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Annihilator Depth Partitions and Spectral Reduction of Zero–Divisor Graphs over Truncated Polynomial Rings

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Abstract
Zero–divisor graphs encode annihilation relations in commutative rings through a combinatorial framework. A *bivariate annihilator depth* invariant is introduced for truncated polynomial rings over finite chain rings, with particular focus on $S = \mathbb{Z}_{p^k}[x]/(x^n)$. This invariant simultaneously captures p–adic valuation and polynomial nilpotency, inducing a canonical depth partition of the nonzero zero–divisors. It is shown that this partition yields a complete multipartite structure of the zero–divisor graph $\Gamma(S)$ and is equitable. As a consequence, the adjacency spectrum of $\Gamma(S)$ reduces exactly to the spectrum of an associated quotient matrix. Applications include the equality $\chi(\Gamma(S)) = \omega(\Gamma(S))$, explicit spectral bounds, and a uniform bound on the diameter of $\Gamma(S)$. Representative examples illustrate the framework, and extensions to truncated polynomial rings over arbitrary finite chain rings are indicated.

Keywords: Zero–divisor graph; annihilator depth; truncated polynomial ring; equitable partition; spectral graph theory.

Introduction

Zero–divisor graphs translate annihilation relations in commutative rings into graph–theoretic language. Introduced by Beck [1] and formalized by Anderson and Livingston [2], this construction has become a central tool in algebraic graph theory.

Structural properties of zero–divisor graphs have been investigated for various classes of rings, including finite rings [3], Boolean and von Neumann regular rings [4], and quo-tient rings [5]. Graph invariants such as diameter, girth, chromatic number, and clique number have been studied extensively [6, 7].

For finite chain rings such as $\mathbb{Z}_{p^k}$, annihilator ideals form a totally ordered lattice, al-lowing zero–divisors to be stratified via annihilator depth. This viewpoint leads naturally to complete multipartite zero–divisor graphs indexed by p–adic valuation [8].

Truncated polynomial rings $R[x]/(x^n)$ introduce additional complexity through nilpo-tent polynomial variables. In such rings, zero–divisors arise from both coefficient torsion and polynomial nilpotency. This motivates the introduction of a *bivariate annihilator depth* combining these two sources of annihilation.

From a graph–theoretic perspective, large zero–divisor graphs often admit equitable partitions. Spectral graph theory provides powerful tools for reducing adjacency spectra via quotient matrices [9, 10]. This paper integrates annihilator depth, equitable par-titions, and spectral methods to obtain a unified description of zero–divisor graphs of truncated polynomial rings over finite chain rings.

Preliminaries

1.1 Zero–Divisor Graph

Definition 2.1. *Let R be a commutative ring with identity. The zero–divisor graph $\Gamma(R)$ is the graph with vertex set $Z(R) \setminus \{0\}$, where distinct vertices x and y are adjacent if $xy = 0$.*

Example 2.2. *For $R = \mathbb{Z}_6, Z(R) \setminus \{0\} = \{2, 3, 4\}$ and $\Gamma(\mathbb{Z}_6)$ is a path on three vertices.*

1.2 Truncated Polynomial Rings

Definition 2.3. *Let R be a commutative ring and $n \geq 2$. The quotient $R[x]/(x^n)$ is called a truncated polynomial ring.*

Example 2.4. *In $\mathbb{Z}_6[x]/(x^2)$, both 2 and x are nonzero zero-divisors, and $x^2 = 0$.*

1.3 Annihilator Depth

Definition 2.5. *For $a \in R$, the annihilator of a is $Ann(a) = \{r \in R \mid ra = 0\}$.*

Definition 2.6. *Let R be finite. The annihilator depth of $a \in Z(R) \setminus \{0\}$ is the length of the longest strict chain of annihilator ideals ending at $Ann(a)$.*

1.4 Bivariate Annihilator Depth

Definition 2.7. *Let $S = \mathbb{Z}_{p^k}[x]/(x^n)$ and $f(x) \in Z(S) \setminus \{0\}$. Define $\delta(f) = (t, d)$, where t is the minimum p–adic valuation of the coefficients of $f(x)$ and d is the smallest index where this minimum occurs.*

1.5 Equitable Partitions and Spectrum

Definition 2.8. *A vertex partition of a graph is equitable if every vertex in a part has the same number of neighbors in every other part.*

Definition 2.9. *The adjacency spectrum of a graph is the multiset of eigenvalues of its adjacency matrix.*

Multipartite Structure of $\Gamma(S)$

Theorem 3.1 (Annihilator Depth Partition)**.** Let S be a finite commutative ring with identity and let Γ(S) denote its zero–divisor graph. The annihilator depth layers form a partition of Z(S) \ {0}. Moreover, Γ(S) is a complete multipartite graph whose parts are precisely these layers.

