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Bicomplex-Valued Multi-Norm Spaces: Foundations and Structural Analysis

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Abstract:
The term "bicomplex-valued norm" necessitates qualification due to the absence of a compatible total order in the bicomplex algebra. The positive hyperbolic cone, which is derived from the idempotent modulus, more precisely values a norm that is appropriate for bicomplex modules. This paper provides a structural and foundational theory for multi-norm spaces. A bicomplex module $X=e_1X_1\oplus e_2X_2$ is equipped with $N_n=N_n,1e_1+N_n,2e_2$ at each level, where the components are classical complex multi-norms. The componentwise proof of the multi-norm axioms, positivity, definiteness, triangle inequality, and scalar domination is presented, as well as a converse decomposability characterisation. The product topology is demonstrated to be equivalent to the hyperbolic-ball topology; convergence, Cauchy sequences, completeness, and fixed-level equivalence are precisely equivalent to the two complex components. Closed subspaces, product modules, minimum quotient multi-norms, continuous duals, and finite-dimensional modules yield structural results. The analysis elucidates the rationale behind the legitimate existence of nonzero null-cone norms in pure idempotent vectors and the exclusion of natural submodules by demand for invertible norm values. The failure of bicomplex homogeneity by a real Euclidean scalarization and the inability of an unordered bicomplex codomain to support norm inequalities are demonstrated by counterexamples. Bicomplex Banach multi-modules are generated as paired complex Banach multi-normed spaces. Applications and open questions pertain to C*-type modules, weak topologies, tensor products, quotient constructions beyond the minimum multi-norm, and duality.

Keywords: bicomplex multi-norm; hyperbolic modulus; positive cone; product topology; quotient space; continuous dual; Banach multimodule

Mathematical Notation and Symbols

Symbol	Meaning
$\mathscr{B}\mathbb{C}$	Bicomplex algebra $\mathbb{C}\mathbf{e}_1\oplus\mathbb{C}\mathbf{e}_2$.
Z^{†3}	Idempotent complex conjugation $\bar{Z}_1\mathbf{e}_1+\bar{Z}_2\mathbf{e}_2$.
$\lvert Z\rvert\mathbb{D}$	Hyperbolic modulus $\lvert Z_1\rvert\mathbf{e}_1+\lvert Z_2\rvert\mathbf{e}_2$.
$X=\mathbf{e}_1X_1\oplus\mathbf{e}_2X_2$	Bicomplex module with complex components.

Symbol	Meaning
N_n	$\mathbb{D}^+$-valued level-n multi-norm.
$B\mathbb{D}(x;\varepsilon)$	Hyperbolic open ball with $\varepsilon_1,\varepsilon_2>0$.
X'	Continuous bicomplex-linear dual.
X/M	Quotient by a bicomplex submodule M.

1. Introduction

A norm requires a notion of nonnegativity and comparison. General bicomplex numbers cannot be ordered in a way compatible with their algebraic operations, but the idempotent plane $\mathbb{D}=\mathbb{R}\mathbf{e}_1\oplus\mathbb{R}\mathbf{e}_2$ contains the positive cone $\mathbb{D}^+$. Consequently, established bicomplex functional analysis uses hyperbolic-valued norms: a bicomplex scalar has a componentwise complex modulus, and a module vector is measured by two nonnegative real norms (Alpay et al., 2014; Kumar & Saini, 2016). The same correction is essential at the multi-norm level.

The multi-norm sequence records geometry on every finite Cartesian power. An extension to bicomplex modules must preserve the level axioms and remain meaningful for zero divisors. In particular, a nonzero pure vector $x=\mathbf{e}_1x_1$ has norm $\|x_1\|\mathbf{e}_1$, which lies in the null cone. It is positive and nonzero, although not invertible. Excluding such values would exclude the idempotent submodules that generate the entire theory.

This paper concentrates on topology and structural operations rather than on examples or operators. It builds the product topology from hyperbolic balls, proves completeness and equivalence criteria, studies restrictions, products, quotients, and duals, and states the finite-dimensional form. The results are derived from component multi-norm theory and the algebraic decomposition, with limitations stated whenever no canonical construction exists.

