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On Near-Perfect Numbers

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ABSTRACT

In this paper, we investigate the concept of near-perfect numbers, a natural generalization of classical perfect numbers in number theory. A near-perfect number is an integer that is equal to the sum of all of its proper divisors except one, which is referred to as the redundant divisor. We examine the fundamental properties of near-perfect numbers and discuss several methods for generating such numbers.

Keywords: Near-perfect numbers, Perfect numbers, divisor sum function.

INTRODUCTION:

Number theory is one of the oldest and most fascinating branches of mathematics, concerned with the study of integers and their properties. Among the many classes of integers investigated by mathematicians, special attention has been devoted to numbers characterized by the behavior of their divisors. Perfect numbers, abundant numbers, and deficient numbers have been studied since ancient times and continue to play an important role in modern mathematical research. In recent years, interest has expanded to include generalizations of perfect numbers, among which near-perfect numbers occupy a significant position.

Pollock and Shevelev introduced near-perfect numbers in 2012 [6]. A positive integer n is near-perfect if $\sigma(n) = 2n + d$, where d is a proper divisor of n, known as redundant divisor of n. They also gave a method for the construction of near-perfect numbers with two prime divisors, and they introduced the concept of k −near-perfect numbers. A positive integer n is a k-near-perfect number when at most, k proper divisors are removed from the summation to make the sum equal to $2n$. When $(n) = 2n - d$, n is called a deficient perfect number with deficient divisor d; also, when $d = 1$, n is called an almost perfect number. Ren and Chen demonstrated that all near-perfect numbers with two prime factors besides 40 fall within one of the three constructions provided by Pollock and Shevelev [6]. In the same year, Tang, Ren and Li proved that there are no odd near-perfect numbers with three prime factors and determined all deficient-perfect numbers with at most two distinct prime factors. Tang and Feng proved that there are no odd deficient perfect numbers with three distinct prime factors. Later in 2016.

In this paper, we present construction rules analogous to Euclid's celebrated theorem for perfect numbers, providing systematic techniques for producing families of near-perfect numbers. The results contribute to a deeper understanding of the structure and distribution of near-perfect numbers and highlight their close relationship with classical perfect numbers.

Preliminaries

A near-perfect number is a positive integer that is very close to being a perfect number. Recall that a perfect number is a number that is equal to the sum of all its proper divisors (divisors excluding the number

itself). 6 is a perfect number because $6 = 1 + 2 + 3$.

Here are some examples of near-perfect numbers:

1. $12 = 1 + 2 + 3 + 6$

   Here, 12 is near-perfect number with redundant divisor 4. 4 is a proper divisor of 12 and

$$\sigma(12) = 28 = 2(12) + 4$$

2. $18 = 1 + 2 + 6 + 9$

Here, 18 is near-perfect number with redundant divisor 3.
3 is a proper divisor of 18 and $\sigma(18) = 39 = 2(18) + 3$ and so on....
Some of the first near-perfect numbers are-
12, 18, 20, 24, 40, 56, 88, 104, 196, 224, 234, 368, ...
with redundant divisors.
4, 3, 2, 12, 10, 8, 4, 2, 7, 56, 78, 8, ...

Euclid like theorem for near-perfect numbers:

For a given integer $k \geq 1$, we consider P_k the set of primes of the form $2^t - 2^k - 1$, where $t \geq k + 1$.

Proposition 1: If $n = 2^{t-1}(2^t - 2^k - 1)$, where $2^t - 2^k - 1 \in P_k$ then n is a near perfect number with redundant divisor 2^k.

Proof: As we know that, $t \geq k + 1$
Therefore, $t - 1 \geq k \; or \; k \leq t - 1$
Then, we have to show that $d = 2^k$, is a proper divisor of n.
Also, $\sigma(n) = (2^t - 1)(2^t - 2^k)$.

