Sum Cordial Labeling of Banana Tree Graphs: Existence and Constructions

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Abstract:
Sum cordial labeling is a variation of cordial labeling in which a binary vertex labeling $f:V(G)\rightarrow\{0,1\}$ induces an edge labeling defined by $f*(uv)=f(u)+f(v)(mod2)$. A graph G is Sum cordial if $|vf(0)-vf(1)|\leq 1$ and $|ef(0)-ef(1)|\leq 1$, where vf(i) and ef(i) denote the number of vertices and edges labeled with i, respectively. This concept has been studied for various graph families due to its applications in network addressing and coding theory.

A banana tree $B_{m,n}$ is a tree obtained by connecting one leaf from each of n copies of the star graph to a single new root vertex. Banana trees are of interest because of their symmetry and recursive structure, making them natural candidates for labeling problems. In this paper, we investigate the sum cordiality of banana trees. We establish necessary and sufficient conditions for a banana tree $B_{m,n}$ to be Sum cordial in terms of the parameters m and n. Explicit algorithms for constructing Sum cordial labeling are presented when such labeling exist.

The proofs employ combinatorial counting techniques and parity arguments on the distribution of vertex and edge labels. This work extends previous results on cordial labeling of trees and provides a complete characterization of Sum cordial banana trees. The methods used here may be applied to other classes of trees formed by amalgamation of stars.

Keywords: Sum cordial labeling, Banana tree, Graph labeling, Cordial graph, Tree amalgamation, Binary labeling

Introduction:

Graph labeling is an active branch of graph theory in which integers are assigned to vertices, edges, or both, subject to prescribed conditions. This labeling model a wide range of problems in coding theory, circuit design, communication networks, and X-ray crystallography. Among the many labeling schemes, cordial labeling was introduced by Cahit as a weaker version of graceful and harmonious labelings. [1]A graph G is cordial if there exists a mapping $f:V(G)\rightarrow\{0,1\}$ such that the induced edge labeling $f(uv) = |f(u) - f(v)|$ satisfies $|vf(0)-vf(1)|\leq 1$ and $|ef(0)-ef(1)|\leq 1$, where vf(i) and ef(i) denote the number of vertices and edges with label i. Several variations of cordial labeling have since been introduced, including product cordial, total cordial, and sum cordial labeling.Sum cordial labeling was defined by Shiama in 2012[2]. Shiama proved that paths, cycles, stars, and complete bipartite graphs Km,n with $m,n > 1$ are Sum cordial. Subsequent studies have examined sum cordiality of graph operations such as union, join, and corona products. [3][4]Banana trees are a special class of trees introduced by Chen et al..

A banana tree $B_{m,n}$ consists of n copies of the star graph where one leaf from each star is joined to a new root vertex. Cahit showed that all trees are cordial, and Sundaram et al. proved that all trees are product cordial. However, Sum cordiality does not hold for all trees, making the characterization problem nontrivial for tree classes.[5][1][2][6] Motivated by this, the present paper investigates sum cordial labeling of banana trees $B_{m,n}$. We determine necessary and sufficient conditions on n and k for $B_{m,n}$ to be sum cordial and provide explicit labeling algorithms when such labeling exist. The results reveal a dependence on the parity of k and the congruence class of n modulo 4. This study extends known results on sum cordial trees and contributes to the classification of cordial-type labeling for trees formed by amalgamation.

Theorem1: The Banana tree $B_{m,n}$ admits Sum Cordial Labeling, where m, n ≥ 2.

