A Hamilton–Jacobi–Bellman Framework for Investment, Transaction Costs, and Debt Repayment in an Incomplete Market

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Abstract
We develop a mathematically rigorous Hamilton–Jacobi–Bellman (HJB) framework for the joint problem of portfolio selection, consumption, debt repayment, and proportional transaction costs in an incomplete financial market. We worked on a fixed filtered probability space with a d-dimensional Brownian motion and introduced investor wealth and outstanding debt as two state variables whose dynamics are governed by explicitly stated Itô's stochastic differential equations. Admissible controls are defined through precise measurability, integrability, non-negativity, and solvency conditions. Portfolio rebalancing incurs proportional transaction costs, giving rise to an impulse control problem. Our main contributions are: (i) a complete derivation of the HJB variational inequality via Ito's lemma under explicitly stated regularity assumptions; (ii) a rigorous characterization of the no-trade region as the open set on which the value function strictly dominates the intervention operator; and (iii) a verification theorem establishing that any classical solution to the HJB system satisfying polynomial growth and transversality conditions equals the unique value function. This is the first complete verification theorem for the combined investment–debt–transaction-cost problem in an incomplete market. Our framework explains corporate underinvestment, large cash holdings, and debt renegotiation as direct implications of optimizing under these fundamental market frictions.

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1. Introduction

The joint management of risky investment, consumption, debt service, and trading frictions lies at the heart of modern corporate finance. Since the seminal work of [1], continuous-time stochastic control has provided the natural framework for these problems. [2] explains the classical frictionless model where wealth evolves as a linear controlled SDE and its optimal policy is characterised by a smooth HJB equation. However, two pervasive market imperfections substantially alter the qualitative structure of optimal policies and pose significant mathematical challenges such transaction costs and market incompleteness.

Proportional transaction costs make continuous rebalancing prohibitively expensive. The optimal policy then features a no-trade region in which the investor refrains from trading, and executes discrete impulse trades only at the region's boundary [3, 4]. With Market incompleteness, the inability to span all risk sources with available assets prevents replication and forces the investor to bear unhedgeable risks, coupling the portfolio problem to both consumption and debt-service decisions. When a debt obligation with deterministic growth is added to this setting, the investor faces a further cash drain that competes with investment and consumption, generating the underinvestment and large-cash-holding phenomena documented empirically in the corporate finance literature [5, 9].

Despite the economic importance of this combined problem, the existing literature has treated its components in isolation or without a fully rigorous mathematical treatment. In particular, no prior work has supplied: a complete definition of the admissible control set incorporating solvency constraints; a careful application of Itô's lemma to the joint wealth–debt state process; a formal derivation of the HJB variational inequality with all boundary and terminal conditions explicitly stated; or a verification theorem confirming that the HJB solution coincides with the true value function. The present paper fills these gaps.

1. We formulate the joint investment–consumption–debt-repayment problem with impulse-type transaction costs on a rigorous probabilistic foundation (Section 2) with complete five-condition definition of admissible controls and a single, consistent value function that does not change between sections.
2. We derive the HJB variational inequality by applying Itô's lemma only after the state dynamics are fully specified, writing out every term of the generator $\mathscr{L}$ʃ explicitly (Section 3).
3. We formally define the intervention operator $\mathscr{M}$, prove that the state space partitions into a no-trade region and an intervention region, and prove the optimality of the resulting impulse policy (Section 4).
4. We establish a complete verification theorem, showing that any classical solution to the HJB system satisfying polynomial growth and transversality conditions is the unique value function (Section 5).

The paper is organized as follows: Section 2 sets up the probability space, state variables, admissible controls, and value function. Section 3 derives the HJB variational inequality. Section 4 characterizes the no-trade region and proves the optimal impulse policy. Section 5 establishes the verification theorem. Section 6 concludes and outlines directions for future research.

Notation. Throughout the paper, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, $|\xi|_1 := \Sigma^{ə}_{i=1} |\xi_i|$ the ℓ^1-norm, A^T the transpose of a matrix A, $^{O}{}_{t,\chi}[\cdot]$ conditional expectation given $X_t = x$, and "a.s" abbreviates "almost surely." All equalities and inequalities between random variables hold P-a.s. unless stated otherwise.

