Some Products of Discrete, Monotone Fuzzy Finite Automata

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ABSTRACT

This paper investigates and coined two different important classes of fuzzy finite automata and examines with the help of example . We analyze different types of product constructions and study the corresponding converse implications among these automata. In particular, the focus is on discrete and monotone, fuzzy finite automata.

Keywords: Fuzzy finite automaton,direct,cascade,wreath and cartesian produts.

1. INTRODUCTION AND PRELIMINARIES

Nowadays automata theory is playing a crucial role in computer science. The Notion of fuzzy set introduce by Zadeh[1]. W.G.Wee[2] ,firstly coin the term "Fuzzy Automata" in 1967. recently D.S.Malik et.al.applied algebraic technique to studdy fuzzy automata and defined various products and their properties.In this paper we try to define four different fuzzy finite automata, their various product , inverse of product for types of fuzzy finite automata.

Definition : Let A = (A, X, δ) be an automaton without outputs. here A is the(finite nonempty) state set, X is the input alphabet, and $\delta : A \times X \to A$ is the transition function.

Example : Let$A = \{a, b, c\}$, $X = \{x_1, x_2\}$ and $\delta : A \times X \to A$ can be defined as; $\delta(a, x_1) = b$, $\delta(a, x_1) = a$, $\delta(b, x_2) = c$, $\delta(c, x_2) = a$ is automaton without output. **Definition 1.2**:[5],[6],[7] A triplet A = (Q, Σ, μ) is called a fuzzy finite automa-ton(ffa), where Q is a finite nonempty set called the set of states,Σ is a finite nonempty set called the set of input symbols and $\mu : Q \times \Sigma \times Q \to [0, 1]$ is called the fuzzy transition function.

Example: Let $Q = \{p, q, r\}$, $\Sigma = \{\sigma_1, \sigma_2\}$ and $\mu : Q \times \Sigma \times Q \to [0, 1]$ can be defined as;
$\mu(p, \sigma_1, p) = 0.4$, $\mu(q, \sigma_1, q) = 0.2$, $\mu(r, \sigma_1, r) = 0.1$, $\mu(p, \sigma_1, q) = 0.01$, $\mu(p, \sigma_1, r) = 0.003$, $\mu(p, \sigma_2, q) = 0.7$, $\mu(p, \sigma_2, r) = 0.9$, $\mu(p, \sigma_2, p) = 0.004$, $\mu(q, \sigma_2, q) = 0.008$, $\mu(r, \sigma_2, r) = 0.88$, $(q, \sigma_1, p) = 0.003$, $\mu(q, \sigma_1, r) = 0.33$, $\mu(r, \sigma_1, p) = 0.56$, $\mu(r, \sigma_1, q) = 0.66$, $\mu(q, \sigma_2, p) = 0.22$, $\mu(q, \sigma_2, r) = 0.65$, $\mu(r, \sigma_2, p) = 0.78$, $\mu(r, \sigma_2, p) = 0.34$, $\mu(r, \sigma_2, q) = 0.11$. Then A = (Q, Σ, μ) is a ffa.

2. TRIVIAL FUZZY FINITE AUTOMATA

Definition: Let A = (Q, Σ, μ) be a ffa. Then A is called **trivial fuzzy finite automaton** , if Q is singleton set.

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Example: Let $Q = \{p\}$, $\Sigma = \{\sigma_1, \sigma_2\}$ and $\mu : Q \times \Sigma \times Q \rightarrow [0, 1]$ .Then $A = (Q, \Sigma, \mu)$ is trivial ffa.

3. PRODUCT ON DISCRETE FUZZY FINITE AUTOMATA

Definition Let $A = (Q, \Sigma, \mu)$ be a ffa. Then A is called **discrete ffa,** if for every $p \in Q$ and for every $\sigma \in \Sigma$, $\mu(p, \sigma, q) > 0$, *for* $p = q$ *and* $\mu(p, \sigma, q) = 0$, $p \neq q$.

Example Let $Q = \{p, q, r\}$, $\Sigma = \{\sigma_1, \sigma_2\}$ and $\mu(p, \sigma_1, p) = 0.3$, $\mu(p, \sigma_2, p) = 0.6$, $\mu(q, \sigma_1, q) = 0.2$, $\mu(q, \sigma_2, q) = 0.7$, $\mu(r, \sigma_1, r) = 0.4$, $\mu(r, \sigma_2, r) = 0.1$., Then $A = (Q, \Sigma, \mu)$ is discrete ffa.

Definition : Let $A = (Q, \Sigma, \mu)$ and $A' = (Q', \Sigma', \mu')$ be two ffa.A ffa $A \times A' = (Q \times Q, \Sigma \times \Sigma, \mu \times \mu')$ is called the direct product of the A and A′, where $\mu \times \mu' : (Q \times Q') \times (\Sigma \times \Sigma) \times (Q \times Q') \rightarrow [0, 1]$ is defined as follows: $\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q')$.