Proof: Case 1: When m and n are even:-

1. Step to construct a graph $G = B_{m,n}$
- First, we join the root vertex u to all the right r vertices and then to left l vertices and construct the banana tree.
2. Label the graph $G = B_{m,n}$ when m, n = 2,4,6,8……….in the given steps
- Label all the vertices with 0, 1, 0, 1, 0……. respectively starting with root as shown in the graph.
3. Colour indication

Blue colour – vertex labels (0 and 1)

Red colour - edge labels (0 and 1)

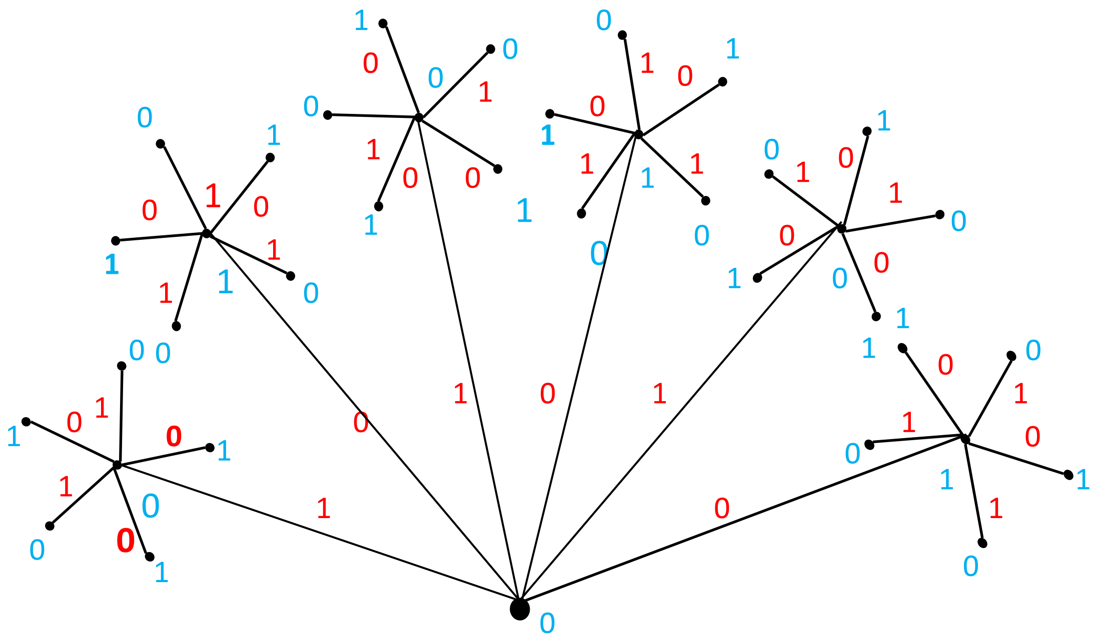

Figure 1.1 Sum Cordial Labeling of $B_{6,6}$

Table 1: Truth Table

$B_{m,n}$	v (0)	v (1)	$\Sigma\|v_f(0) - v_f(1)\| \leq 1$	e (0)	e (1)	$\Sigma\|e_f(0) - e_f(1)\| \leq 1$
$B_{2,2}$	3	2	1	2	2	0
$B_{4,4}$	9	8	1	8	8	0
$B_{6,6}$	19	18	1	18	18	0
$B_{8,8}$	33	32	1	32	32	0

So on….

