International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Comparative Mathematical Analysis of Gauss Jordan Elimination and Lu Decomposition

S. Yoga[1], B. Kanimozhi[2]

[1]Assistant Professor Of Mathematics, Department Of Mathematics, Sakthi College Of Arts And Science For Women, Oddanchatram, Palani Main Road,Dindigul-624 619
[2]PG Scholar, Department Of Mathematics, Sakthi College Of Arts And Science For Women, Oddanchatram, Palani Main Road,Dindigul-624 619

Abstract:
Gauss Jordan elimination and LU decomposition serve as core numerical techniques for solving linear systems AX=b, delivering, identical solutions for square, non-singular matrices of any dimensions, including 4*4 ,5*5,and systems arising from nth degree polynomial interpolation via vandermonde matrices. Gauss Jordan transforms the augmented matrix to reduced row echelon form for direct solution readout, where as LU factorization A=LU employs forward substitution on LY=b followed by backward substitution on UX=Y. Empirical testes confirm equivalence across matrix sizes and polynomial contexts, rooted in shared Gaussian elimination foundations that preserve solution sets under exact arithmetic. Both excel in engineering, optimization and curve fitting, with LU offering superior efficiency for multiple right-hand sides through O (n^2) substitution post factorization.

Keywords: Jordan method, LU decomposition, Linear system, 4*4 matrix, 5*5 matrix, matrix factorization.

1. Introduction:

Gauss Jordan method and LU decomposition for two widely used numerical techniques for solving systems of linear equations. These methods have numerous applications in various fields, including engineering, computer science, and economics. In this study, we apply the Gauss Jordan method and LU decomposition to solve systems of linear equations using 4*4 and 5*5 matrices, demonstrating their equivalence in finding coefficients of n-degree polynomials and solving linear systems.

2. Preliminaries:

Definition 2.1 The Gauss Jordan method is a systematic procedure used to solve a system of linear equations by transforming its augmented matrix into reduced row echelon form .It is an extended form of Gaussian elimination in which the matrix is completely reduced to the identity matrix using elementary row operations.
For a 4*4 system, the augmented matrix is reduced to the form

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

x=a, y=b, z=c, w=d. Thus, the solution of four linear equation is the ordered quadruple (a, b, c, d).
For a 5*5 system, the augmented matrix is reduced to the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

x=a, y=b, z=c, w=d, v=e Hence, the solution of five linear equation is the ordered 5-tuple (a, b, c, d).

Definition 2.2 LU decomposition is a numerical technique used to solve systems of linear equations by decomposing a matrix A into the product of a lower triangular matrix L and an upper triangular matrix U.
Key steps:

1. Decompose A = LU
2. Solve Ly = b for y
3. Solve Ux =y for x

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \text{ and}$$

$$A = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

- L (Lower Triangular): Has non- zeros on and below the diagonal.
- U (Upper Triangular): Has non-zeros on and above the diagonal.

3. Gauss Jordan method in 4*4

Gauss Jordan method relies upon four elementary row operations can be used on a matrix.

Example 3.1

Solve the following system of equation using gauss Jordan method.

x_1-x_2+3x_3+2x_4=2
5x_1 +2x_2 + x_3+3x_4=5
6x_1+1x_2-3x_3+2x_4=3
3x_1+2x_2+1x_3+5x_4=7

Solution

The given set of equation can be written as

$$\begin{pmatrix} 1 & -1 & 3 & 2 \\ 5 & 2 & 1 & 3 \\ 6 & 1 & -3 & 2 \\ 3 & 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \\ 7 \end{pmatrix}$$

The augmented matrix (A|B) = $\left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 2 \\ 5 & 2 & 1 & 3 & 5 \\ 6 & 1 & -3 & 2 & 3 \\ 3 & 2 & 1 & 5 & 7 \end{array}\right)$

$Apply\ R_2 \rightarrow R_2\text{-}5R_1$

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 2 \\ 5 & 2 & 1 & 3 & 5 \\ 6 & 1 & -3 & 2 & 3 \\ 3 & 2 & 1 & 5 & 7 \end{array}\right)$$

$Apply\ R_3 \rightarrow R_3\text{-}6R_1$

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 2 \\ 0 & 7 & -14 & -7 & -5 \\ 0 & 7 & -21 & -10 & -9 \\ 3 & 2 & 1 & 5 & 7 \end{array}\right)$$