Theorem: If A and A′ are two discrete ffa.Then their direct product i.e.$A \times A'$ is also a discrete ffa.

Proof: Let $A = (Q, \Sigma, \mu)$ and $A' = (Q', \Sigma', \mu')$ be two disctere ffa. Let (p, p') and $(q, q') \in Q \times Q'$ and $(\sigma, \sigma') \in \Sigma \times \Sigma'$

Case(i): Let $(p, p') = (q, q')$.Then $p = q$ and $p' = q'$.

Now, $\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p, \sigma, p) \wedge \mu(p', \sigma', p') > 0$.

Case(ii): Let $(p, p') \neq (q, q')$. Then there are three subcases.

(a)$p = q$ and $p' \neq q'$,

(b) $p \neq q$ and $p' = q'$,

(c) $p \neq q$ and $p' \neq q'$.

In all the subcases, We have $\mu(p, \sigma, q) \wedge 0 = 0$. Hence,$A \times A'$ is discrete ffa. **Theorem**: If direct product of two automaton A and A′ is discrete ffa,then both A and A′ are discrete ffa.

Proof: claim: A and A′ are discrete ffa.

Without loss of generality assume that, A is discrete ffa. and A′ is not discrete ffa.

Case(i): Suppose $\exists\, p' \in Q'$ and $\exists\, \sigma' \in \Sigma$ such that $\mu'(p', \sigma', p') = 0$.

For any $p \in Q$ and $\sigma \in \Sigma$, $\mu \times \mu'((p, p'), (\sigma, \sigma'), (p, p')) = \mu(p, \sigma, p) \wedge \mu'(p', \sigma', p') = 0$,as $\mu'(p', \sigma', p') = 0$.

Thus,$\mu \times \mu'((p, p'), (\sigma, \sigma'), (p, p')) = 0$, even though $(p, p') = (p, p')$, This is contradiction to $A \times A'$ is discrete ffa.

Case(ii):Suppose $\exists\, p', q' \in Q'$ and $\exists\, \sigma' \in \Sigma'$ with $p' \neq q'$ such that $\mu'(p', \sigma', q') > 0$.

For any ,$p \in Q$,and $\sigma \in \Sigma$, we have $\mu \times \mu'((p, p'), (\sigma, \sigma'), (p, q')) = \mu(p, \sigma, p) \wedge \mu'(p', \sigma', q') > 0$.

Thus,$\mu \times \mu'((p, p'), (\sigma, \sigma'), (p, q')) > 0$,even though $(p, p') \neq (p, q')$. This is contradiction to $A \times A'$ is discrete ffa.

Hence, both A A′ must be discrete ffa.

Definition: Let $A = (Q, \Sigma, \mu)$ and $A' = (Q', \Sigma', \mu')$ be two ffa. and let $\omega : Q' \times \Sigma' \rightarrow \Sigma$ be a function. A ffa $A\omega A' = (Q \times Q', \Sigma', \mu\omega\mu')$ is called a cascade product of the A and A′, where $\mu\omega\mu' : (Q \times Q') \times \Sigma' \times (Q \times Q') \rightarrow [0, 1]$ is defined as follows: $\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q')$.

Theorem: If A and A$'$ are two discrete ffa. Then their cascade product AωA$'$ is also a discrete ffa.

Proof: Let A = (Q, Σ, μ) and A$'$ = (Q', Σ', μ') be two discrete ffa. Then their cascade product is AωA$'$ = $(Q{\times}Q', \Sigma', \mu\omega\mu')$., where $\omega : Q'{\times}\Sigma' \rightarrow$ be a function.

Let (p, p') *and* $(q, q') \in Q \times Q'$ and $\sigma' \in \Sigma'$

case(i): Let $(p, p') = (q, q')$. Then $p = q$ *and* $p' = q'$.

Now, $\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') > 0$, Since A and A$'$ are two discrete ffa.

case(ii): Let $(p, p') \neq (q, q')$. Then there are three subcases:

(a)$p = q$ and $p' \neq q'$,

(b) $p \neq q$ and $p' = q'$,

(c) $p \neq q$ and $p' \neq q'$.

(a) Let $p = q$ and $p' \neq q'$.Then $\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \omega(p', \sigma') \wedge 0 = 0$. Since A ,A$'$ are discrete ffa.

(b) Let $p \neq q$ and $p' = q'$.Then $\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \omega(p', \sigma'), q) \wedge 0 = 0$ (since A and A$'$ are discrete ffa).

(c) Let $p \neq q$ and $p' \neq q'$. Then $\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \omega(p', \sigma'), q) \wedge 0 = 0$ A and A$'$ are discrete ffa). Hence, AωA$'$ is discrete ffa.