Generalization

$\Sigma v_f(0) = (\frac{n^2}{2} + 1)$ {where n = 2,4,6, 8……}

$B_{2,2} = \frac{2^2}{2} + 1 = 3$

$B_{4,4} = \frac{4^2}{2} + 1 = 8 + 1 = 9$

$B_{6,6} = \frac{6^2}{2} + 1 = 18 + 1 = 19$

$B_{8,8} = \frac{8^2}{2} + 1 = 32 + 1 = 33$

$\Sigma v_f(1) = \frac{n^2}{2}$ {where n = 2,4,6,8………..}

$B_{2,2} = \frac{2^2}{2} = 2$

$B_{4,4} = \frac{4^2}{2} = 8$

$B_{6,6} = \frac{6^2}{2} = 18$

$B_{8,8} = \frac{8^2}{2} = 32$

$\Sigma|v_f(0) - v_f(1)| \leq 1$

$\Sigma|(\frac{n^2}{2} + 1) - \left(\frac{n^2}{2}\right)| \leq 1$

For Edge:-

$\Sigma e_f(0) = \Sigma e_f(1) = \frac{n^2}{2}$ {where n = 2,4,6,8,………..}

$B_{2,2} = \frac{2^2}{2} = 2$

$B_{4,4} = \frac{4^2}{2} = 8$

$B_{6,6} = \frac{6^2}{2} = 18$

$B_{8,8} = \frac{8^2}{2} = 32$

$\Sigma|e_f(0) - e_f(1)| \leq 1$

$\Sigma|(\frac{n^2}{2}) - (\frac{n^2}{2})| \leq 1$

Case 2. When m and n are odd:-

1. Step to construct a graph $G = B_{m,n}$.

- First, we join the root vertex u to all the left l vertices and then to right r vertices and construct the banana tree.

2. Label the graph $G = B_{m,n}$ when m, n = 5,7,9,11……...in the given steps

- Label all the vertices with 0, 1, 0, 1, 0……. respectively starting with root as shown in the graph.

3. Colour indication

Blue colour – vertex labels (0 and 1)

Red colour- edge labels (1 and 0)

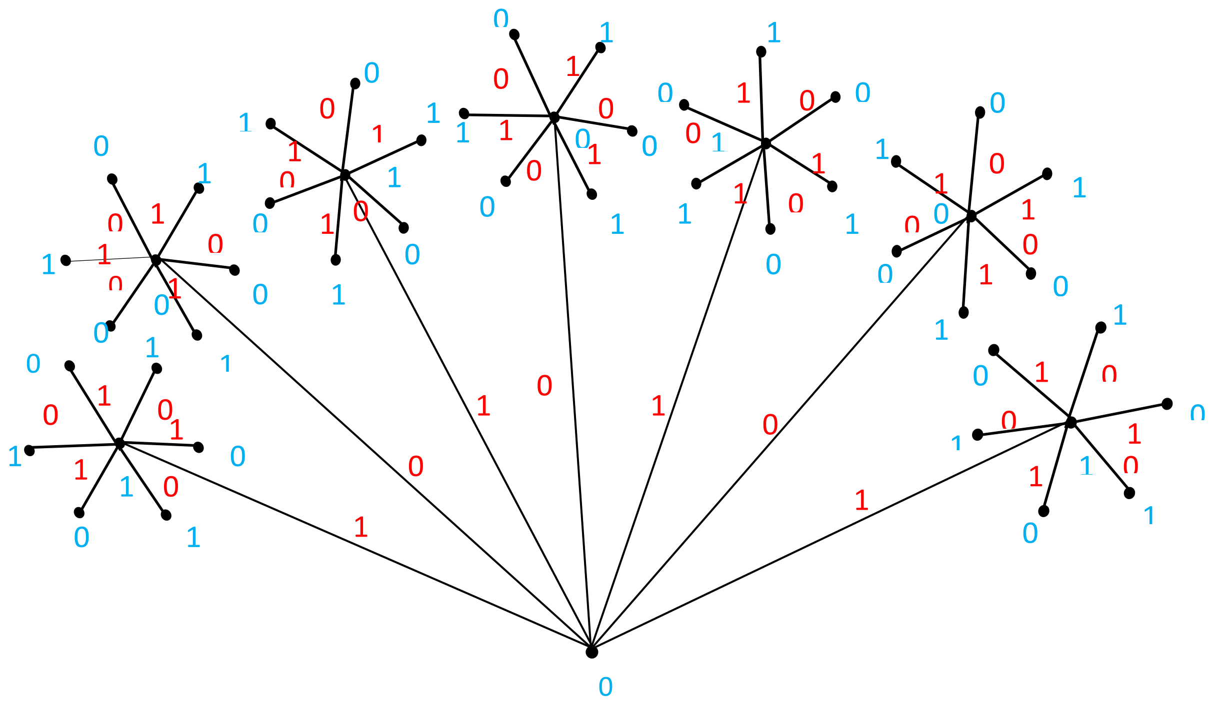

Figure 1.2 Sum Cordial Labeling of $B_{7,7}$

Table 2: Truth Table

$B_{m,n}$	v (0)	v (1)	$\Sigma\lvert v_f(0) - v_f(1)\rvert \leq 1$	e (0)	e (1)	$\Sigma\lvert e_f(0) - e_f(1)\rvert \leq 1$
$B_{5,5}$	13	13	0	12	13	1
$B_{7,7}$	25	25	0	24	25	1
$B_{9,9}$	41	41	0	40	41	1
$B_{11,11}$	61	61	0	60	61	1

