alpha_\phi(fx,x) = 0$ Therefore fx=x hence x is a fixed point of f. by lemma 3.1, a sequence $\{x_n\}$ converges uniquely at point $x \in X$.

4 Conclusion

This paper builds upon the relative recent concept of rectangular b-metric spaces by introducing an extension that generalizes the concept $s \geq 1$ to a function $\phi(x,y)$ in the quadrilateral in equality. The primary contribution, theorem (3.1), is a variant of the Banach contraction principle that incorporates the concept of orbit.

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