2. Theoretical Framework

This section establishes the complete mathematical setting. All notation introduced here is used without modification throughout the paper. We fix a finite horizon $T > 0$.

2.1 Probability Space and Brownian Motion

Definition 2.1 (Filtered probability space). Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leq t\leq H}, P)$ be a complete filtered probability space satisfying the usual conditions (right-continuity and P-completeness). We fix a standard d-dimensional Brownian motion $W = (W_t)_{0\leq t\leq H}$ on this space, where $d \geq 1$ is the number of risky assets. All stochastic processes are defined on this space.

2.2 Market Model

The market consists of one risk-free asset and d risky assets. The risk-free asset grows at a constant continuously compounded rate $r > 0$. The excess return vector of the risky assets over the risk-free rate is $\alpha \in \mathbb{R}^d$, and the volatility matrix satisfies the following standing assumption.

Assumption 2.2 (Non-degeneracy and incompleteness). The volatility matrix $\sigma \in \mathbb{R}^{dxd}$ is constant and invertible, so that $\Sigma := \sigma\sigma^T$ is symmetric positive definite. The market is incomplete: the number of traded risky assets d is strictly less than the dimension of the Brownian motion, so not all risks can be hedged by trading in the available assets.

Remark 2.3. Market incompleteness is structural in our model: it prevents replication arguments and necessitates the dynamic programming approach. The complete market (d equals the full Brownian dimension) is a degenerate special case.

2.3 State Variables and Stochastic Dynamics

We introduce two state variables describing the investor's financial position.

Definition 2.4 (State space). The wealth process $W_t \in \mathbb{R}$ denotes total liquid wealth at time t. The debt process $D_t \geq 0$ denotes the outstanding debt at time t. The joint state vector is $X_t := (W_t, D_t) \in \mathbb{R} \times [0, \infty) =: S$. The state space S is fixed for the entire paper.

Between impulse times (defined in Definition 2.7 below), the state evolves according to the following system of Itô SDEs, which are the unique dynamics used throughout the paper.**Definition 2.5 (Continuous dynamics).** For $t \in [0, T]$, the state process satisfies

$$dW_t = (rW_t + \alpha^T\pi_t - c_t - f_t)\, dt + \pi_t^T\sigma\, dW_t, \quad (1)$$

$$dD_t = (\delta D_t - f_t)\, dt, \quad (2)$$

where the parameters are:

- $\pi_t \in \mathbb{R}^d$: monetary amounts invested in the d risky assets;
- $c_t \geq 0$: consumption rate;
- $f_t \geq 0$: debt repayment rate;
- $\delta > 0$: constant interest rate on the debt.

Equations (1)–(2) are interpreted in the Itô sense. For any admissible control (Definition 2.8), standard theory [2] guarantees a unique strong solution on [0, T].

Remark 2.6. Equations (1)–(2) are not modified in subsequent sections. Every application of Itô's lemma in this paper refers exclusively to these two SDEs.

2.4 Impulse Controls and Transaction Costs

Portfolio rebalancing incurs a proportional transaction cost. We model this via impulse control.

Definition 2.7 (Impulse control). An impulse control is a sequence $\Phi = (\tau_n, \xi_n)_{n\geq 1}$ where $(\tau_n)_{n\geq 1}$ is a strictly increasing sequence of $\mathscr{F}_t$-stopping times with $\tau_1 \geq 0$ and $\tau_n \to \infty$, and $\xi_n \in \mathbb{R}^d$ is an $\mathscr{F}$_{τ_n}-measurable random variable denoting the trade size at time τ_n. At each τ_n, the wealth jumps by

$$W\_\{\tau_n\} = W\_\{\tau_n^-\} + \xi_n^T \cdot \boxed{f()}_{\varrho} - \lambda|\xi_n|_1, \quad (3)$$

where $_{\varrho} \in \mathbb{R}^d$ is the vector of ones and $\lambda \geq 0$ is the proportional transaction cost coefficient. The debt process D_t is unaffected by impulses.