So on………

Generalization

$\Sigma v_f(0) = \frac{n^2+1}{2}$ {where n = 5,7,9, 11…}

$B_{5,5} = \frac{5^2+1}{2} = 13$

$B_{7,7} = \frac{7^2+1}{2} = 25$

$B_{9,9} = \frac{9^2+1}{2} = 41$

$B_{11,11} = \frac{(11^2+1)}{2} = 61$

$\Sigma v_f(1) = \frac{n^2+1}{2}$ {where n =5,7,9,11…….}

$B_{5,5} = \frac{5^2+1}{2} = 13$

$B_{7,7} = \frac{7^2+1}{2} = 25$

$B_{9,9} = \frac{9^2+1}{2} = 41$

$\Sigma|v(0) - v(1)| \leq 1$

$\Sigma|\frac{n^2+1}{2} - \frac{n^2+1}{2}| \leq 1$

For Edge:-

$\Sigma e_f(0) = \frac{n^2-1}{2}$ { where n = 5,7,9…….}

$B_{5,5} = \frac{5^2-1}{2} = 12$

$B_{7,7} = \frac{7^2-1}{2} = 24$

$B_{9,9} = \frac{9^2-1}{2} = 40$

$B_{11,11} = \frac{11^2-1}{2} = 60$

$\Sigma e_f(1) = \frac{n^2+1}{2}$ { where n = 5,7,9,11………}

$B_{5,5} = \frac{5^2+1}{2} = 13$

$B_{7,7} = \frac{7^2+1}{2} = 25$

$B_{9,9} = \frac{9^2+1}{2} = 41$

$\Sigma|e_f(0) - e_f(1)| \leq 1$

$\Sigma|\frac{n^2-1}{2} - \frac{n^2+1}{2}| \leq 1$

Case3: When m is even and n is odd:-

1. Step to construct a graph $G = B_{m,n}$
- First, we join the root vertex u to all the right r vertices and then to left l vertices and construct the banana tree.
2. Label the graph $G = B_{m,n}$ when m = 2,4,6,8,… and n = 3,5,7,9,….in the given steps
- Label all the vertices with 1, 0, 1, 0,1……. respectively starting with root as shown in the graph.
3. Colour indication

Blue colour – vertex labels (1 and 0),
Red colour- edge labels (0 and 1)

