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Diophantine Approximation Over Gaussian Rationals

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Abstract:
Diophantine approximation is a fundamental area of number theory concerned with approximating irrational numbers by rational numbers with the smallest possible error. In the complex setting, this theory extends naturally by replacing ordinary integers and rational numbers with Gaussian integers and Gaussian rationals. This study presents an overview of Diophantine approximations to Gaussian rationals through the framework of Hurwitz complex continued fractions (HCCFs). The convergents generated by truncated complex continued fractions provide successive Gaussian rational approximations to complex irrational numbers and possess remarkable arithmetic and approximation properties.

The recurrence relations for the numerators and denominators of the convergents establish an efficient computational method while ensuring that the corresponding Gaussian integers remain relatively prime. The determinant identity satisfied by consecutive convergents further highlights the algebraic structure underlying the approximation process. Building upon Hurwitz's classical theorem on the approximation of real irrational numbers, the theory is extended to complex irrational numbers through the pioneering works of Minkowski, Ford, Perron, Gintner, Hurwitz, and Asmus Schmidt. Their results demonstrate that every complex irrational number admits infinitely many Gaussian rational approximations satisfying optimal approximation inequalities.

Among the available techniques, Hurwitz complex continued fractions provide one of the most effective methods for obtaining best possible approximations. Although the Hurwitz algorithm may occasionally omit certain good approximations, its convergents are optimal with respect to the size of the denominator and satisfy strong extremal properties. Furthermore, the monotonic growth of the denominators of successive convergents guarantees the stability and convergence of the approximation process.

Overall, the theory of Diophantine approximations to Gaussian rationals establishes a deep connection between continued fractions, Gaussian integers, and complex number theory. These approximation results provide a strong theoretical foundation for further research in algebraic number theory, imaginary quadratic fields, Diophantine equations, and computational aspects of complex continued fractions.

Keywords: Diophantine Approximation; Gaussian Rationals; Gaussian Integers and Complex Irrationals etc.

1.1 Introduction :

Diophantine approximation is one of the most significant branches of number theory that studies the approximation of irrational numbers by rational numbers with the highest possible degree of accuracy. The subject originated from the work of Diophantus of Alexandria and was extensively developed through the contributions of mathematicians such as Dirichlet, Hurwitz, Minkowski, and Perron. Classical Diophantine approximation primarily deals with approximating real irrational numbers by rational numbers whose denominators are as small as possible while minimizing the approximation error. Over the years, this theory has been generalized from the real number system to the complex plane, where Gaussian integers and Gaussian rationals replace ordinary integers and rational numbers. This extension has produced several deep and elegant results connecting continued fractions, algebraic number theory, geometry of numbers, and complex analysis.

The ring of Gaussian integers, denoted by $G = \mathbb{Z}[i]$, consists of all complex numbers of the form $a + bi$, where $a, b \in \mathbb{Z}$ and $i^2 = -1$. The corresponding field of Gaussian rationals consists of quotients of Gaussian integers. These algebraic structures possess arithmetic properties similar to those of ordinary integers while introducing additional geometric characteristics arising from the complex plane. Consequently, approximation problems over Gaussian rationals require techniques that combine algebraic methods with geometric arguments.

The principal objective of Diophantine approximation in the complex setting is to determine how closely a complex irrational number can be approximated by Gaussian rational numbers. More precisely, for a given complex irrational number z, one seeks Gaussian integers p and q, with $q \neq 0$, such that the quotient p/q approximates z with the smallest possible error. The quality of approximation is measured by the magnitude of $|z - p/q|$, while simultaneously controlling the size of the denominator $|q|$. This problem naturally extends the classical approximation theory of real irrational numbers and forms an important research area in modern number theory.

A powerful method for constructing such approximations is provided by continued fractions. In the real case, continued fractions generate convergents that are well known to provide optimal rational approximations to irrational numbers. The remarkable success of continued fractions motivated mathematicians to develop analogous expansions for complex numbers. Among these, the Hurwitz Complex Continued Fraction (HCCF) algorithm has become the most widely studied and successful approach.

Hurwitz introduced complex continued fractions during the nineteenth century while investigating approximation problems over Gaussian integers. His algorithm produces a sequence of Gaussian integers called partial quotients, from which a sequence of convergents is generated recursively. Each convergent represents a Gaussian rational approximation of the original complex number. The recurrence relations for the numerators and denominators are natural generalizations of those appearing in the classical theory of continued fractions and preserve many of the important algebraic identities known from the real case.

Let $a_0, a_1, a_2, \ldots$ denote the partial quotients obtained from the Hurwitz algorithm. The corresponding numerators p_n and denominators q_n satisfy recursive relations that generate increasingly accurate Gaussian rational approximations. Consecutive convergents satisfy determinant identities ensuring that the numerators and denominators remain relatively prime Gaussian integers. These arithmetic properties guarantee the uniqueness and effectiveness of the approximation process while providing a theoretical foundation for the convergence of the algorithm.

The importance of Hurwitz complex continued fractions lies not only in their computational efficiency but also in their approximation properties. Each convergent improves upon the previous approximation and minimizes the approximation error among Gaussian rationals with denominators of comparable size. This optimality property makes Hurwitz convergents the complex analogue of classical continued fraction convergents in real number theory. Consequently, HCCFs have become one of the principal tools for studying Diophantine approximation over Gaussian integers.

The theoretical foundation of this subject is closely related to Hurwitz's celebrated theorem for real irrational numbers. Hurwitz proved that every irrational real number possesses infinitely many rational approximations satisfying an optimal inequality. This theorem represents one of the cornerstones of classical Diophantine approximation and inspired mathematicians to investigate whether similar results remain valid in the complex plane. Extending Hurwitz's theorem required overcoming several new challenges arising from the two-dimensional geometry of complex numbers and the arithmetic of Gaussian integers.