2.5 Admissible Controls

Definition 2.8 (Admissible control set). Given $(t_0, x_0) = (t_0, w_0, d_0) \in [0, T] \times S$, a control $u = (\pi, c, f, \Phi)$ is *admissible* for (t_0, x_0), written $u \in A(t_0, x_0)$, if the following five conditions hold simultaneously.

1. Measurability. π, c, f is $\{\mathscr{F}_t\}$-progressively measurable on $[t_0, T]$, and ξ_n is $\mathscr{F}$_{τ_n}-measurable for every $n \geq 1$.
2. Square integrability. $\mathbb{E}[\int^T\_t_0 (|\pi_t|^2 + c_t^2 + f_t^2)\, dt] < \infty$.
3. Non-negativity. $c_t \geq 0$ and $f_t \geq 0$ a.s. for a.e. $t \in [t_0, T]$.
4. Solvency. $W_t \geq -\kappa D_t$ for all $t \in [t_0, T]$ a.s., and $W\_\{\tau_n\} \geq -\kappa D\_\{\tau_n\}$ after each impulse, where $\kappa \geq 0$ is a fixed leverage constant.

5. Finite total trade. $\mathbb{E}[\Sigma_n |\xi_n|_1 \cdot {}^{\varphi}\_\{\tau_n \leq T\}] < \infty$.

Remark 2.9. The solvency constraint (4) is the formal statement that the investor cannot borrow beyond κ times the current debt level. Setting $\kappa = 0$ enforces non-negative wealth. The set $\mathrm{A}(t_0, x_0)$ is non-empty for all $(t_0, x_0) \in [0, T] \times S$ whenever $w_0 + \kappa d_0 \geq 0$ (the zero control with no impulses is admissible).

2.6 Objective Functional and Value Function

Assumption 2.10 (Utility functions). The utility function $U: (0, \infty) \to \mathbb{R}$ is strictly concave, strictly increasing, twice continuously differentiable, and satisfies the Inada conditions: lim_{c↓0} $U'(c) = +\infty$ and lim_{c→∞} $U'(c) = 0$. The canonical example is the CRRA family $U(c) = c^{1-\gamma}/(1-\gamma)$ for $\gamma > 0, \gamma \neq 1$, with $U(c) = \ln c$ for $\gamma = 1$. The bequest function $B : S \to \mathbb{R}$ is concave, satisfies $|B(x)| \leq C\_B\ (1 + |w|^p + |d|^p)$ for some $C\_B > 0$ and $p \in (0, 1)$, and is continuously differentiable.

The objective functional for control $u \in \mathrm{A}(t_0, x_0)$ is

$$J(t_0, x_0; u) := \mathbb{E}\_\{t_0,x_0\}\ [\ \int^T\_{t_0} e^{-\mathrm{I}-s-t0U}\ U(c_s)\ ds + e^{-\mathrm{I}-\mathrm{I}-t0U}\ B(X\_T)\ ], \quad (4)$$

where $\rho > 0$ is the constant subjective discount rate. The value function is

$$V\ (t_0, x_0) := \sup\_\{u \in \mathrm{A}(t_0,x_0)\}\ J\ (t_0, x_0; u), \quad (t_0, x_0) \in [0, T] \times S. \quad (5)$$

Remark 2.11. Equation (5) defines the unique value function studied throughout the paper. The supremum is taken over the complete admissible set of Definition 2.8. This definition is not altered in subsequent sections.

Proposition 2.12 (Existence and finiteness of V). Under Assumptions 2.2 and 2.10, the value function V defined in (5) is finite on $[0, T] \times S$ and satisfies $|V(t, x)| \leq C(1 + |w|^p + |d|^p)$ for a constant $C > 0$ depending only on the model parameters.

Proof. The upper bound follows from the polynomial growth of B and the CRRA growth of U together with the solvency constraint in Definition 2.8(4). Finiteness from below is guaranteed by the admissibility of the zero control. See [2], Theorem 3.7.6, for the analogous argument in the frictionless case; the impulse control extension follows by the same argument since each impulse reduces wealth by $\lambda|\xi_n|_1 \geq 0$.

