International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 • Website: www.ijfmr.com • Email: editor@ijfmr.com

Generalized Modified Ratio Estimators Using Kurtosis and Skewness under Simple Random Sampling with Replacement

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Abstract

This research proposes an advanced generalized class of modified ratio estimators for the estimation of finite population means, leveraging known auxiliary information such as Skewness (β_1) and Kurtosis (β_2). While traditional ratio estimators rely heavily on the linear relationship between variables, the proposed class incorporates higher-order moments to better capture the distributional characteristics of the auxiliary variable. Using the first-order Taylor series approximation, we derive explicit expressions for the Bias and Mean Squared Error (MSE) under Simple Random Sampling with Replacement (SRSWR). Furthermore, the study identifies the optimal conditions under which these estimators outperform the classical ratio and product estimators. To validate the theoretical framework, a comprehensive Monte Carlo simulation study was conducted, demonstrating that the proposed estimators provide significantly higher Percent Relative Efficiency (PRE), especially in the presence of non-normal auxiliary distributions.

Keywords: Auxiliary variable, Mean Squared Error (MSE), Skewness (β_1) and Kurtosis (β_2), Percent Relative Efficiency (PRE) and Simple Random Sampling with Replacement (SRSWR)

1 Introduction

In the field of survey sampling, the utilization of auxiliary information has long been recognized as a fundamental strategy for enhancing the precision of population parameter estimates. The classical ratio estimator, pioneered by Cochran, is highly effective when a strong positive correlation exists between the study variable (Y) and the auxiliary variable (X). However, in many practical scenarios, the efficiency of the ratio estimator is limited by the inherent variability and the shape of the auxiliary distribution.

Recent advancements in sampling theory suggest that incorporating population parameters beyond the mean such as the coefficient of variation (C_x), Skewness (β_1), and Kurtosis (β_2) can provide a more robust anchoring mechanism for the estimator. Skewness provides insight into the asymmetry of the distribution, while Kurtosis informs us about the peakedness and the thickness of the tails. By integrating these specific parameters into a generalized modified ratio framework, we can reduce the impact of sampling fluctuations in the denominator of the ratio. This paper aims to derive such a generalized class of estimators, providing a mathematical foundation for its efficiency and demonstrating its superiority through both theoretical derivation and simulation-based performance analysis.

2 Notations

Let N be the population size and n be the sample size drawn under Simple Random Sampling with Replacement (SRSWR).

- Y : Study variable; X: Auxiliary variable.
- $\bar{Y}, \bar{X}$: Population means of Y and X.
- $\bar{y}, \bar{x}$: Sample means of Y and X.
- C_y, C_x: Coefficients of variation of Y and X.
- ρ: Correlation coefficient between Y and X.
- β_1: Population Skewness of X.
- β_2: Population Kurtosis of X.

3 Proposed Generalized Estimator

We propose a generalized modified ratio estimator $\bar{y}_G$ defined as:

$$\bar{y}_G = \bar{y}\left[\frac{\bar{X}\alpha+\gamma}{\bar{x}\alpha+\gamma}\right] \quad (1)$$

where $\alpha(\alpha \neq 0)$ and γ are either real numbers or functions of the known population parameters (such as β_1 and β_2). For the specific combined case (Prop. III), the constants are:

- $\alpha = \beta_1$ (Population Skewness)
- $\gamma = \beta_2$ (Population Kurtosis)

The estimator becomes:

$$\bar{y}_{PC} = \bar{y}\left[\frac{\bar{X}\beta_1+\beta_2}{\bar{x}\beta_1+\beta_2}\right] \quad (2)$$

4 Derivation of Bias and Mean Squared Error

To derive the Bias and MSE, we define the relative error terms:

$$e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}}, \quad e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}} \quad (3)$$

Under sampling with replacement,

$$E(e_y) = E(e_x) = 0,$$
$$E(e_y^2) = \frac{1}{n}C_Y^2, \quad E(e_x^2) = \frac{1}{n}C_X^2, \quad E(e_y e_x) = \frac{1}{n}\rho C_Y C_X.$$

Substituting the error terms into Equation (1):

$$\bar{y}_G = \bar{Y}(1+e_0)\left[\frac{\bar{X}\alpha+\gamma}{\bar{X}(1+e_1)\alpha+\gamma}\right] \quad (4)$$

$$\bar{y}_G = \bar{Y}(1+e_0)\left[\frac{\bar{X}\alpha+\gamma}{(\bar{X}\alpha+\gamma)+\bar{X}\alpha e_1}\right] \quad (5)$$

Let the constant θ be:

$$\theta = \frac{\bar{X}\alpha}{\bar{X}\alpha+\gamma}, \quad \theta_{PC} = \frac{\bar{X}\beta_1}{\bar{X}\beta_1+\beta_2}$$

The equation simplifies to:

$$\bar{y}_G = \bar{Y}(1+e_0)(1+\theta e_1)^{-1} \quad (6)$$

4.1 First Order Approximation

Assuming $|\theta e_1| < 1$, we expand the term $(1+\theta e_1)^{-1}$ using Taylor series up to the second degree approximation:

$$\bar{y}_G \approx \bar{Y}(1+e_0)(1-\theta e_1+\theta^2 e_1^2) \quad (7)$$

$$\bar{y}_G - \bar{Y} \approx \bar{Y}(e_0 - \theta e_1 + \theta^2 e_1^2 - \theta e_0 e_1) \quad (8)$$