As, we know $\sigma(n) = 2n + d$

$$\Rightarrow d = \sigma(n) - 2n$$
$$= (2^t - 1)(2^t - 2^k) - 2.2^{t-1}(2^t - 2^k - 1)$$
$$= (2^t - 1)(2^t - 2^k) - 2^t(2^t - 2^k - 1)$$
$$= 2^{2t} - 2^t - 2^{t+k} + 2^k - 2^{2t} + 2^{t+k} + 2^t$$
$$= 2^k$$

The proposition follows.
We illustrate the above proposition by following example:
Let us take $k = 4$ then $t = 6$
Now, $2^{t-1}(2^t - 2^k - 1)$

$$= 2^{6-1}(2^6 - 2^4 - 1)$$
$$= 2^5(2^6 - 2^4 - 1)$$
$$= 32(64 - 16 - 1)$$
$$= 1504$$

which is a near-perfect number

$$1504 = 1 + 2 + 4 + 8 + 32 + 47 + 94 + 188 + 376 + 752$$

with redundant divisor $16 = 2^4$.
Also, $\sigma(1504) - 2(1504) = (2^6 - 1)(2^6 - 2^4) - 2^6(2^6 - 2^4 - 1)$

$$= 63(64 - 16) - 64(64 - 16 - 1)$$

$$= 3024 - 3008$$
$$= 16$$
$$= 2^4$$

This follows the proposition.

In contrast to the case of perfect numbers, there exist even near-perfect numbers which have not of the form $n = 2^{t-1}(2^t - 2^k - 1)$. For example, let us consider a near-perfect number, $n = 650$ with redundant divisor 2 but is not expressible from the Proposition 1.

Near-Perfect numbers generated by perfect numbers:

Proposition 2: A number n of the form $n = 2^x m$, where m is even perfect, is a near-perfect number if and only if either $x = 1$ or $x = p$, where p is that prime for which $2^{p-1}|m$.

Proof: Since $2^{p-1}|m$, then we have $m = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are prime. Therefore, $n = 2^{p+x-1}(2^p - 1)$.

We have $\sigma(n) = 2n + d$

$$\Rightarrow d = \sigma(n) - 2n$$
$$= (2^{x+p} - 1)2^p - 2^{x+p}(2^p - 1)$$
$$= 2^p(2^x - 1).$$

This is a proper divisor of n if and only if either $x = p$ *or* $x = 1$.

So, every even perfect number $m = 2^{p-1}(2^p - 1)$ generates two distinct near-perfect numbers $n_1 = 2m$ and $n_2 = 2^p m$.

Example: Let us take $m = 28$, an even perfect number. Then, $x = p = 3$ and $m = 2^2(2^3 - 1) = 28$

Thus, $n = 2^3.28 = 224$, which is a near-perfect number

Also

$$224 = 1 + 2 + 4 + 7 + 8 + 14 + 16 + 28 + 32 + 112$$

with redundant divisor $d = 56 = 2^3(2^3 - 1)$

which is of the form $2^p(2^x - 1)$ is a proper divisor of n. Or$n = 2^{p+x-1}(2^p - 1)$

$= 2^{3+3-1}(2^3 - 1)$

$$= 32(8 - 1)$$
$$= 224$$

Proposition 3: The number n of the form $n = 2^{p-1}(2^p - 1)^2$, if both p and $2^p - 1$ are prime numbers, is near-perfect with redundant divisor $2^p - 1$.

Proof: We have, $\sigma(n) = 2n + d$

$$\Rightarrow d = \sigma(n) - 2n$$
$$= (2^p - 1)((2p - 1)^2 + (2^p - 1) + 1) - 2^p(2^p - 1)^2$$
$$= 2^p - 1$$

Since, $2^p - 1$ is a proper divisor of n, then n is a near-perfect number.

From the above proposition let us take $p = 3$ then

$$n = 2^{3-1} - (2^3 - 1)^2$$
$$= 4(8 - 1)^2$$
$$= 196$$

Which is a near-perfect number.

Also $196 = 1 + 2 + 4 + 14 + 28 + 49 + 98$

With redundant divisor, $d = 7 = 2^3 - 1$ of the form $d = 2^p - 1$.

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