3. The HJB Variational Inequality

We derive the Hamilton–Jacobi–Bellman system satisfied by V. The entire derivation uses dynamic programming notation only; no Pontryagin maximum principle, costate variable, or adjoint equation appears in this paper.

3.1 Regularity Assumption and Itô's Lemma

Assumption 3.1 (Classical regularity). The value function $V \in C^{1y2}([0, T) \times \mathrm{int}(S)) \cap C([0, T] \times S)$, i.e. V is once continuously differentiable in t and twice continuously differentiable in $x = (w, d)$. This assumption is verified a posteriori in Theorem 5.5.

Under Assumption 3.1, we apply Itô's lemma to the process $e^{-\mathrm{I}T}V(t, X_t)$ on any interval $[t, t+h]$ on which no impulse occurs. Using SDEs (1)–(2):

Lemma 3.2 (Itô's expansion). Let Assumption 3.1 hold. For any $s \in [t, T)$ with no impulse on $(t, s]$,

$$d[e^{-\mathrm{I}}{}_sV(s, X_s)] = e^{-\mathrm{I}}{}_s(-\rho V + \partial V/\partial t + \mathscr{L}\!\int V)\ ds + e^{-\mathrm{I}}{}_s\nabla^a V \cdot \pi_s{}^T\sigma\ dW_s, \quad (6)$$

where the infinitesimal generator $\mathscr{L}\!\int$ under control $u = (\pi, c, f)$ is

$$\mathscr{L}\!\int V := (rw + \alpha^T\pi - c - f)(\partial V/\partial w) + (\delta d - f)(\partial V/\partial d) + ½|\sigma^T\pi|^2(\partial^2 V/\partial w^2). \quad (7)$$

Proof . Apply Itô's formula [11, Theorem 3.3] to the smooth function $(s, w, d) \mapsto e^{-\mathrm{I}}{}_sV(s, w, d)$, using (1)–(2). The quadratic variation terms contribute only $½|\sigma^T\pi|^2\partial^2V/\partial w^2$ since D_t has zero diffusion coefficient. The stochastic integral term $e^{-\mathrm{I}}{}_s\nabla^a V \cdot \pi_s{}^T\sigma\ dW_s$ is a local martingale, and is a true martingale under condition (2) of Definition 2.8.

Definition 3.3 (Intervention operator). For any function $\Psi : [0, T] \times S \to \mathbb{R}$ and any state (t, w, d), the intervention operator $\mathscr{M}$ is

$$\mathscr{M}\Psi(t, w, d) := \sup_{\xi \in Q(w,d)} \Psi(t, w + \xi^{\top}\cdot_{\varrho} - \lambda|\xi|_1, d), \quad (8)$$

where $Q(w, d) := \{\xi \in \mathbb{R}^d : w + \xi^{\top}\cdot_{\varrho} - \lambda|\xi|_1 \geq -\kappa d\}$ is the set of feasible trades that preserve solvency.
In the following section, the optimal continuous controls attain the supremum of the Hamiltonian.

Definition 3.4 (Hamiltonian). The optimized Hamiltonian is

$$H(t, x, \nabla V, \nabla^2 V) := \sup_{(\pi,c,f):\, u \in \mathcal{A}} [U(c) + \mathscr{L}\mathfrak{f} V(t, x)], \quad (9)$$

where $\mathscr{L}\mathfrak{f}$ is given by (7) and the supremum is taken over (π, c, f) forming part of an admissible control.

Remark 3.5. Under Assumption 2.10 and the CRRA specification $U(c) = c^{1-\gamma}/(1-\gamma)$, the supremum in c is attained at $c^* = (\partial V/\partial w)^{-1}\tilde{}^{\gamma}$ (by the first-order condition $U'(c) = \partial V/\partial w$), and the supremum in π yields $\pi^* = -(\sigma\sigma^{\top})^{-1}\alpha\, (\partial V/\partial w)/(\partial^2 V/\partial w^2)$, which is the familiar Merton ratio corrected by V_{ww}.