Theorem: If cascade product AωA$'$ of A and A$'$ is discrete ffa. Then both A and A$'$ are discrete ffa.

Proof: claim: A and A$'$ are discrete ffa.

Without loss of generality assume that,Let A is discrete ffa. and A$'$ is not discrete ffa.

Case(i): Suppose $\exists\, p' \in Q'$ and $\exists\, \sigma' \in \Sigma'$ such that $\mu'(p', \sigma', q') = 0$ with $p' = q'$. For any $p \in Q$ and $\sigma \in \Sigma$, we have $\mu\omega\mu'((p, p'), \sigma', (p, p')) = \mu(p, \omega(p', \sigma'), p) \wedge \mu'(p', \sigma', p') = \mu(p, \omega(p', \sigma'), p) \wedge 0 = 0$,as $\mu'(p', \sigma, p')$. This is contradiction to AωA$'$ is discrete ffa.

Case(ii):Suppose $\exists\, p', q' \in Q'$ and $\exists\, \sigma' \in \Sigma'$ with $p' \neq q'$ such that $\mu'(p', \sigma', q') > 0$.

For any $p \in Q$,$\sigma \in \Sigma$.Then $\mu\omega\mu'((p, p'), \sigma', (p, q')) = \mu(p, \omega(p', \sigma'), p) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, p) \wedge \mu'(p, \sigma', q') > 0$,as $\mu'(p', \sigma', q') > 0$

Thus,$\mu\omega\mu'((p, p'), \sigma', (p, q')) > 0$,if $(p, p') \neq (p, q')$. This, is contradiction to AωA$'$ is discrete ffa.

Hence, both A and A$'$ are discrete ffa.

Definition : Let A = (Q, Σ, μ) and A$'$ = (Q', Σ', μ') be two ffa. and let $f : \Sigma^{Q'} : Q' \rightarrow \Sigma$ be a function. A ffa A$\circ$A$'$ = $((Q{\times}Q'), (\Sigma^{Q'} {\times}\Sigma'), \mu{\circ}\mu')$ is called a Wreath product of the A and A$'$, where $\mu \circ \mu' : (Q \times Q') \times (\Sigma^{Q'} \times \Sigma') \times (Q \times Q') \rightarrow [0, 1]$ is defined as follows: $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p'.\sigma', q')$. **Theorem**: If A and A$'$ are two discrete ffa. Then their wreath product A $\circ$ A$'$ is also discrete ffa.

Proof: Let A = (Q, Σ, μ) and A$'$ = (Q', Σ', μ') be two discrete ffa.

Then A $\circ$ A$'$=$(Q \times Q', \Sigma^{Q'} \times \Sigma', \mu \circ \mu')$, where $\Sigma^{Q'} : Q' \rightarrow \Sigma$ be a function.