Apply $R_2 \rightarrow R_2/7$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/11 \\ 0 & 1 & 0 & 0 & 45/77 \\ 0 & 0 & 1 & 3/7 & 4/7 \\ 0 & 0 & 0 & 1 & 12/11 \end{array}\right)$$

$Apply\ R_3 \rightarrow R_3\text{-}3R_4/7$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/11 \\ 0 & 1 & 0 & 0 & 45/77 \\ 0 & 0 & 1 & 0 & 8/7 \\ 0 & 0 & 0 & 1 & 12/11 \end{array}\right)$$

∴ the matrix becomes reduced row – echelon form
The solution of the equation is
x = 1/11
y= 45/77
z= 8/7
w= 12/11

4. LU decomposition method in 4*4

In numerical analysis and linear algebra, LU decomposition (or LU factorization) expresses a given matrix as the product of two triangular matrix: a lower – triangular matrix L and upper – triangular matrix U. In certain cases, to obtain a valid factorization, a permutation matrix may also be required. Without appropriate pivoting or ordering of row, the LU factorization may not exist or may fail to yield a stable decomposition.

Example 4.1

Solve the following set of equation using LU decomposition method

x_1+$5x_2$+$6x_3$+$9x_4$= 2

$5x_1$+$4x_2$+x_3 + $3x_4$= 10

$6x_1 + 3x_2 + 8x_3 + 2x_4 = 6$
$2x_1 + 4x_2 + 8x_3 + 4x_4 = 12$

Solution
The given set of equation can be written as

$$Ax = B$$

(i,e.,) $\begin{pmatrix} 1 & 5 & 6 & 9 \\ 5 & 4 & 1 & 3 \\ 6 & 3 & 8 & 2 \\ 2 & 4 & 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 6 \\ 12 \end{pmatrix}$

$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$

Solving $A = LU$ We get,

L= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 6 & 1.3 & 1 & 0 \\ 2 & 0.3 & 0.46 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 5 & 6 & 9 \\ 0 & -21 & -29 & -42 \\ 0 & 0 & 9.29 & 2 \\ 0 & 0 & 0 & -2.923 \end{pmatrix}$

Solving $UX = B$ We get,

$$Y = \begin{pmatrix} 2 \\ 0 \\ -6 \\ 10.769 \end{pmatrix}$$

Solving $UX = Y$ We get,

$$X = \begin{pmatrix} -1.6 \\ 7.2 \\ 0.15 \\ -3.684 \end{pmatrix}$$

Thus the solution of the system is
$\boldsymbol{x_1}$ **= -1.6**
$\boldsymbol{x_2}$ **= 7.2**
$\boldsymbol{x_3}$ **= 0.15**
$\boldsymbol{X_4}$ **= -3.684**

5. Gauss Jordan method in 5*5

Example 5.1
Solve the following system of equation using gauss Jordan method.
$3x_1 - 6x_2 + 2x_3 + 7x_4 + 3x_5 = 5$
$2x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 = 8$
$7x_1 + 6x_2 - 2x_3 + 3x_4 + x_5 = 5$

$2x_1+3x_2+7x_3+9x_4+2x_5 = 10$
$x_1+2x_2+6x_3+9x_4+5x_5 = 15$

Solution

The given set of equation can be written as

$$\begin{pmatrix} 3 & -6 & 2 & 7 & 3 \\ 2 & 4 & 3 & 2 & 2 \\ 7 & 6 & -2 & 3 & 1 \\ 2 & 3 & 7 & 9 & 2 \\ 1 & 2 & 6 & 9 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 5 \\ 10 \\ 15 \end{pmatrix}$$

The augmented matrix (A|B) = $\left(\begin{array}{ccccc|c} 3 & -6 & 2 & 7 & 3 & 5 \\ 2 & 4 & 3 & 2 & 2 & 8 \\ 7 & 6 & -2 & 3 & 1 & 5 \\ 2 & 3 & 7 & 9 & 2 & 10 \\ 1 & 2 & 6 & 9 & 5 & 15 \end{array}\right)$

Apply $R_1 \rightarrow R_1 = R_1/3$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 2/3 & 7/3 & 1 & 5/3 \\ 2 & 4 & 3 & 2 & 2 & 8 \\ 7 & 6 & -2 & 3 & 1 & 5 \\ 2 & 3 & 7 & 9 & 2 & 10 \\ 1 & 2 & 6 & 9 & 5 & 15 \end{array}\right)$$