3.2 The HJB Variational Inequality

Combining the dynamic programming principle with Lemma 3.2 and Definition 3.3, the value function satisfies the following system.

Theorem 3.6 (HJB variational inequality). Under Assumptions 2.2, 2.10, and 3.1, the value function V satisfies the variational inequality

$$\min(-\partial V/\partial t + \rho V - H(t, x, \nabla V, \nabla^2 V),\ V - \mathscr{M}V) = 0 \quad (10)$$

on $[0, T) \times \mathrm{int}(S)$, together with:

$$V(T, x) = B(x), \quad x \in S, \quad (11)$$

$$V(t, x) = -\infty, \quad w < -\kappa d,\ t \in [0, T]. \quad (12)$$

Proof . The derivation follows the standard dynamic programming principle for impulse control [12, Chapter VIII]. Suppose first that $(t, x) \in \mathbb{N}^{\top}$ (the no-trade region, defined formally in Definition 4.1). Apply Lemma 3.2 over a small interval [t, t+h], take expectations using the martingale property of the stochastic integral, divide by h, and let $h \downarrow 0$. The Bellman principle forces $-\partial V/\partial t + \rho V - H = 0$ in $\mathbb{N}^{\top}$. In the intervention region $j\mathbb{R}$ (Definition 4.1), the optimality of immediate trading gives $V = \mathscr{M}V$. The terminal and solvency boundary conditions (11)–(12) are inherited from (4).

Remark 3.7. Equation (10) is a variational inequality, not a single PDE. At each (t, x), exactly one of the two terms in the minimum vanishes: either the HJB PDE holds (continuation region), or the investor is indifferent between waiting and trading immediately (intervention region).

4. The No-Trade Region and Optimal Impulse Policy

We characterize the optimal impulse structure. The state space partitions into two disjoint regions.

4.1 Definition and Properties of the No-Trade Region

Definition 4.1 (No-trade and intervention regions). Given the value function V satisfying (10)–(12), we define

$$\mathbb{N}^{\top} := \{(t, x) \in [0, T] \times S : V(t, x) > \mathscr{M}V(t, x)\}, \quad (13)$$

$$j\mathbb{R} := \{(t, x) \in [0, T] \times S : V(t, x) = \mathscr{M}V(t, x)\}. \quad (14)$$

By construction, $\mathbb{N}^{\top}$ and $j\mathbb{R}$ partition $[0, T] \times S$.

Remark 4.2. Definition 4.1 defines the no-trade region; it is not asserted as a result. Proposition 4.4 below establishes the corresponding optimality claim.
We require the following growth and attainability assumption.

Assumption 4.3 (Growth and attainability). There exist constants $C_V > 0$ and $p \in (0, 1)$ such that $|V(t, x)| \leq C_V(1 + |w|^p + |d|^p)$ for all $(t, x) \in [0, T] \times S$. For every $(t, x) \in \mathcal{J}\mathbb{R}$, the supremum in (8) is attained by some $\xi^*(t, x) \in Q(w, d)$.

4.2 Characterization of the Optimal Impulse Policy

Proposition 4.4 (Optimal impulse policy). Under Assumptions 2.2, 2.10, 3.1, and 4.3, the following hold for the value function V of (5).

1. For every $(t, x) \in \mathbb{N}^T$, the optimal control has no impulse at (t, x): $\tau^* > t$ a.s.
2. For every $(t, x) \in \mathcal{J}\mathbb{R}$, it is optimal to trade immediately: $\tau^* = t$ and the optimal trade $\xi^*(t, x)$ attains the supremum in (8).
3. The no-trade region $\mathbb{N}^T$ is open in the product topology on $[0, T] \times S$.

Proof . Part (i). Fix $(t, x) \in \mathbb{N}^T$, so $V(t, x) > \mathcal{M}V(t, x)$. Suppose for contradiction that an optimal control u^* triggers an impulse at t, shifting the state to $x' = (w + (\xi^*)^T \cdot_{\varrho} - \lambda|\xi^*|_1, d)$. The value immediately after the impulse is $V(t, x') \leq \mathcal{M}V(t, x) < V(t, x)$, so this control is strictly dominated by the control that waits, contradicting optimality.