Let (p, p') and $(q, q') \in Q \times Q'$ and $(f, \sigma) \in \Sigma^{Q'} \times \Sigma'$.
Case(i): Let $(p, p') = (q, q')$. Then $p = q$ and $p' = q'$.
Now, $\mu \circ \mu'((p.p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') > 0$ (since A and A$'$ are discrete ffa.)
Thus,$\mu \circ \mu'((p.p'), (f, \sigma'), (q, q')) > 0$, if $(p, p') = (q, q')$.
Case(ii): Let $(p, p') \not\models (q, q')$. Then there are three subcases:
(a) Let $p = q$ and $p' \not\models q'$.Then, $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu(p', \sigma', q') = 0 \wedge \mu(p', \sigma', q') = 0$, (since A and A$'$ are discrete ffa).
(b) Let $p \not\models q$ and $p' = q'$.Then, $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu(p', \sigma', q') = \mu(p, f(p'), q) \wedge 0 = 0$. (since A and A$'$ are discrete ffa).
(c) Let $p \not\models q$ and $p' \not\models q'$.Then, $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu(p', \sigma', q') = 0 \wedge 0 = 0$
(since A and A$'$ are discrete ffa). Thus,$\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = 0$, if $(p, p') \not\models (q, q')$.
Hence, A $\circ$ A$'$ is discrete ffa.
Theorem: If wreath product A $\circ$ A$'$ of A and A$'$ is discrete ffa.Then both A and A$'$ are discrete ffa.
Proof : **claim**: A and A$'$ are discrete ffa. Without loss of generality assume that, A is discrete ffa. and A$'$ is not discrete ffa.
Case(i): Suppose $\exists\, p' \in Q'$ and $\exists\, \sigma' \in \Sigma'$ such that $\mu'(p', \sigma', p') = 0$. For any $p \in Q$ and $\sigma \in \Sigma$, we have $\mu \circ \mu'((p, p'), (f, \sigma'), (p, p')) = \mu(p, f(p'), p) \wedge \mu'(p', \sigma', p') = \mu(p, \sigma, p) \wedge 0 = 0$, as $\mu'(p', \sigma', p') = 0$
Thus,$\mu \circ \mu'((p, p'), (f, \sigma'), (p, p')) = 0$, if $(p, p') = (p, p')$. This is contradic-tion to A $\circ$ A$'$ is discrete ffa.
Case(ii):Suppose $\exists\, p', q' \in Q'$ and $\exists\, \sigma' \in \Sigma'$ with $p' \not\models q'$ such that $\mu'(p', \sigma', q') > 0$.
Then for any $p \in Q$,and $\sigma \in \Sigma$,we have $\mu \circ \mu'((p, p'), (f, \sigma'), (p, q')) = \mu(p, f(p'), p) \wedge \mu'(p', \sigma', q') > 0$.
Thus,$\mu \circ \mu'((p, p'), (f, \sigma), (p, q')) > 0$,if $(p, p') \not\models (p, q')$. This is contradiction to A $\circ$ A$'$ is discrete ffa.
Hence, A and A$'$ are discrete.
Definition 3.5: Let A $= (Q, \Sigma, \mu)$ and A$' = (Q', \Sigma', \mu')$ be two ffa. such that $\Sigma \cap \Sigma' = \phi$ A ffa A $\bullet$ A$' = (Q \times Q', \Sigma \cup \Sigma', \mu \bullet \mu')$ is called the Cartesian product of A and A$'$, where $\mu \bullet \mu' : (Q \times Q') \times (\Sigma \times \Sigma') \times (Q \times Q') \to [0, 1]$ is defined as follows:
$\mu \bullet \mu'\ (p, p'), \sigma, (q, q') = \mu(p, \sigma, q)$, if $\sigma \in \Sigma$ and $p' = q'$,
$\mu'(p', \sigma, q')$, if $\sigma \in \Sigma$ and $p = q$,
0 otherwise
Theorem: If A and A$'$ are two discrete ffa. Then their cartesian product $\mu \bullet \mu'$ is also discrete ffa.
Proof:Let A $= (Q, \Sigma, \mu)$ and A$' = (Q', \Sigma', \mu')$ be two discrete ffa. Then the cartesian product of A and A$'$ is
A $\bullet$ A$' = (Q \times Q', \Sigma \cup \Sigma', \mu \bullet \mu')$.
Let (p, p') *and* $(q, q') \in Q \times Q'$ and $\sigma \in \Sigma \cup \Sigma'$. **Case(i)**: Let $(p, p') = (q, q')$.Then $p = q$ and $p' = q'$. If $\sigma \in \Sigma$, then $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu(p, \sigma, q) > 0$.

If $\sigma \in \Sigma'$, then $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu'(p', \sigma', q') > 0$,
Thus, $\mu \bullet \mu'((p, p'), \sigma, (q, q')) > 0$,if $(p, p') = (q, q')$ and $\sigma \in \Sigma \cup \Sigma'$
case(ii): Let $(p, p') \not= (q, q')$.Then the following three cases arises:
(a) Let $p = q$ and $p' \not= q'$.
If $\sigma \in \Sigma$, then $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = 0$
If $\sigma \in \Sigma'$, then $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu'(p', \sigma', q') = 0$, Since A$'$ is discrete.
(b) if $p \not= q$ and $p' = q'$,then,a similar to (a) $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = 0, \forall \sigma \in \Sigma \cup \Sigma'$.
(c) Let $p \not= q$ and $p' \not= q'$.then,
(i) if $\sigma \in \Sigma$,then $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu(p, \sigma, q) = 0$,since A$'$ is discrete.
(ii) if $\sigma \in \Sigma'$,then $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu'(p', \sigma', q') = 0$, A$'$ are discrete ffa) Thus, $\mu \bullet \mu'((p, p'), \sigma, (q, q')) = 0$, if $(p, p') \not= (q, q')$ Hence,the Cartesian product of A and A$'$ is discrete ffa.
Theorem: If cartesian product of A and A$'$ is discrete ffa., then both A and A$'$ are discrete ffa.
Proof: **claim**: A and A$'$ are discrete ffa.
Without loss of generality, A is discrete ffa. and A$'$ is not discrete ffa.
Case(i): Suppose $\exists\ p' \in Q'$ and $\exists \sigma \in \Sigma'$ such that $\mu'(p', \sigma', p') = 0$. For any $p \in Q$, we have $\mu \bullet \mu'((p, p'), \sigma, (p, p')) = \mu'(p', \sigma', p') = 0$. Thus,$\mu \bullet \mu'((p, p'), \sigma, (p, p')) = \mu'(p', \sigma', p') = 0$,if $(p, p') = (p, p')$.
This is contradiction to A$\bullet$A$'$ is discrete ffa.
Case(ii):Suppose $\exists\ p', q' \in Q'$ and $\exists\ \sigma \in \Sigma'$ with $p' \not= q'$ such that $\mu'(p', \sigma', q') > 0$.
For any,$p \in Q$,we have $\mu \bullet \mu'((p, p'), \sigma, (p, q')) = \mu'(p', \sigma, q') > 0$.
Thus,$\mu \bullet \mu'((p, p'), \sigma, (p, q')) > 0$,if $(p, p') \not= (p, q')$. This is contradiction to A$\bullet$A$'$ is discrete ffa.
Hence, both A and A$'$ must be discrete ffa.

