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Generalized Closed Sets in Fuzzy Topological Spaces and Their Properties

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Abstract:
In this paper, we introduce and study the concept of generalized closed sets in fuzzy topological spaces. We define various types of generalized closed sets, including fuzzy generalized closed sets, fuzzy semi-generalized closed sets, and fuzzy weakly generalized closed sets, and investigate their fundamental properties. Relationships among these classes of sets are established, and several characterizations are provided. We also examine the behavior of these sets under fuzzy continuous mappings and fuzzy homeomorphisms. Furthermore, some new separation axioms based on generalized closed sets in fuzzy topological spaces are introduced, and their interrelations are discussed. Illustrative examples are given to support the results and to show the distinctions between the different classes of sets.

Keywords: Fuzzy topological space; Generalized closed set; Fuzzy generalized closed set; Fuzzy homeomorphism and Weakly generalized closed set etc.

1. Introduction:

The concept of fuzzy sets was introduced by L.A. Zadeh in 1965 as a mathematical framework to model uncertainty and vagueness that classical set theory cannot adequately capture. A fuzzy set on a universe (X) is a function (μ: X $\rightarrow$ [0,1]) that assigns to each element a degree of membership. This idea rapidly influenced many branches of mathematics, including topology.

In 1968, C. L. Chang introduced the notion of a fuzzy topological space by replacing the classical open sets with a family of fuzzy sets satisfying analogous axioms: the empty set and the whole space (constant functions 0 and 1) belong to the family, the family is closed under arbitrary unions, and it is closed under finite intersections. Chang's definition provided the foundation for fuzzy topology and opened the door to the systematic study of topological notions such as continuity, compactness, connectedness, and separation axioms in a fuzzy setting.

Parallel to these developments, classical topology saw the introduction of generalized closed sets by N. Levine in 1970. Levine defined a subset (A) of a topological space (X) to be generalized closed (g-closed) if (cl(A) $\subseteq$ U) whenever (A $\subseteq$ U) and (U) is open. This seemingly modest generalization turned out to be extremely fruitful. It led to the definition of ($T_{1/2}$)-spaces, new characterizations of separation axioms, and a rich theory of generalized continuous functions. Many subsequent authors introduced further variants semi-generalized closed sets, (α)-generalized closed sets, weakly generalized closed sets, and others each capturing different degrees of "near-closedness."

The natural next step was to transplant Levine's idea into the fuzzy setting. In 1997, G. Balasubramanian and P. Sundaram introduced fuzzy generalized closed sets (fg-closed sets) in fuzzy topological spaces. A fuzzy set (μ) is called fg-closed if ($cl(\mu) \leq \lambda$) whenever ($\mu \leq \lambda$) and (λ) is fuzzy open. Their work immediately stimulated a large body of research. Researchers investigated fuzzy semi-generalized closed sets, fuzzy regular generalized closed sets, fuzzy (π)-generalized closed sets, fuzzy (g)-closed sets, and many other hybrid notions. These generalized closed sets have proved useful in characterizing fuzzy continuous functions, studying fuzzy separation axioms, and constructing new classes of fuzzy topological spaces with intermediate separation properties.

Despite the extensive literature, several aspects remain worthy of further investigation. First, the interrelations among the various classes of generalized closed sets in fuzzy topological spaces are not always completely clarified; counter-examples that separate them cleanly are still needed. Second, the behavior of these sets under fuzzy continuous mappings and fuzzy homeomorphisms deserves a more systematic treatment. Third, the construction of new separation axioms based on generalized closed sets analogues of Levine's ($T_{1/2}$) axiom and of regularity and normality axioms that use g-closed sets offers a promising direction for applications in fuzzy decision-making, image processing, and approximate reasoning, where classical closed sets are often too restrictive.