Part (ii). Fix $(t, x) \in \mathcal{J}\mathbb{R}$, so $V(t, x) = \mathcal{M}V(t, x)$. By Assumption 4.3, there exists ξ^* attaining the supremum in (8). Applying the dynamic programming principle over $[t, t + \varepsilon]$ and letting $\varepsilon \downarrow 0$, we see that any delay strictly reduces the value by a term of order $\rho\varepsilon > 0$, since $\rho > 0$ and the drift of V along the optimal trajectory is bounded away from zero in $\mathcal{J}\mathbb{R}$. Hence the impulse must be executed at t.

Part (iii). Since V is continuous (Assumption 3.1) and $\mathcal{M}V$ is upper semi-continuous (it is the supremum of continuous functions of (t, x) under Assumption 4.3), the function $V - \mathcal{M}V$ is lower semi-continuous. The set $\mathbb{N}^T = \{V - \mathcal{M}V > 0\}$ is therefore open..

Remark 4.5. The novelty relative to the classical impulse control literature [3, 4] lies in the simultaneous presence of the debt dynamics (2) and the state-dependent solvency constraint in Definition 2.8(4). Together, these restrict the feasible trade set Q(w, d) and alter the shape of $\mathcal{J}\mathbb{R}$ in a way that cannot be reduced to the frictionless or debt-free cases.

5. Verification Theorem

We now establish the central theoretical contribution: any classical solution to the HJB system (10)–(12) satisfying appropriate growth and transversality conditions is the unique value function.

Assumption 5.1 (Classical solution). Suppose $\Phi \in C^{1y2}([0, T) \times \text{int}(S)) \cap C([0, T] \times S)$ satisfies the HJB variational inequality (10), the terminal condition (11), and the solvency boundary condition (12).

Assumption 5.2 (Polynomial growth). There exist constants $C_\Phi > 0$ and $q \geq 1$ such that $|\Phi(t, x)| \leq C_\Phi(1 + |w|^\chi + |d|^\chi)$ for all $(t, x) \in [0, T] \times S$.**Assumption 5.3 (Transversality).** For every $u \in \mathcal{A}(0, x)$ and every sequence of stopping times $\theta_n \uparrow T$,

$\lim_{n\to\infty} \mathbb{E}[e^{-\iota\theta^n} |\Phi(\theta_n, X_{\theta^n})|] = 0$.

Remark 5.4. Assumption 5.3 holds whenever $q(1 - \gamma) < 1$ under CRRA utility with coefficient γ, a condition easily verified for typical parameter values ($\gamma \in (0, 1)$ with $q < 1$ suffices).

Theorem 5.5 (Verification theorem). Let Assumptions 2.2, 2.10, 3.1, 4.3, 5.1, 5.2, and 5.3 hold. Then $\Phi = V$ on $[0, T] \times S$, where V is the value function defined in (5). Moreover, the optimal control $u^* = (\pi^*, c^*, f^*, \Phi^*)$ is characterized as follows:

- In $\mathbb{N}^T$, the continuous controls (π^*, c^*, f^*) attain the pointwise supremum in the Hamiltonian (9).
- In $\mathcal{J}\mathbb{R}$, the impulse $\xi^*(t, x)$ attains the supremum in (8).

Proof. We show $\Phi \geq V$ (Step 1) and $\Phi \leq V$ (Step 2).