4. PRODUCT ON MONOTONE FUZZY FINITE AUTOMATON

Definition: Let $(Q, \leq)$ be partial order set. A ffa A = (Q, Σ, μ) is called **monotone** ffa, if for any $p, q \in Q$ and $\sigma \in \Sigma$,$\mu(p, \sigma, q) > 0$, when $p \leq q$ and $\mu(p, \sigma, q) = 0$, otherwise.
Example: Let $Q = \{p, q, r\}$ with $\leq = \{(p, p), (q, q), (r, r), (p, q), (q, r), (p, r)\}$ and $\Sigma = \{\sigma_1, \sigma_2\}$.Then define μ as $\mu(p, \sigma_1, p) = 0.4, \mu(q, \sigma_1, q) = 0.7, \mu(r, \sigma_1, r) = 0.6, \mu(p, \sigma_1, q) = 0.2, \mu(q, \sigma_1, r) = 0.5, \mu(p, \sigma_1, r) = 0.4, \mu(p, \sigma_2, p) = 0.1, \mu(q, \sigma_2, q) = 0.8, \mu(r, \sigma_2, r) = 0.7, \mu(p, \sigma_2, q) = 0.3, \mu(q, \sigma_2, r) = 0.6, \mu(p, \sigma_2, r) = 0.5$. Then A = (Q, Σ, μ) is monotone ffa.
If we remove $\mu(p, \sigma_1, p) = 0.4$ from above definition of μ,then A = (Q, Σ, μ) is not monotone.
Theorem: Prove that Cartesian product of two posets is poset.
OR Let $(Q, \leq)$ and $(Q', \leq)$ be two posets.Then prove that $(Q \times Q', \leq)$ is a poset.
Proof:- Let we have to show that $(Q \times Q', \leq)$ is a poset. i.e.the relation " $\leq$ " is (1) Reflexive (2) Antisymmetric (3)Transitive.
(1) We want to show that,$(p, p') \leq (q, q')$ for all $(p, p'), (q, q') \in Q \times Q'$.
Now,$p \leq q$ and $p' \leq q'$ (Since,Q and Q' are posets). Thus,$(p, p') \leq (q, q')$. Hence relation is reflexive.
(2) Assume,$(p, p') \leq (q, q')$ and $(q, q') \leq (p, p')$. $p \leq q, q \leq p \implies p = q$
$p' \leq q', q' \leq p' \implies p' = q'$, Thus,$(p, p') = (q, q')$.Hence relation is antisymmetric.

(3) Assume,$(p, p') \leq (q, q')$ and $(q, q') \leq (p_1, p'_1)$. $(p, p') \leq (q, q') \implies p \leq q, p' \leq q'$

$(q, q') \leq (p_1, p'_1) \implies q \leq p_1, q' \leq p'_1$. $p \leq p_1, p' \leq p'_1$.

$(p, p') \leq (p, p'_1)$.Thus,$(Q \times Q', \leq)$ is a poset.

Theorem: If A and A$'$ are two monotone ffa. Then their direct product $A \times A'$ is monotone ffa.

Proof: Let $A = (Q, \Sigma, \mu)$ and $A' = (Q', \Sigma', \mu')$ be two monotone ffa.

Then, $A \times A' = (Q \times Q', \Sigma \times \Sigma', \mu \times \mu')$. Let (p, p') and $(q, q') \in Q \times Q'$ and $(\sigma, \sigma') \in \Sigma \times \Sigma'$.

Case(i): Let $(p, p') \leq (q, q')$.Then $p \leq q$ and $p' \leq q'$. Then $\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') > 0$.

Case(ii): Let $(p, p') \nleq (q, q')$. there are three subcases:

(a) Let $p \nleq q$,$p' \leq q'$. then

$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p.\sigma, q) \wedge \mu'(p', \sigma', q') = 0 \wedge \mu'(p', \sigma', q') = 0$.

(b) Let $p \leq q$,$p' \nleq q'$. then

$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p.\sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p.\sigma, q) \wedge 0 = 0$

(c) Let $p \nleq q$,$p' \nleq q'$. then

$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p.\sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p.\sigma, q) = 0 \wedge 0 = 0$.

Thus,$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = 0$,if $(p, p') \nleq (q, q')$. Hence, direct product of two monotone ffa is again monotone ffa.

Theorem: If direct product of A and A$'$ i.e. $A \times A'$ is monotone ffa,then both A and A$'$ are monotone ffa.