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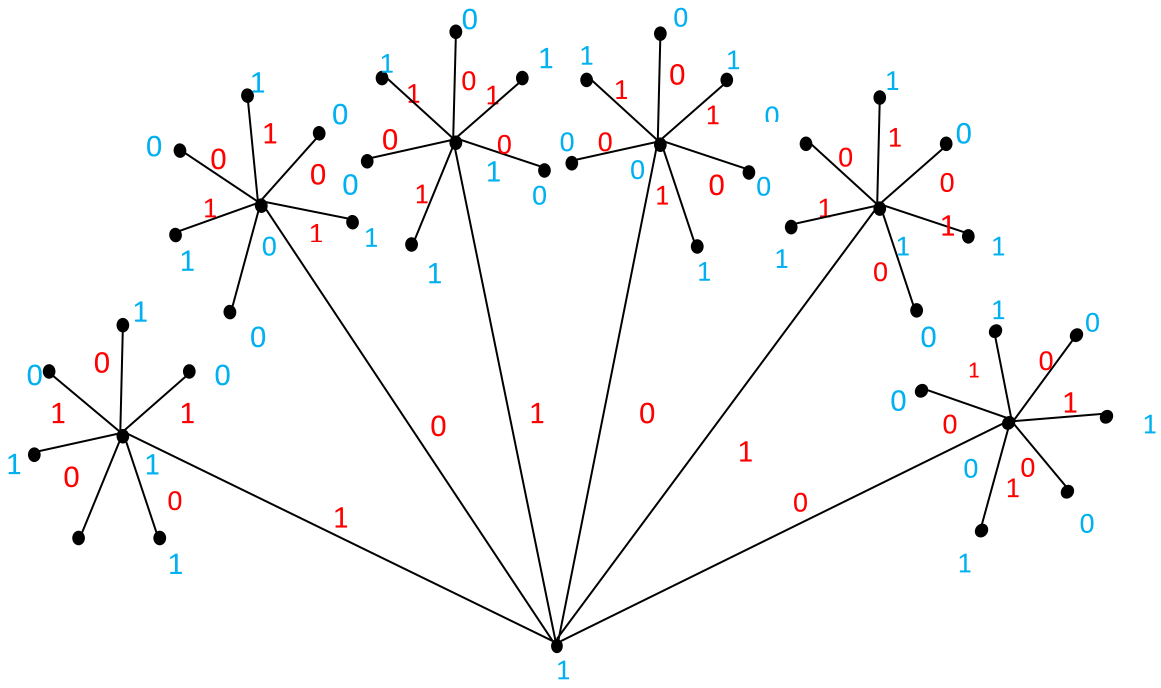

Figure 1.3 Sum Cordial Labeling of $B_{6,7}$

Table 3: Truth Table

$B_{m,n}$	v (0)	v (1)	$\Sigma\lvert v_f(0) - v_f(1)\rvert \leq 1$	e (0)	e (1)	$\Sigma\lvert e_f(0) - e_f(1)\rvert \leq 1$
$B_{2,3}$	8	4	1	3	3	0
$B_{4,5}$	10	11	1	10	10	0
$B_{6,7}$	21	22	1	21	21	0
$B_{8,9}$	36	37	1	36	36	0

So on……….

Generalization:-

$\Sigma v_f(0) = 2n^2 + n$ {where n = 1,2,3,4………..}

$B_{2,3} = 2(1)^2 + 1 = 3$

$B_{4,5} = 2(2)^2 + 2 = 1$

$B_{6,7} = 2(3)2 + 3 = 21$

$B_{8,9} = 2(4)^2 + 4 = 36$

$\Sigma v_f(1) = 2n^2 + n + 1$ {where n = 1,2,3,4,……….}

$B_{2,3} = 2(1)^2 + 1 + 1 = 4$

$B_{4,5} = 2(2)^2 + 2 + 1 = 11$

$B_{6,7} = 2(3)^2 + 3 + 1 = 22$

$\Sigma\lvert v_f(0) - v_f(1)\rvert \leq 1$

$\Sigma|(2n^2 + n) – (2n^2 + n + 1)|\leq 1$

For Edge:-

$\Sigma e_f(0) = \frac{n(n+1)}{2}$ { where n = 2,4,6, 8……..}

$B_{2,3} = \frac{2(2+1)}{2} = 3$

$B_{4,5} = \frac{4(4+1)}{2} = 10$

$B_{6,7} = \frac{6(6+1)}{2} = 21$

$B_{8,9} = \frac{8(8+1)}{2} = 36$

$\Sigma e_f(1) = \frac{n(n+1)}{2}$ { where n = 2,4,6, 8………}

$B_{2,3} = \frac{2(2+1)}{2} = 3$

$B_{4,5} = \frac{4(4+1)}{2} = 10$

$B_{6,7} = \frac{6(6+1)}{2} = 21$

$B_{8,9} = \frac{8(8+1)}{2} = 36$

$\Sigma|e_f(0) – e_f(1)| \leq 1 \Sigma|\frac{n(n+1)}{2} - \frac{n(n+1)}{2}| \leq 1$