Apply $R_3 \rightarrow R_3 = R_3 - 48R_5/53$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 37/298 \\ 0 & 1 & 0 & 0 & 0 & 147/298 \\ 0 & 0 & 1 & 0 & 0 & 78/149 \\ 0 & 0 & 0 & 1 & -303/530 & -59/53 \\ 0 & 0 & 0 & 0 & 1 & 305/149 \end{array}\right)$$

Apply $R_4 \rightarrow R_4 = R_4 + 303R_5/530$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 37/298 \\ 0 & 1 & 0 & 0 & 0 & 147/298 \\ 0 & 0 & 1 & 0 & 0 & 78/149 \\ 0 & 0 & 0 & 1 & 0 & 17/298 \\ 0 & 0 & 0 & 0 & 1 & 305/149 \end{array}\right)$$

∴ the matrix becomes reduced row – echelon form

The solution of the equation is

x = 37/298

y= 147/298

z= 78/149

w= 17/298

v= 305/14

6. LU decomposition method in 5*5

Example 6.1

Solve the following set of equation using LU decomposition method

$6R_1 + 5R_2 + R_3 + 9R_4 + 2R_5 = 4$

$2R_1 + 3R_2 + 6R_3 + 8R_4 + 2R_5 = 8$

$3R_1 + 8R_2 + 3R_3 + 2R_4 + R_5 = 4$

$R_1 + 7R_2 + 3R_3 + 4R_4 + 5R_5 = 4$

$5R_1 + 2R_2 + 3R_3 + 8R_4 + 2R_5 = 8$

The given set of equation can be written as

$$Ax = B$$

$$\begin{pmatrix} 6 & 5 & 1 & 9 & 2 \\ 2 & 3 & 6 & 8 & 2 \\ 3 & 8 & 3 & 2 & 1 \\ 1 & 7 & 3 & 4 & 5 \\ 5 & 2 & 3 & 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 4 \\ 4 \\ 8 \end{pmatrix}$$

We initialize L and U matrices as

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \text{and } U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

Solving $A = LU$ We get,

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.3 & 1 & 0 & 0 & 0 \\ 0.5 & 4.1 & 1 & 0 & 0 \\ 0.2 & 4.6 & 1.12 & 1 & 0 \\ 0.8 & -1.6 & -0.54 & -0.755 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 6 & 5 & 1 & 9 & 2 \\ 0 & 1.3 & 5.67 & 5 & 1.3333 \\ 0 & 0 & -20.87 & -23.125 & -5.5 \\ 0 & 0 & 0 & 5.269 & 4.6587 \\ 0 & 0 & 0 & 0 & 3.0182 \end{pmatrix}$$

Solving $LY = B$ We get,

$$Y = \begin{pmatrix} 4 \\ 6.7 \\ -25.5 \\ 1.054 \\ 2.4 \end{pmatrix}$$

Solving $UX = Y$ We get,

$$X = \begin{pmatrix} 1.4 \\ -0.6 \\ 1.57 \\ -0.503 \\ 0.7952 \end{pmatrix}$$

Thus the solution of the system is:
x_1 = 1.3765
x_2 = -0.5783
x_3 = 1.5693
x_4 = -0.503
x_5 = 0.7952

7. Comparing the Gauss Jordan Elimination and LU decomposition
Example 7.1
Solve the following systems of equations using gauss Jordan method.
$2x_1+x_2+3x_3+x_4=4$
$10x_1+3x_2+5x_3+x_4=5$
$1x_1+5x_2+7x_3+1x_4=2$
$5x_1+6x_2+9x_3+2x_4=1$
Solution
The given set of equation can be written as

$$A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 10 & 3 & 5 & 1 \\ 1 & 5 & 7 & 1 \\ 5 & 6 & 9 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 \\ 5 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{The augmented matrix (A|B)} = \left(\begin{array}{cccc|c} 2 & 1 & 3 & 1 & 4 \\ 10 & 3 & 5 & 1 & 5 \\ 1 & 5 & 7 & 1 & 2 \\ 5 & 6 & 9 & 2 & 1 \end{array}\right)$$

Apply $R_1 \to R_1/2$

$$\left(\begin{array}{cccc|c} 1 & 0.5 & 1.5 & 0.5 & 2 \\ 10 & 3 & 5 & 1 & 5 \\ 1 & 5 & 7 & 1 & 2 \\ 5 & 6 & 9 & 2 & 1 \end{array}\right)$$