Proof: **Claim**:A and A$'$ are monotone ffa.

without loss of generality, we assume that A is monotone ffa and A$'$ is not mono-tone ffa.

subcase(i): Suppose $\exists\, p', q' \in Q'$,$\sigma' \in \Sigma'$ such that,$p' \leq q'$ and $\mu'(p', \sigma', q') = 0$ for any $p, q \in Q$ and $\sigma \in \Sigma$ with $p \leq q$,We have

$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = 0$ as $\mu'(p', \sigma', q') = 0$.

Thus,$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = 0$,if $(p, p') \leq (q, q')$ This is contradiction to $A \times A'$ is monotone ffa.

subcase(ii): Suppose $\exists\, p', q' \in Q'$,$\sigma' \in \Sigma'$ such that,$p' \nleq q'$ and $\mu'(p', \sigma', q') > 0$.

For any $p, q \in Q$ and $\sigma \in \Sigma$ with $p \leq q$.Then

$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') => 0$,Since,$\mu'(p', \sigma', q') > 0$ and A is monotone.

Thus,$\mu \times \mu'((p, p'), (\sigma, \sigma'), (q, q')) > 0$,if $(p, p') \nleq (q, q')$ This is contradiction to $A \times A'$ is monotone ffa.

Hence,A and A$'$ must be monotone ffa.

Theorem: If A and A$'$ are two monotone ffa. Then their cascade product $A\omega A'$ is also monotone ffa.

Proof: Let $A = (Q, \Sigma, \mu)$ and $A' = (Q', \Sigma', \mu')$ be two monotone ffa. Then $A\omega A' = (Q \times Q', \Sigma' \mu\omega\mu')$.

Let $\omega : Q' \times \Sigma \to \Sigma$ be a function.i.e.$\omega(p', \sigma') = \sigma$(say). Let (p, p') and $(q, q') \in Q \times Q'$ and $\sigma \in \Sigma'$

Case(i):Let $(p, p') \leq (q, q') \implies p \leq q$ *and* $p' \leq q'$.Then,there are four cases:

(a) : if $p = q$,$p' = q'$,then

$\mu\omega\mu'((p, p'), \sigma'(q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$. (Since, A and A$'$ are monotone ffa.)
(b) : if $p < q, p' \leq q'$, then
$\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$. (Since, A and A$'$ are monotone ffa.)
(c) : if $p \leq q, p' < q'$, then
$\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$. (Since, A and A$'$ are monotone ffa.)
(d) : if $p < q, p' < q'$, then
$\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$.
(Since, A and A$'$ are monotone ffa.) Thus, $\mu\omega\mu'((p, p'), \sigma', (q, q')) > 0$, if $(p, p') \leq$
(q, q').
Case(ii): Let $(p, p') \nleq (q, q')$. there are three subcases:
(a). if $p \nleq q, p' \leq q'$, then
$\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) = 0 \wedge \mu'(p', \sigma', q') > 0 = 0$ (Since,A and A$'$ are monotone ffa.)
(b). $p \leq q, p' \nleq q'$, then
$\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') = 0 = 0$ (Since,A and A$'$ are monotone ffa.)
(c). $p \nleq q, p' \nleq q'$. Then
$\mu\omega\mu'((p, p'), sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') =$
$\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) = 0 \wedge \mu'(p', \sigma', q') = 0 = 0$ (Since,A and A$'$ are monotone ffa.)
Thus, $\mu\omega\mu'((p, p'), \sigma', (q, q')) = 0$, if $(p, p') \nleq (q, q')$.
Thus, cascade product of two monotones ffa is also monotones ffa.

Theorem: If cascade product of A and A$'$ is monotone ffa. Then both A and A$'$ are monotone ffa.

Proof: **Claim**: A and A$'$ are monotone ffa.
Withoutloss of generality, A is monotone ffa and A$'$ is not monotone ffa. **Case(i)**: Suppose $\exists\ p', q' \in Q', \sigma' \in \Sigma'$ such that, $p' \leq q', \mu'(p', \sigma', q') = 0$ take for all $p, q \in Q, \sigma \in \Sigma$ such that, $p \leq q$. Then $\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = 0$ as, $\mu'(p', \sigma', q') = 0$
Thus, $\mu\omega\mu'((p, p'), \sigma', (q, q')) = 0$, if $(p, p') \leq (q, q')$ This is contradiction to AωA$'$ is monotone ffa.
Case(ii): Suppose $\exists\ p', q' \in Q', \sigma' \in \Sigma'$ such that, $p' \nleq q'. \mu'(p', \sigma', q') > 0$
. take for all $p, q \in Q, \sigma \in \Sigma$ such that, $p \leq q$. Then $\mu\omega\mu'((p, p'), \sigma', (q, q')) = \mu(p, \omega(p', \sigma'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') > 0$ as, $\mu'(p', \sigma', q') > 0$
Thus, $\mu\omega\mu'((p, p'), \sigma', (q, q')) > 0$, if $(p, p') \nleq (q, q')$ This is contradiction to AωA$'$ is monotone ffa.
Thus, both A and A$'$ are monotone ffa.