Case 4: When m is odd and n is even:-

1. Step to construct a graph $G = B_{m,n}$
- First, we join the root vertex u to all the right r vertices and then to left l vertices and construct the banana tree.
2. Label the graph $G = B_{m,n}$ when m, = 3, 5,7…. And n = 4, 6, 8….in the given steps
- Label all the vertices with 1, 0, 1, 0, 1……. respectively starting with root as shown in the graph.
3. Colour indication

Blue colour – vertex labels (1 and 0)

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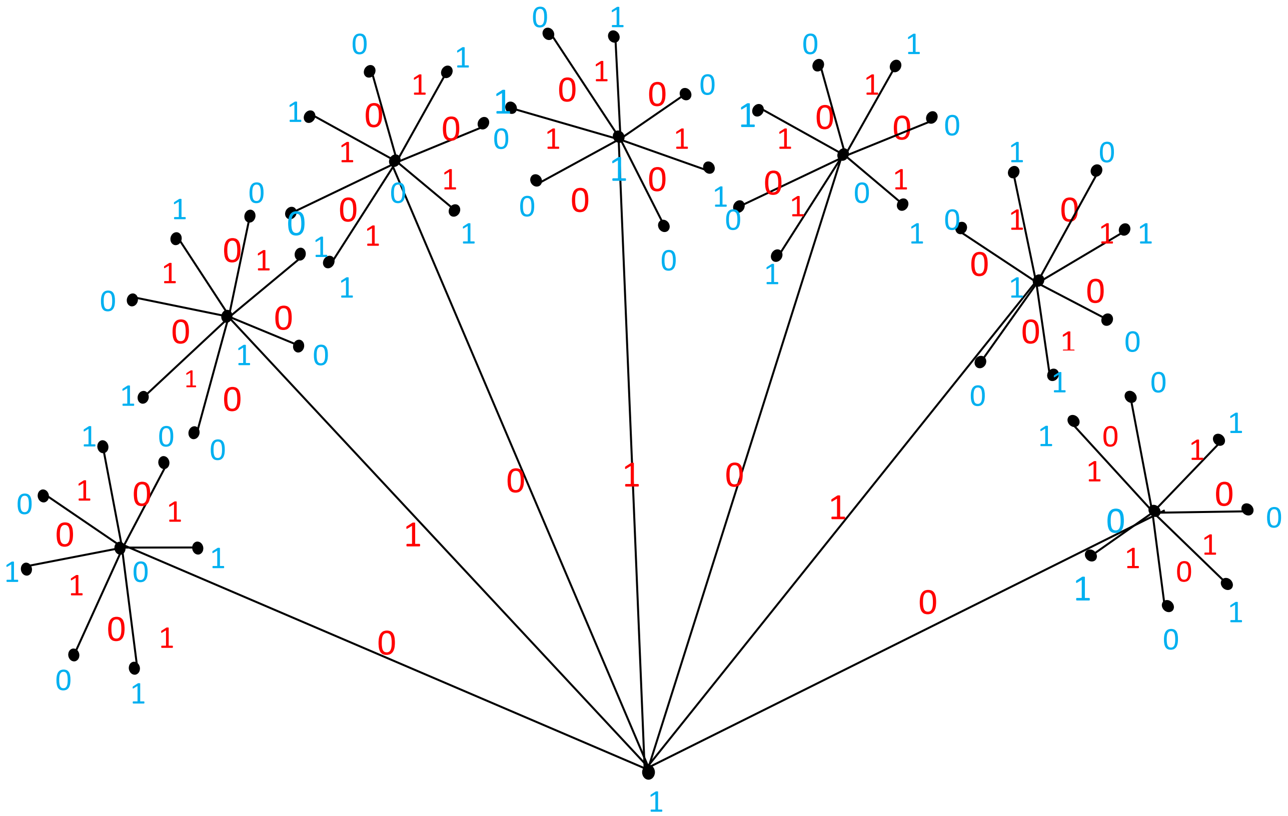

Figure 1.4 Sum Cordial Labeling of $B_{7,8}$

Table 4: Truth Table

$B_{m,n}$	v (0)	v (1)	$\Sigma\lvert v_f(0) - v_f(1)\rvert \le 1$	e (0)	e (1)	$\Sigma\lvert e_f(0) - e_f(1)\rvert \le 1$
$B_{3,4}$	6	7	1	6	6	0
$B_{5,6}$	15	16	1	15	15	0
$B_{7,8}$	28	29	1	28	28	0
$B_{9,10}$	45	46	1	45	45	0

So on …….