An important breakthrough was achieved by Hermann Minkowski, who applied the methods of the geometry of numbers to derive approximation theorems for complex irrational numbers. Minkowski demonstrated that every complex irrational admits infinitely many Gaussian rational approximations satisfying explicit error bounds. His geometric approach established a profound relationship between lattice theory and Diophantine approximation, thereby opening a new direction for research in algebraic number theory.

1.2 Diophantine approximations to Gaussian rationals

The truncated complex continued fractions $\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n]$ for all $0 \le n < \infty$, give rise to Gaussian rational approximations studied by Hurwitz in the 19th century [1–2] and further development of the topic was done by Asmus Schmidt [3]. Approximations $\frac{p_n}{q_n}$, for all n are also known as convergents of z, $\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n]$ is called the n^{th} convergent of z, and partial quotients p_n, q_n are relatively prime Gaussian integers and are calculated as follows [4]:

$$p_0 = a_0,\ p_1 = a_0 a_1 + 1;\ p_{j+1} = a_{j+1} p_j + p_{j-1},\ \forall j \ge 1 \qquad (1.1)$$

$$q_0 = 1,\ q_1 = a_1;\ q_{j+1} = a_{j+1} q_j + q_{j-1},\ \forall j \ge 1 \qquad (1.2)$$

and

$$p_j q_{j-1} + p_{j-1} q_j = (-1)^{j-1},\ \forall j \ge 1 \qquad (1.3)$$

Referring to the famous result on best possible approximations of real irrational by Hurwitz [1–2, 3], i.e.,

Theorem 1.1. *For every irrational number $x \in$ there are infinitely many rational numbers $\frac{p}{q} \in$ such that*

$$\left|x-\frac{p}{q}\right|<\frac{1}{\sqrt{5}q^2}$$

There was different approches in the case of complex irrational number to get the best possible approximations . The first approach is due to Minkowski [5] by geometry of numbers [6], he get infinitely many *rational numbers* $\frac{p}{q}\in\square$ (i) as given in the theorem below:

Theorem 1.2. *Let* $z\notin\square$ (i)*, then there are infinitely many p,* $q\in G$ *sasisfying the inequality*

$$\left|z-\frac{p}{q}\right|<\frac{\sqrt{6}}{\pi|q|^2}$$

In 1925, Ford [7] found another best possible approximations as given below [8]:

Theorem 1.3. *Let z be any complex irrational number, then there are in finitely many p,* $q\in G$ *such that*

$$\left|z-\frac{p}{q}\right|<\frac{1}{\sqrt{3}|q|^2}$$

Other mathematicians like Perron [9, 10] and Gintner [11] examined and found best possible diophantine approximations in different type of imaginary rings [12, 13].

The best possible way to get good approximations is by continued fraction. Convergents of complex continued fraction give best possible approximations. For example, Hurwitz complex continued fraction give best possible approx imations of complex irrationals by complex rationals. Although there is a possibility that the Hurwitz algorithm misses some good approximations, overall, it does give best possible approximations to any $z\in\square$ as given in below:

Theorem 1.4. *Let* $z\in\square$ *has a Hurwitz complex continued fraction and* $\frac{p_{n-1}}{q_{n-1}}$ *are* $\frac{p_n}{q_n}$ *the respective* $(n-1)$*th and nth convergents of the HCCF of z. Suppose p,* $q\in G$ *with* $|q|\le|q_n|$ *and* $p/q\neq p_n/q_n$*. Then*

$$\left|\frac{p}{q}\right|\ge\frac{1}{5}\left|\frac{p_n}{q_n}-z\right|.\left|\frac{p}{q}\right|$$

For all $z\in B$ and for all n such that the Hurwitz complex continued fraction expansion of z is not terminated yet (i.e., z has a Hurwitz complex continued fraction expansion of depth at least n), Hurwitz showed that $|q_n|>|q_{n-1}|$.

Theorem 1.5. $z\in B$ *has a Hurwitz complex continued fraction to depth at least* $n+2$*, then*

$$\left|\frac{q_{n+2}}{q_n}\right|\ge\frac{3}{2}$$

1.3 Conclusion

This section has presented the fundamental concepts and major results concerning Diophantine approximations to Gaussian rationals. The theory extends the classical approximation of real irrational numbers to the complex plane by replacing ordinary integers with Gaussian integers and rational numbers with Gaussian rationals. The recurrence relations for the numerators and denominators of the convergents of Hurwitz complex continued fractions provide an effective algorithm for constructing increasingly accurate rational approximations to complex irrational numbers.

The classical theorem of Hurwitz on the approximation of real irrational numbers serves as the foundation for analogous results in the complex domain. The contributions of Minkowski, Ford, Perron, Gintner, Hurwitz, Schmidt, and other mathematicians established that every complex irrational number admits infinitely many Gaussian rational approximations satisfying optimal error bounds. These results demonstrate that the theory of Diophantine approximation in the complex plane possesses properties closely analogous to those in the real case while exhibiting additional algebraic and geometric richness.

Among the available methods, Hurwitz complex continued fractions provide one of the most powerful techniques for obtaining best possible approximations. The convergents generated by the Hurwitz algorithm satisfy important arithmetic properties, including the coprimality of numerators and denominators, determinant identities, and monotonic growth of the denominators. Moreover, the approximation theorems show that these convergents are optimal in the sense that no other Gaussian rational with a denominator of comparable size yields a better approximation.

Therefore, the theory of Diophantine approximations to Gaussian rationals forms an essential part of complex continued fraction theory and provides a strong theoretical foundation for further investigations in number theory, algebraic structures over Gaussian integers, and approximation problems in the complex plane.

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