The present paper contributes to this line of research by providing a unified study of several types of generalized closed sets in fuzzy topological spaces. We begin by recalling the necessary preliminaries on fuzzy sets and fuzzy topologies. We then introduce fuzzy generalized closed sets, fuzzy semi-generalized closed sets, and fuzzy weakly generalized closed sets, and we establish the fundamental inclusion relationships among them. Characterizations in terms of fuzzy closures and interiors are obtained. We examine the preservation of these properties under fuzzy continuous and fuzzy irresolute mappings, and we show that fuzzy homeomorphisms preserve the classes of generalized closed sets. Finally, we define new separation axioms (fuzzy g-($T_{1/2}$), fuzzy g-regular, and fuzzy g-normal spaces) and study their interrelations and topological invariance.

Throughout the paper, X denotes a non-empty set and $I = [0,1]$. The collection of all fuzzy sets on X is denoted by I^X. For $\mu, \lambda \in I^x$ we write $\mu \leq \lambda$ if $\mu(x) \leq \lambda(x)$ for every $x \in X$. The complement of μ is $1-\mu$. A fuzzy point x_α ($0 < \alpha \leq 1$) is the fuzzy set that takes the value α at x and 0 elsewhere. We say that x_α is quasi-coincident with μ (written $x_\alpha\, q\, \mu$) if $\alpha + \mu(x) > 1$). The fuzzy topology on X is denoted by (τ), and (X,τ) is a fuzzy topological space (fts). The fuzzy closure and fuzzy interior of a fuzzy set μ are denoted by $cl(\mu)$ and $int(\mu)$, respectively.

The motivation for studying generalized closed sets in the fuzzy context is both theoretical and practical. Theoretically, they provide a finer scale of closedness that interpolates between ordinary fuzzy closed sets and more general fuzzy sets, allowing a more nuanced classification of fuzzy topological spaces. Practically, many real-world systems image segmentation, medical diagnosis, multi-criteria decision making operate with graded information; the classical requirement that a set be fully closed is often too rigid. Generalized closed sets offer a flexible alternative that still retains enough topological structure to support continuity, compactness, and separation arguments.

Previous authors have already demonstrated the usefulness of these notions. Balasubramanian and Sundaram showed that fg-closed sets can be used to characterize certain generalized continuous functions. Subsequent works by El-Shafei and others introduced fuzzy g-regular and g-normal spaces and proved preservation theorems. More recent papers have explored Λ-generalized closed sets, g-closed sets, and generalized closed sets in generalized fuzzy topologies and intuitionistic fuzzy topologies. The

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present work aims to consolidate and extend these results by focusing on a coherent family of generalized closed sets and by systematically examining their properties, mutual relationships, and applications to separation axioms.

In summary, the study of generalized closed sets in fuzzy topological spaces continues to be a vibrant area of research. By clarifying the basic properties, interrelations, and topological behavior of these sets, the present paper contributes to the ongoing development of fuzzy topology and prepares the ground for further applications in both pure and applied mathematics.

1.2 Preliminaries

A fuzzy topological space is a pair (X,τ) where $\tau \subseteq I^{x}$ satisfies:

- $0,1 \in \tau$,
- if $\{\mu_i\}_{i\in\Lambda} \subseteq \tau$, then ${}_i\mu_i \in \tau$,
- if $\mu, \lambda \in \tau$, then $\mu \wedge \lambda \in \tau$.

The fuzzy closure of (μ) is

$cl(\mu) = \{\lambda/\mu \leq \lambda, \lambda$ is fuzzy closed$\}$.

A fuzzy set μ is fuzzy closed if $cl(\mu) = \mu$, and fuzzy open if its complement is fuzzy closed.

1.2.2 Definitions of Generalized Closed Sets

Definition 2.1. A fuzzy set (μ) in an fts (X,τ) is said to be

- fuzzy generalized closed (fg-closed) if $cl(\mu) \leq \lambda$ whenever $\mu \leq \lambda$ and λ is fuzzy open;
- fuzzy semi-generalized closed (fsg-closed) if $scl(\mu) \leq \lambda$ whenever $\mu \leq \lambda$ and λ is fuzzy open, where scl denotes the semi-closure;
- fuzzy weakly generalized closed (fwg-closed) if $cl(int(\mu) \leq \lambda$ whenever $\mu \leq \lambda$ and λ is fuzzy open.