Generalization:-

$\Sigma v_f(0) = \frac{n(n+1)}{2}$ {where n = 3,5,7, 9……...}

$B_{3,4} = \frac{3(3+1)}{2} = 6$

$B_{5,6} = \frac{5(5+1)}{2} = 15$

$B_{7,8} = \frac{7(7+1)}{2} = 28$

$\Sigma v_f(1) = \frac{(n+1)(n+2)}{2} - n$

$B_{3,4} = \frac{(3+1)(3+2)}{2} - 3 = \frac{(4)(5)}{2} - 3 = 7$

$B_{5,6} = \frac{(5+1)(5+2)}{2} - 5 = \frac{(6)(7)}{2} - 5 = 16$

$B_{7,8} = \frac{(7+1)(7+2)}{2} - 7 = \frac{(8)(9)}{2} - 7 = 29$
$\Sigma \mid v_f(0) - v_f(1) \mid \leq 1$
$\Sigma \mid \frac{n(n+1)}{2} - \left\{\frac{(n+1)(n+2)}{2} - n\right\} \mid \leq 1$

For Edge
$\Sigma e_f(0) = n(2n - 1)$ { where n = 2,3,4,5,…….}
$B_{3,4} = 2\{2(2) - 1\} = 2(3) = 6$
$B_{5,6} = 3\{2(3) - 1\} = 3(5) = 15$
$B_{7,8} = 4\{2(4) - 1\} = 4(7) = 28$
$\Sigma e_f(1) = n(2n - 1)$ {where n = 2,3,4,5……..}
$B_{3,4} = 2\{2(2) - 1\} = 2(3) = 6$
$B_{5,6} = 3\{2(3) - 1\} = 3(5) = 15$
$B_{7,8} = 4\{2(4) - 1\} = 4(7) = 28$
$\Sigma \mid e_f(0) - e_f(1) \mid \leq 1$
$\Sigma \mid n(2n - 1) - n(2n - 1) \mid \leq 1$
Hence the Banana tree $B_{m,n}$ admits Sum Cordial labeling where m, n ≥ 2.

APPLICATION

Digital Coding: Famous math puzzle called "Graceful Labeling" gives every point and line in graph a unique number.
Network Security (Network Science): Large warehouse or a city square that needs security cameras.
Chemistry: Used to math formulas called indices on these shapes to represent the molecule.

Conclusion:

In this paper, we investigated sum cordial labeling of banana tree graphs $B_{m,n}$, a class of trees formed by joining one leaf of each of n copies of the star graph to a new root vertex. Sum cordial labeling, defined by the induced edge function f∗(uv)=f(u)+f(v)(mod2), imposes stronger parity restrictions than classical cordial labeling, making the characterization of sum cordial trees a nontrivial problem.
We established that the sum cordiality of $B_{m,n}$ depends on the parameters m and n. Constructive algorithms were provided for all cases where sum cordial labeling exist, and non existence was confirmed through parity arguments on vertex and edge counts for the remaining cases. These results demonstrate that banana trees exhibit a complete, yet conditional, behavior under sum cordial labeling, contrasting with the fact that all trees are cordial and product cordial. The dependence on congruence classes highlights the role of modular arithmetic in cordial-type problems for recursively constructed graphs. Future work may extend this study in several directions: examining sum cordial labeling of generalized banana trees with non-uniform star sizes, investigating n-banana trees where multiple leaves per star are joined to the root, and exploring sum cordial labelings of banana trees. The techniques developed here may also apply to other tree amalgamations and rooted products. This work contributes to the broader classification of Sum cordial graphs and provides new insights into how local structure and global parity constraints interact in graph labeling problems.

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