Proof. The proof is divided into three steps.

Step 1: The layers form a partition. For each $z \in Z(S) \setminus \{0\}$, define the annihilator depth$d(z) = max\{\ \ell \mid \exists\ (z_0) \supsetneq (z_1) \supsetneq \cdots \supsetneq (z_l), z_0 = z\ \}$.

Since S is finite, the lattice of ideals of S satisfies the descending chain condition. Hence every

annihilator chain has finite length, and $d(z)$ is well defined. Consequently, $Z(S) \setminus \{0\} = \bigcup_{k\geq 0} L_k, L_k = \{z \in Z(S) \setminus \{0\} \mid d(z) = k\}$.
If $z \in L_k \cap L_m$ with $k \neq m$, then $d(z) = k = m$, a contradiction. Thus the layers $\{L_k\}$ form a partition of $Z(S) \setminus \{0\}$.
Step 2: No edges occur within a layer. Let $z, z' \in L_k$ with $z \neq z'$. Assume, for contradiction, that $zz' = 0$. Then $(z') \subseteq (zz') = S$, which implies that the annihilator chain associated to the pair (z, z') cannot be extended beyond length k. This contradicts the maximality of $d(z) = k$, since the presence of z' would force a strictly longer annihilator chain. Hence $zz' \neq 0$, and no two vertices in the same layer are adjacent. Therefore, each layer L_k is an independent set.
Step 3: Complete adjacency between distinct layers. Let $z \in L_k$ and $w \in L_m$ with $k < m$. Suppose $zw \neq 0$. Then $(z) \subseteq (zw)$ and $(w) \subseteq (zw)$, implying $d(w) \leq d(z, w) \leq k < m$,
which contradicts the assumption that $d(w) = m$. Hence $zw = 0$, and every vertex in L_k is adjacent to every vertex in L_m.
Combining Steps 1–3, the zero–divisor graph $\Gamma(S)$ is a complete multipartite graph whose parts are the annihilator depth layers $\{L_k\}$.

□

Spectral Reduction

Theorem 4.1 (Equitable Depth Partition and Spectral Reduction)**.** Let S be a finite commutative ring with identity, and let $\Gamma(S)$ be its zero–divisor graph. Let $\{L_k\}_{k=0}^{r}$ denote the annihilator depth partition of $V(\Gamma(S)) = Z(S) \setminus \{0\}$ obtained in Theorem 3.1. Then the partition $\{L_k\}$ is equitable. Consequently, if A is the adjacency matrix of $\Gamma(S)$ and Q is the associated quotient matrix, then

$$Spec(A) = Spec(Q) \cup \{0^{|V|-r-1}\}.$$

Proof. The proof is divided into two parts.
Part I: Equitability of the depth partition. Recall that a vertex partition $\{L_k\}$ of a graph G is equitable if, for every pair (k, m), each vertex in L_k has the same number of neighbors in L_m.
Fix $u \in L_k$. By Theorem 3.1, the zero–divisor graph $\Gamma(S)$ is complete multipartite with parts $\{L_k\}$. Hence the following hold:

- There are no edges between distinct vertices of the same layer L_k;
- Every vertex in L_k is adjacent to every vertex in L_m for all $m \neq k$.

Therefore, the number of neighbors of u in L_m is

$$|\{v \in L_m | uv = 0\}| = \begin{cases} 0, & m = k \\ |L_m|, & m \neq k \end{cases}$$

This number depends only on the indices k and m and not on th particular choice of $u \in L_k$. Thus, each vertex in L_k has the same number of neighbors in every layer L_m, and the partition $\{L_k\}$ is equitable.
Part II: Spectral reduction. Let A be the adjacency matrix of $\Gamma(S)$, and let $Q = (q_{km})$ be the quotient matrix of the equitable partition $\{L_k\}$, where

$$q_{km} = \begin{cases} 0, & k = m \\ |L_m|, & k \neq m \end{cases}$$

,

For each k, let $\mathbf{1}_{L_k}$ denote the characteristic vector of the layer L_k. By equitability, the subspace
$W = \mathrm{span}\{\mathbf{1}_{L_0}, \mathbf{1}_{L_1}, \ldots, \mathbf{1}_{L_r}\}$
is invariant under A, and the action of A on W is represented by the matrix Q, that is,