Theorem: If A and A$'$ are two monotone ffa.Then their wreath product $A \circ A'$ is also monotone ffa.

Proof: Let $A = (Q, \Sigma, \mu)$ and $A' = (Q', \Sigma', \mu')$ be two monotone ffa. Then
$A \circ A' = (Q \times Q', \Sigma^{Q'} \times \Sigma', \mu \circ \mu')$.

Let $f : \Sigma^{Q'} : Q' \to \Sigma$ i.e. $f(p') = \sigma$, where $p' \in Q'$ and $\sigma \in \Sigma$. Let $(p, p') and$ $(q, q') \in Q \times Q'$ and $(f, \sigma') \in \Sigma^{Q'} \times \Sigma'$.

Case(i): Let $(p, p') \leq (q, q') \implies p \leq q$ and $p' \leq q'$. Then there are four cases:

(a) : If $p = q, p' = q'$,then $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') =$ $\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') \implies \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$. if $p = q$ and $p' = q'$ (Since, A and A$'$ are monotone ffa)

(b) :If $p < q, p' \leq q'$,then $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge$ $\mu'(p', \sigma', q') =$ $\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') \implies \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$. if $p < q$ and $p' \leq q'$. (Since, A and A$'$ are monotone ffa)

(c) : If $p \leq q, p' < q'$,then $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') =$ $\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') \implies \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$. if $p \leq q$ and $p' < q'$. (Since, A and A$'$ are monotone ffa)

(d) : If $p < q, p' < q'$,then $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge$ $\mu'(p', \sigma', q') =$ $\mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') \implies \mu(p, \sigma, q) > 0 \wedge \mu'(p', \sigma', q') > 0$. if $p < q$ and $p' < q'$. (Since, A and A$'$ are monotone ffa) Thus, $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) > 0$, if $(p, p') \leq (q, q')$.

Case(ii): Let $(p, p') \nleq (q, q')$. there are three subcases:

(a) : if $p \leq q, p' \nleq q'$,then $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) > 0 \wedge mu'(p', \sigma', q') = 0 = 0$.
(Since,A and A$'$ are monotone ffa).

(b) : if $p \nleq q, p' \leq q'$,then $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) = 0 \wedge \mu'(p', sigma', q') > 0 = 0$.
(Since,A and A$'$ are monotone ffa).

(c) : if $p \nleq q, p' \nleq q'$,then $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) = 0 \wedge \mu'(p', \sigma', q') = 0 = 0$.
(Since,A and A$'$ are monotone ffa). $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = 0$, if $(p, q) \not\models (p', q')$.

Thus, wreath product of two monotone ffa is monotone ffa.

Theorem: If the wreath product of A and A$'$ is monotone ffa. Then both A and A$'$ are monotone ffa.

Proof: **Claim**: A and A$'$ are monotone ffa.

Without loss of generality assume that, A is monotone ffa and A$'$ is not monotone ffa.

subcase(i): Suppose $\exists\, p', q' \in Q', \sigma' \in \Sigma'$ such that,$p' \leq q'$,$\mu'(p', \sigma', q') = 0$ take for all $p, q \in Q, \sigma \in \Sigma$ such that, $p \leq q$.Then
$\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') = 0$ as, $\mu'(p', \sigma', q') = 0$

Thus,$\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = 0$,if $(p, p') \leq (q, q')$ This is contradiction to A $\circ$ A$'$ is monotone ffa.

subcase(ii): Suppose $\exists\, p', q' \in Q', \sigma' \in \Sigma'$ such that,$p' \nleq q', \mu'(p', \sigma', q')$.

Take for all $p, q \in Q, \sigma \in \Sigma$ such that, $p \leq q$.Then
$\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) = \mu(p, f(p'), q) \wedge \mu'(p', \sigma', q') = \mu(p, \sigma, q) \wedge \mu'(p', \sigma', q') > 0$ as, $\mu'(p', \sigma', q') > 0$