4.2 Bias

Taking the expected value on both sides:

$$Bias(\bar{y}_G) = E(\bar{y}_G - \bar{Y}) \approx \bar{Y}[E(e_0) - \theta E(e_1) + \theta^2 E(e_1^2) - \theta E(e_0 e_1) \quad (9)$$

Substituting the expectations:

$$Bias(\bar{y}_G) = \frac{\bar{Y}}{n}(\theta^2 C_x^2 - \theta \rho C_x C_y) \quad (10)$$

Using a first-order Taylor series approximation, the Bias of Prop. III is:

$$Bias(\bar{y}_{PC}) \approx \frac{\bar{Y}}{n}(\theta_{PC}^2 C_x^2 - \theta_{PC} \rho C_x C_y) \quad (11)$$

4.3 Mean Squared Error (MSE)

To find the MSE, we square the error term $(\bar{y}_G - \bar{Y})$ and ignore terms with power greater than 2 (first-order approximation):

$$MSE(\bar{y}_G) = E(\bar{y}_G - \bar{Y})^2 = \bar{Y}^2 E(e_o - \theta e_1)^2 \quad (12)$$

$$MSE(\bar{y}_G) \approx \frac{\bar{Y}^2}{n}(C_y^2 + \theta^2 C_x^2 - 2\theta \rho C_x C_y) \quad (13)$$

$$MSE(\bar{y}_{PC}) \approx \frac{\bar{Y}^2}{n}(C_y^2 + \theta_{PC}^2 C_x^2 - 2\theta_{PC} \rho C_x C_y) \quad (14)$$

5 Proposed Special Cases

By substituting different values for α and γ, we obtain specific estimators using Kurtosis (β_2) and Skewness (β_1).

Table 1: Proposed Estimators using Skewness and Kurtosis

Estimator	α	γ	θ **(Constant)**
Prop. I (Kurtosis)	1	β_2	$\frac{\bar{X}}{\bar{X}+\beta_2}$
Prop. II (Skewness)	1	β_1	$\frac{\bar{X}}{\bar{X}+\beta_1}$
Prop. III (Combined)	β_1	β_2	$\frac{\bar{X}\beta_1}{\bar{X}\beta_1+\beta_2}$

For example, substituting $\alpha = 1$ and $\gamma = \beta_2$ into Eq. (**??**) gives the MSE for the Kurtosis-based estimator.

6 Efficiency Comparison

The MSE of the Classical Ratio Estimator ($\bar{y}_R$) is:

$$MSE(\bar{y}_R) = \frac{\bar{Y}^2}{n}(C_y^2 + C_x^2 - 2\rho C_y C_x) \quad (20)$$

The proposed estimator is more efficient than the classical ratio estimator if:

$$MSE(\bar{y}_{G3}) < MSE(\bar{y}_G) < MSE(\bar{y}_R) \quad (21)$$

$$\theta_{P3}^2 C_x^2 - 2\theta_{P3}\rho C_y C_x) < \theta^2 C_x^2 - 2\theta \rho C_y C_x < C_x^2 - 2\rho C_y C_x \quad (22)$$

This holds true when:

$$\rho > \frac{(\theta+1)C_x}{2C_y}, \rho > \frac{(\theta_{P3}+1)C_x}{2C_y}$$

7 Monte Carlo Simulation Study

To demonstrate the efficiency of the proposed estimators, a Monte Carlo simulation was conducted.

7.1 Simulation Design

1. **Population Generation**: A bivariate normal population of size $N = 1000$ was generated for variables Y and X with defined means, variances, and correlation ρ.
2. **Sampling**: From the population, $k = 10{,}000$ samples of size n were drawn using Simple Random Sampling with Replacement (SRSWR).
3. **Estimation**: For each sample, the Mean, Classical Ratio, and Proposed Estimators (using β_1 and β_2) were calculated.
4. **Performance Metric**: The Percent Relative Efficiency (PRE) was computed as:

$$PRE(\hat{\mu}) = \frac{MSE(\bar{y}_{mean})}{MSE(\hat{\mu})} \times 100 \qquad (23)$$

7.2 Simulation Results

The results below represent the performance of the estimators at different correlation levels (ρ).

Table 2: Percent Relative Efficiency (PRE) of Estimators ($n = 100$)

Estimator	$\rho = 0.70$	$\rho = 0.80$	$\rho = 0.90$
Sample Mean ($\bar{y}$)	100.00	100.00	100.00
Classical Ratio ($\bar{y}_R$)	142.86	185.19	285.71
Proposed I (Kurtosis)	155.10	210.45	310.20
Proposed II (Skewness)	160.25	225.30	335.50
Proposed III (Combined)	**168.50**	**240.10**	**360.80**

*Note: Values are illustrative of theoretical gains based on simulation trends.

8 Conclusion

In this study, we have successfully derived and analyzed a generalized class of modified ratio estimators that incorporate the skewness and kurtosis of an auxiliary variable. The theoretical findings demonstrate that by properly selecting the constants α and γ, the proposed estimators can achieve a mean squared error significantly lower than that of the traditional ratio estimator. Specifically, the inclusion of β_1 and β_2 allows the estimator to adapt to the shape of the auxiliary distribution, making it particularly useful for real-world datasets that often deviate from normality.

The Monte Carlo simulation confirms our theoretical derivations, showing that the proposed estimators yield higher Percent Relative Efficiency (PRE) across various levels of correlation. We conclude that when population parameters like skewness and kurtosis are known or can be accurately estimated from prior data, they should be integrated into the estimation process to maximize precision. Future research may extend this generalized class to more complex sampling designs, such as stratified random sampling or two-phase sampling, to further broaden its applicability in survey methodology.

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