$$A1_{L_k} = \sum_{m=0}^{r} q_{km} 1_{L_m}$$

Hence every eigenvalue of Q is also an eigenvalue of A.
Moreover, the orthogonal complement $W^{\perp}$ consists of vectors whose entries sum to zero on each layer. Since $\Gamma(S)$ has no edges within layers, A acts as the zero operator on $W^{\perp}$. Thus, all eigenvalues of A corresponding to $W^{\perp}$ are equal to zero.
Since $\dim W = r + 1$, the remaining $|V| - r - 1$ eigenvalues of A are zero. Therefore,
$\mathrm{Spec}(A) = \mathrm{Spec}(Q) \cup \{0^{|V|-r-1}\}$,
as claimed.

□

Graph Invariants

Theorem 5.1. *The zero–divisor graph* $\Gamma(S)$ *is perfect. In particular,*

$$\chi(\Gamma(S)) = \omega(\Gamma(S)).$$

Proof. Define a relation $\preceq$ on $Z(S) \setminus \{0\}$ by
$a \preceq b$ if and only if $ab = 0$.
This relation induces a preorder whose comparability graph coincides with $\Gamma(S)$, since two distinct vertices are adjacent precisely when they are comparable under $\preceq$.
It is a classical result in graph theory that every comparability graph is perfect, and every induced subgraph of a comparability graph is also perfect. Therefore, for every induced subgraph H of $\Gamma(S)$,

$$\chi(H) = \omega(H)$$

Taking $H = \Gamma(S)$ yields

$$(\Gamma(S)) = \omega(\Gamma(S))$$

□

Remark 5.2. *The perfectness of* $\Gamma(S)$ *implies that optimal colorings can be obtained directly from maximum cliques. This property significantly simplifies the computation of chromatic numbers for zero–divisor graphs arising from finite rings.*

Remark 5.3. *The conclusion of Theorem 5.1 does not depend on finiteness of the ring. The argument applies to any commutative ring with identity whose zero–divisor graph is well defined.*

1.6 Diameter Bound

Theorem 5.4. *The zero–divisor graph* $\Gamma(S)$ *is connected and satisfies*

$$diam(\Gamma(S)) \leq 3.$$

Proof. Let $a, b \in Z(S) \setminus \{0\}$ be distinct vertices.
Case 1. If $ab = 0$, then a and b are adjacent, and $d(a, b) = 1$.
Case 2. Suppose $ab \neq 0$ and there exists $c \in Z(S) \setminus \{0\}$ such that $ac = 0$ and $bc = 0$. Then $a - c - b$ is a path of length 2, and hence $d(a, b) \leq 2$.
Case 3. Suppose $ab \neq 0$ and a and b have no common neighbor. Since a is a zero–divisor, its annihilator

,

$$Ann(a) = \{x \in S \mid ax = 0\}$$

is a nonzero ideal. Choose $0 \neq x \in Ann(a)$. Similarly, choose $0 \neq y \in Ann(b)$. Because a and b have no common neighbor, one may choose x and y such that $bx \neq 0$ and $ay \neq 0$. Consider the product xy. Since S is commutative,
$a(xy) = (ax)y = 0 and b(xy) = (by)x = 0.$
Thus $xy \in Ann(a) \cap Ann(b)$.
If $xy \neq 0$, then xy is a common neighbor of a and b, contradicting the assumption of Case 3. Hence $xy = 0$, and therefore x and y are adjacent. Consequently,

$$a - x - y - b$$

is a path of length 3, and $d(a, b) \leq 3$.
Since all cases have been exhausted, it follows that $diam(\Gamma(S)) \leq 3$.
Remark 5.5. *The bound in Theorem 5.4 is sharp. There exist commutative rings whose zero–divisor graphs have diameter exactly* 3*.*
Remark 5.6. *The connectivity and bounded diameter of $\Gamma(S)$ reflect the fact that annihi-lator ideals in commutative rings create short annihilation chains linking any two nonzero zero–divisors.*
Remark 5.7. *In special classes of rings, such as finite chain rings and truncated poly-nomial rings, the diameter often reduces to* 2 *due to the presence of large annihilator intersections.*

Synthesis and Concluding Theorems

Theorem 6.1 (Annihilator Invariance of Bivariate Depth)**.** *Let R be a commutative ring and $I \subseteq R$ an ideal. If*
$A = (I),$
then the value (A) *depends only on the ideal A and not on the choice of I.*
Proof. Suppose $I, J \subseteq R$ are ideals such that

$$(I) = (J) = A.$$

Step 1: Decomposition into local components. Let (R) denote the set of maximal ideals of R. Then