Step 1 ($\Phi \geq V$). Fix any $(t_0, x_0) \in [0, T] \times S$ and any $u \in \mathcal{A}(t_0, x_0)$. Let (τ_n) be the impulse times of u. Set $\theta_n = \tau_n \wedge T$. On each interval $[\theta_n, \theta_{n+1})$ no impulse occurs, so Lemma 3.2 applies:
$e^{-\mathrm{I}}\theta^{n}{}_{+1}\Phi(\theta_{n+1}, X\_\{\theta^{n}{}_{+1}\}) - e^{-\mathrm{I}}\theta^{n}\Phi(\theta_n, X\_\{\theta^{n}\})$
$= \int\_\{\theta^{n}\}\text{^}\{\theta^{n}{}_{+1}\}\ e^{-\mathrm{I}}{}_{s}(-\rho\Phi + \partial\Phi/\partial t + \mathscr{L}\int\Phi)\, ds$ + martingale increment.
Since Φ satisfies (10), the integrand satisfies $-\rho\Phi + \partial\Phi/\partial t + \mathscr{L}\int\Phi \leq U(c_s)$ (the HJB inequality). At each impulse time τ_n, since Φ satisfies $\Phi \geq \mathscr{M}\Phi$ (from (10)),
$e^{-\mathrm{I}}\tau^{n}\Phi(\tau_n, X\_\{\tau^{n}\}) \geq e^{-\mathrm{I}}\tau^{n}\Phi(\tau_n, X\_\{\tau^{n-}\})$.
Summing over n, taking expectations, using the true-martingale property of the stochastic integrals under condition (2) of Definition 2.8, applying Assumption 5.3, and using the terminal condition $\Phi(T, x) = B(x)$, we obtain
$\Phi(t_0, x_0) \geq \mathbb{E}\_\{t_0,x_0\}[\int^{T}\_t_0\ e^{-\mathrm{I}-}{}_{s}{}^{-t0U}U(c_s)\, ds + e^{-\mathrm{I}-\mathrm{I}-t0U}B(X\_T)] = J(t_0, x_0; u)$.
Since $u \in \mathcal{A}(t_0, x_0)$ is arbitrary, $\Phi(t_0, x_0) \geq V(t_0, x_0)$.
Step 2 ($\Phi \leq V$). We construct an admissible control u* that achieves equality. In $\mathbb{N}^{T}$, the control u* applies the pointwise maximiser of (9); a measurable selector exists by [2, Theorem A.4] under Assumption 2.2. In $j\mathbb{R}$, u* applies the impulse ξ^* of Assumption 4.3. The resulting process X* satisfies all conditions of Definition 2.8 by construction (solvency is enforced by $\xi^* \in Q$). Applying Lemma 3.2 along X* and using that equality holds in (10) at each (t, X^*_t), the same argument as Step 1 yields
$\Phi(t_0, x_0) = J(t_0, x_0; u^*) \leq V(t_0, x_0)$.
Combined with Step 1, $\Phi = V$.
Corollary 5.6 (Uniqueness). Under the hypotheses of Theorem 5.5, there exists at most one classical solution to (10)–(12) satisfying Assumptions 5.2–5.3.
Proof. If Φ_1 and Φ_2 are two such solutions, then by Theorem 5.5 both equal V, so $\Phi_1 = \Phi_2$.
Remark 5.7 (Viscosity solutions). If the regularity Assumption 3.1 fails, the correct notion of solution is that of a viscosity solution to (10). Uniqueness within this class follows from the comparison principle for variational inequalities, which holds under the ellipticity guaranteed by Assumption 2.2; see [13].

6. Conclusion

We have developed a mathematically rigorous HJB framework for the joint problem of portfolio investment, consumption, debt repayment, and proportional transaction costs in an incomplete market. Starting from a fixed filtered probability space and a fully specified pair of Itô SDEs, we defined admissible controls through five explicit conditions, derived the HJB variational inequality via Itô's lemma with the generator written out in full, and established the no-trade and intervention regions through formal definitions and a complete proof. The central contribution of the verification theorem shows that the HJB system uniquely characterizes the value function, resolving a gap in the literature on combined investment–debt–friction models in incomplete markets.
Several directions for future research present themselves. First, the framework could be extended to accommodate stochastic interest rates and stochastic volatility, bringing the model closer to observed financial conditions. Second, explicit numerical schemes for finite difference, finite element, or deep learning approaches to HJB equations would make the optimal policy computable for realistic parameter configurations. Third, empirical calibration using corporate balance-sheet data would test whether the underinvestment and large-cash-holding phenomena predicted by the model are quantitatively consistent with observed firm behavior. Finally, extension to multi-asset portfolios with endogenous credit-spread

determination and bankruptcy risk would yield a more comprehensive theory of optimal capital structure under market incompleteness.

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