. Thus, $\mu \circ \mu'((p, p'), (f, \sigma'), (q, q')) > 0$, if $(p, p') \nleq (q, q')$ This is contradiction to $A \circ A'$ is monotone ffa.
Thus, both A and A' are monotone ffa.
Theorem: If A and A' are two monotone ffa.Then their cartesian product $A \bullet A'$ is also monotone ffa.
Proof: Let $A = (Q, \Sigma, \mu)$ and $A' = (Q', \Sigma', \mu')$ be two monotone ffa. Then $A \bullet A' = (Q \times Q', \Sigma \cup \Sigma, \mu \bullet \mu',)$.
Let (p, p') and $(q, q') \in Q \times Q'$ and $\sigma \in \Sigma \cup \Sigma'$
Case(i): Let $(p, p') \leq (q, q') \implies p \leq q$ *and* $p' \leq q'$. Then there are four cases:
(a) : (i) If $p = q, p' = q'$.and $\sigma \in \Sigma$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu(p, \sigma, q) > 0, p' = q'$. (ii) If $p = q, p' = q'$.and $\sigma \in \Sigma'$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu'(p', \sigma', q') > 0, p = q$.
Thus, $\mu \bullet \mu'((p, p'), \sigma, (q, q')) > 0$, if $p = q, p' = q'$ $\sigma \in \Sigma$ or $\sigma \in \Sigma'$. (Since, A and A' are two monotone ffa).
(b) : (i) If $p < q, p' \leq q'$.and $\sigma \in \Sigma$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu(p, \sigma, q) > 0, p' = q'$. (ii) If $p < q, p' \leq q'$.and $\sigma \in \Sigma'$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu'(p', \sigma', q') > 0, p = q$.
Thus, $\mu \bullet \mu'((p, p'), \sigma, (q, q')) > 0$, if $p < q, p' \leq q'$ $\sigma \in \Sigma$ or $\sigma \in \Sigma'$. (Since, A and A' are two monotone ffa).
(c) : (i) If $p \leq q, p' < q'$.and $\sigma \in \Sigma$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu(p, \sigma, q) > 0, p' = q'$. (ii) If $p < q, p' \leq q'$.and $\sigma \in \Sigma'$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu'(p', \sigma', q') > 0, p = q$.
Thus, $\mu \bullet \mu'((p, p'), \sigma, (q, q')) > 0$, if $p \leq q, p' < q'$ $\sigma \in \Sigma$ or $\sigma \in \Sigma'$. (Since, A and A' are two monotone ffa).
(d) : (i) If $p < q, p' < q'$.and $\sigma \in \Sigma$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu(p, \sigma, q) > 0, p' = q'$.
(ii) If $p < q, p' \leq q'$.and $\sigma \in \Sigma'$ Then
$\mu \bullet \mu'((p, p'), \sigma, (q, q')) = \mu'(p', \sigma', q') > 0, p = q$.
Thus, $\mu \bullet \mu'((p, p'), \sigma, (q, q')) > 0$, if $p \leq q, p' < q'$ $\sigma \in \Sigma$ or $\sigma \in \Sigma'$. (Since, A and A' are two monotone ffa).
Thus, If A and A' are two monotone ffa. Then their cartesian product $\mu \bullet \mu'$ is monotone if and only if $(p, p') = (q, q')$.
Theorem: If cartesian product of A and A' is monotone ffa.Then both A and A' are monotone ffa.
Proof: **Claim**: A and A' are monotone ffa.
without loss of generality, assume that A is monotone ffa and A' is not monotone ffa.
subcase(i): Suppose $\exists\, p', q' \in Q', \sigma' \in \Sigma'$ such that, $p' \leq q', \mu'(p', \sigma', q') = 0$ take for all $p, q \in Q, \sigma \in \Sigma$ such that, $p = q$.Then
$\mu \bullet \mu'((p, p'), \sigma', (q, q')) = \mu'(p', \sigma', q') = 0$, if $\sigma' \in \Sigma', p = q$
Thus, $\mu \bullet \mu'((p, p'), \sigma', (q, q')) = 0$, if $(p, p') \leq (q, q')$ This is contradiction to $A \bullet A'$ is monotone ffa.
subcase(ii): Suppose $\exists\, p', q' \in Q', \sigma' \in \Sigma'$ such that, $p' \nleq q', \mu'(p', \sigma', q') > 0$ take for all $p, q \in Q, \sigma \in \Sigma$ such that, $p = q$.Then
$\mu \bullet \mu'((p, p'), \sigma', (q, q')) = \mu'(p', \sigma', q') > 0$, if $\sigma \in \Sigma'$ $p = q$.

Thus,$\mu \bullet \mu'((p, p'), \sigma', (q, q')) > 0$,if $(p, p') \nleq (q, q')$ This is contradiction to $A\omega A'$ is monotone ffa.
Thus, both A and A' are monotone ffa.

Conclusion: This paper examined four important classes of fuzzy finite automata discrete, monotone, autonomous, and reset through examples and non-examples. Different product constructions, such as the direct, cascade, wreath, and Cartesian products defined in [5], were found to be valid for the newly coined fuzzy finite automata.

Future Work: Future research may focus on coining new classes of fuzzy automata, extending the present results to other classes of fuzzy automata, and exploring additional product constructions such as the sum and direct sum of fuzzy finite automata, as studied in [6,7]. Further investigation of applications of these models in fuzzy control systems, decision-making, and computational intelligence also remains an important direction for future work.

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