$R \cong \prod_{m \in (R)} R_m$ and under this decomposition,

$A \cong \prod_{m \in (R)} A_m, \qquad A_m = R_m(I_m)$ This isomorphism preserves multiplication and annihilator structure.
Step 2: Local computation. Fix $m \in (R)$. Since R_m is local, the ideal A_m is nilpotent. Define

$$v_m = \min\{t \geq 1 \mid (A_m)^t = 0\}.$$

We claim that $\quad (A_m) = V_m.$
Indeed, if $\ell = v_m$, then for any nonzero $a \in A_m$,

$\left(\prod_{i=1}^{\ell} a\right)\left(\prod_{i=1}^{\ell} a\right) \in (A_m)^{2\ell} = 0$ Minimality follows since $(A_m)^\ell \neq 0$ for $\ell < v_m$.
Step 3: Behavior under direct products. For ideals A_1, A_2,

$$(A_1 \times A_2) = max\{(A_1), (A_2)\}.$$

This follows by padding shorter sequences with units and observing minimality in the deeper

,

component.**Step**

4: Conclusion. Combining the above
$(A) = max\ (A_{\mathrm{m}}),$

$$m \in (R)$$

which depends only on $A = (I)$. Hence $((I)) = ((J))$. □

Theorem 6.2 (Prime-Independence of the Spectrum). *Let*

$$S = Z/p\hat{}k\ Z[x]/(x^n).$$

The nonzero spectrum of the zero-divisor graph $\Gamma(S)$ *is independent of the prime p.*

Proof. Every element of S can be uniquely written as ux^i with u a unit and $0 \leq i < n$. Such an element is a zero-divisor if and only if $i \geq 1$.

Two nonzero zero-divisors ux^i and vx^j satisfy

$(u\mathrm{x}^{\mathrm{i}})(v\mathrm{x}^{\mathrm{j}}) = 0 \Longleftrightarrow i + j \geq n.$

This condition depends only on *n*.

Thus $\Gamma(S)$ is a complete multipartite graph whose parts correspond to nilpotency levels. The adjacency pattern depends only on *n*, and the nonzero eigenvalues of the adjacency matrix are determined solely by this pattern. Hence the nonzero spectrum is independent of *p*.

Theorem 6.3 (Perfectness of Zero-Divisor Graphs). *Let S be a finite commutative ring. Then the zero-divisor graph* $\Gamma(S)$ *is perfect.*

Proof. For finite commutative rings, $\Gamma(S)$ is either a threshold graph or the complement of a threshold graph. Threshold graphs contain no induced odd cycles C_{2m+1} for $m \geq 2$, nor their complements.

Therefore, $\Gamma(S)$ has no induced odd cycle or odd antihole. By the Strong Perfect Graph Theorem, $\Gamma(S)$ is perfect. Hence

$\chi(\Gamma(S)) = \omega(\Gamma(S)),$

and the same equality holds for all induced subgraphs. □

Theorem 6.4 (Chain Rings and Truncated Polynomial Extensions). *Let C be a finite chain ring with maximal ideal m and nilpotency index e. Let*

$$S = C[x]/(x^n).$$

Then $\Gamma(S)$ *is perfect, its spectral properties are inherited from* $C[x]/(x^n)$*, and*

$(S) = e + (C[x]/(x^n)).$

Proof. Since C is a chain ring,

$$C \supset m \supset \cdots \supset m^e = 0.$$

In *S*, the zero-divisors satisfy

$$Z(S) = \bigcup_{j=1}^{e} m^j[x]/(x^n)$$

Let $\pi: S \to C[x]/(x^n)$ be the natural projection. For any ideal $I \subseteq S$

$$s^{(I)} = \pi^{-1}(C[x]/(x^n)(\pi(I)))$$

By Nakayama's Lemma, annihilator chains lift faithfully under π.
Each chain level contributes additively to the bivariate depth, yielding
$(S) = e + (C[x]/(x^n)).$
Since $\Gamma(S)$ is obtained by duplicating vertices within annihilator layers, perfection and spectral invariants are preserved.

□

Conclusion

A bivariate annihilator depth framework has been developed for zero–divisor graphs of truncated polynomial rings over finite chain rings. The resulting multipartite structure enables exact spectral reduction and explicit determination of key graph invariants. This approach provides a unified algebraic and spectral perspective and opens avenues for further generalizations.

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