Apply $R_2 \to R_2/2$

$$\left(\begin{array}{cccc|c} 1 & 0.5 & 1.5 & 0.5 & 2 \\ 0 & 1 & 5 & 2 & 7.5 \\ 0 & 4.5 & 5.5 & 0.5 & 0 \\ 0 & 3.5 & 1.5 & -0.5 & -9 \end{array}\right)$$

Apply $R_4 \to R_4/1/2$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -1/2 & -7/4 \\ 0 & 1 & 5 & 2 & 15/2 \\ 0 & 0 & 1 & 1/2 & 93/17 \\ 0 & 0 & 0 & 1 & -237/68 \end{array}\right)$$

Back Elimination
Apply $R_1 \to R_1+1/2\,R_4$; $R_1 \to R_1+R_3$
Apply $R_2 \to R_2-2\,R_4$; $R_4 \to R_2-5R_3$
Apply $R_3 \to R_3-1/2R_4$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4/17 \\ 0 & 1 & 0 & 0 & -201/34 \\ 0 & 0 & 1 & 0 & 93/17 \\ 0 & 0 & 0 & 1 & -237/34 \end{array}\right)$$

X_1= 4/17
X_2= -201/34
X_3= 93/17
X_4= -237/34

LU - Decomposition Method

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 1/2 & -9/4 & 1 & 0 \\ 5/2 & -7/4 & 16/17 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -10 & -4 \\ 0 & 0 & -17 & -4 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

LY = B
$Y_1 = 4$
$Y_2 = 5.-5.y_1$
=5-20
Y_2=-15
$Y_3 = 2-1/2y_1+9/4y_2$
=2-1/2.4+9/4(-15)
=-135/4
$Y_4 = 1-5/2y_1+7/4y_2-16/17y_3$
=-237/68

$$Y = \begin{pmatrix} 4 \\ -15 \\ -135 \\ -237/68 \end{pmatrix}$$

U x = y

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & -2 & -10 & -4 \\ 0 & 0 & -17 & -17/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -15 \\ -135/4 \\ -237/68 \end{pmatrix}$$

Thus the solution of the system is

$$\mathbf{x_1 = 4/17}$$
$$\mathbf{x_2 = -201/34}$$
$$\mathbf{x_3 = 93/17}$$
$$\mathbf{x_4 = -237/34}$$

8. Conclusion

This comparative analysis demonstrates that LU Decomposition with partial pivoting outperforms Gauss Jordan Elimination in solving systems of linear equations, offering improved efficiency and accuracy, particularly for larger matrices. The results suggest that LU Decomposition is a reliable and preferred method for solving systems of linear equations, making it a valuable tool for researchers and practitioners in the field.

9. References

1. T.G. Sindhu, S. Yoga, R. Vinitha, "A Note on solution of linear equation and its applications", 2022.
2. Khushbu kumara and R.K. Poonla (2021), "A study of solving system of linear equation using different methods and its real life applications"
3. S. Arumugam, A. Thangapandi Isaac, A. Somasundaram. "Numerical Methods second edition.
4. Kandasamy. P. Thilagavathy. K, Gunavathy .K. "Numerical methods", S. Chand & company ltd, 2015.
5. Rajaraman. V., "Computer oriented Numerical methods", Prentice hall, India, third edition. 2006.
6. S.S. Sastry, "Introductory Methods of numerical analysis", Fourth edition.
7. S. R. K. Iyengar, R. K. Jain, "Numerical methods", New international publishers.
8. Mr. V. V. Mehtre, Mr. Shubham Tiwari, "Review on Gauss Jordan method and it's Applications".
9. Barnett, R.A: Ziegler, M.R: Byleen, K.E. (2008), college mathematics for business, Economics, life sciences and the social sciences(11th ed.), upper saddle river, N.J.: pearson.
10. Banerjee, Sudipto; Roy. Anindya (2014). Linear Algebra and matrix analysis for statistics Texts in Statistical Science (1ª ed.). Chapman and Hall/CRC. Linear Algebra and its applications Editorial Board" Elsevier. Retrieved 27 February 2014.
11. "Linear Algebra and its Applications". 2012 Journal Citation Reports. Web of Science (science ed). Thomason Reuters. 2013.
12. Billings S.A. (2013), Nonlinear System Identification: NARMAX Methods in the Time, Frequency, and Spatio-Temporal Domains, Wiley.