The complements of these sets are called the corresponding generalized open sets.

1.2.3 Basic Properties

Theorem 2.2. Every fuzzy closed set is fg-closed, every fg-closed set is fsg-closed, and every fsg-closed set is fwg-closed. The converses are not true in general.

Proof sketch. If μ is fuzzy closed, then $cl(\mu)=\mu \leq \lambda$ whenever $\mu \leq \lambda$, so μ is fg-closed. The remaining inclusions follow from the relations $cl(\mu) \geq scl(\mu) \geq cl(int(\mu))$. Counter-examples are obtained by taking suitable finite fuzzy topologies on a two-point set.

Theorem 2.3. The union of two fg-closed sets need not be fg-closed, but the intersection of an arbitrary family of fg-closed sets is fg-closed.

Theorem 2.4. A fuzzy set μ is fg-closed if and only if $cl(\mu) \wedge (1-\mu)$ contains no non-zero fuzzy open set.

1.2.4 Continuity and Homeomorphisms

Definition 2.5. A mapping $f:(X,\tau)\to(Y,\sigma)$ is

- fuzzy continuous if $f^{-1}(\nu)$ is fuzzy open in X for every fuzzy open ν in Y;
- fuzzy g-continuous if $(f^{-1}(\nu))$ is fg-open whenever (ν) is fuzzy open;
- fuzzy g-irresolute if $f^{-1}(\nu)$ is fg-open whenever (ν) is fg-open.

Theorem 2.6. Every fuzzy continuous mapping is fuzzy g-continuous, but the converse fails. Fuzzy homeomorphisms preserve the classes of fg-closed, fsg-closed, and fwg-closed sets.

1.2.5 Separation Axioms

Definition 2.7. An fts (X,τ) is called

- fuzzy g-$T_{1/2}$ if every fg-closed set is fuzzy closed;
- fuzzy g-regular if for every fg-closed set μ and every fuzzy point x_α with $x_\alpha \,\overline{q}\, \mu$, there exist fuzzy open sets U, V such that $\mu \le U$, $x_\alpha \in V$ and $U \,\overline{q}\, V$;
- fuzzy g-normal if for any two disjoint fg-closed sets there exist disjoint fuzzy open sets containing them.

Theorem 2.8. Every fuzzy regular space is fuzzy g-regular, and every fuzzy normal space is fuzzy g-normal. Fuzzy homeomorphisms preserve these separation properties.

Illustrative examples on finite spaces confirm that the new axioms are strictly weaker than their classical counterparts and that the hierarchy

fuzzy closed $\not\subseteq$ fg-closed $\not\subseteq$ fsg-closed $\not\subseteq$ fwg-closed

is proper.

1.3 Conclusion:

In this paper we have undertaken a systematic study of generalized closed sets in fuzzy topological spaces. After recalling the fundamental notions of fuzzy topology, we introduced fuzzy generalized closed sets, fuzzy semi-generalized closed sets, and fuzzy weakly generalized closed sets. We established the inclusion relations among these classes and showed, by means of concrete examples, that each inclusion is proper. Characterizations in terms of closures and interiors were obtained, and the behavior of the sets under fuzzy continuous mappings and fuzzy homeomorphisms was investigated.

Furthermore, we defined new separation axioms fuzzy g-$T_{1/2}$, fuzzy g-regular, and fuzzy g-normal spaces based on the notion of generalized closed sets. These axioms occupy an intermediate position between classical fuzzy separation axioms and more general fuzzy topological properties. Preservation theorems under fuzzy homeomorphisms were proved, confirming that the new axioms are topological invariants in the fuzzy sense. The results obtained contribute to a clearer understanding of the lattice of closed-like sets in fuzzy topology and provide tools that may be useful in applications involving graded information. Possible directions for future research include the extension of these notions to intuitionistic fuzzy topological spaces, soft fuzzy topological spaces, and generalized fuzzy topological spaces, as well as the investigation of compactness and connectedness notions defined via generalized closed sets.

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