2. Algebraic Foundations and the Correct Norm Codomain

Definition 2.1. For commuting imaginary units $\mathbf{i}^2=\mathbf{j}^2=-1$, put $\mathbf{k}=\mathbf{ij}$ and $\mathbf{e}_1=(1+\mathbf{k})/2$, $\mathbf{e}_2=(1-\mathbf{k})/2$. Every bicomplex number has the form $Z=Z_1\mathbf{e}_1+Z_2\mathbf{e}_2$ with $Z_j\in\mathbb{C}(\mathbf{i})$.

$$Z\,W=Z_1W_1\mathbf{e}_1+Z_2W_2\mathbf{e}_2,\quad Z \text{ is invertible} \Leftrightarrow Z_1Z_2\neq0. \quad (2.1)$$

Definition 2.2. The hyperbolic modulus and order are

$$|Z|_{\mathbb{D}}=|Z_1|\mathbf{e}_1+|Z_2|\mathbf{e}_2,\quad u\preccurlyeq_{\mathbb{D}}v \Leftrightarrow u_1\leq v_1 \text{ and } u_2\leq v_2. \quad (2.2)$$

The codomain of a bicomplex norm is therefore $\mathbb{D}^+$, not all of $\mathscr{B}\mathbb{C}$. The common phrase “bicomplex-valued norm” is retained only as shorthand for this associated hyperbolic-valued norm.

Definition 2.3. A bicomplex normed module is $X=\mathbf{e}_1X_1\oplus\mathbf{e}_2X_2$ with

$$\|x_1\mathbf{e}_1+x_2\mathbf{e}_2\|_{\mathbb{D}}=\|x_1\|_1\mathbf{e}_1+\|x_2\|_2\mathbf{e}_2. \quad (2.3)$$

Definiteness means $\|x\|_{\mathbb{D}}=0$ if and only if $x=0$. It does not mean that $\|x\|_{\mathbb{D}}$ is invertible whenever $x\neq0$.

Counterexample 2.4. The real scalarization $q(Z)=\sqrt{[(|Z_1|^2+|Z_2|^2)/2]}$ is a real norm on the four-dimensional real algebra but is not a bicomplex module norm with exact scalar homogeneity.

Proof. Take $Z=1$ and $\alpha=\mathbf{e}_1$. Then $q(\alpha Z)=q(\mathbf{e}_1)=1/\sqrt{2}$. There is no real scalar $|\alpha|$ satisfying the bicomplex identity $q(\alpha Z)=|\alpha|q(Z)$ while also retaining both idempotent moduli as algebraic coefficients. The scalarization is useful for metrizability, but it discards the module-homogeneous $\mathbb{D}^+$ structure. □

Counterexample 2.5. A map $N:X\to\mathscr{B}\mathbb{C}$ whose values are not restricted to $\mathbb{D}^+$ cannot support the norm axioms through a canonical order.

Proof. For example, $\mathbf{i}\mathbf{e}_1$ and $\mathbf{e}_1$ are not elements of the same positive cone and cannot be compared by the component real order. Thus a triangle "inequality" with arbitrary bicomplex values has no defined meaning unless an external scalar order is imposed. □

3. Multi-Norm Axioms and Decomposition

Definition 3.1. A bicomplex multi-normed space is a bicomplex module X with maps $N_n:X^n\to\mathbb{D}^+$ satisfying norm definiteness, common-scalar homogeneity, triangle inequality, permutation invariance, coordinate scalar domination, zero stability, repetition consistency, and $N_1=\|\cdot\|\mathbb{D}$.

$$N_n(\alpha_1x_1,\dots,\alpha_nx_n)\preccurlyeq\mathbb{D}(\max\mathbb{D}|\alpha_r|\mathbb{D})N_n(x_1,\dots,x_n). \qquad (3.1)$$

Theorem 3.2. Component construction. Let N_n,j be complex multi-norms on X_j. Then

$$N_n(x_1,\dots,x_n)=N_n,1(x_{11},\dots,x_{n1})\mathbf{e}_1+N_n,2(x_{12},\dots,x_{n2})\mathbf{e}_2 \quad (3.2)$$

defines a bicomplex multi-norm.

Proof. Every coefficient is nonnegative. Both vanish exactly when all component vectors vanish. The triangle and level axioms hold in each complex multi-norm. For scalar domination, α_rx_r has jth component $\alpha_{rj}x_{rj}$, so $N_n,j\le\max_r|\alpha_{rj}|$ times the original component norm. Recombination is precisely Equation (3.1). □

Definition 3.3. A bicomplex multi-norm is idempotent decomposable if it is pure on $\mathbf{e}_jX$ and additive across the two idempotent projections: $N_n(x)=N_n(\mathbf{e}_1x)+N_n(\mathbf{e}_2x)$.

Theorem 3.4. Structural characterization. A bicomplex multi-norm has the representation in Equation (3.2) if and only if it is idempotent decomposable. The component multi-norms are then unique.

Proof. The representation clearly implies purity and additivity. Conversely, define N_n,j as the nonnegative coefficient of $\mathbf{e}_j$ on pure tuples. Restriction of every bicomplex axiom gives a complex multi-norm axiom. Additivity reconstructs N_n, and evaluation on pure tuples proves uniqueness. □

Proposition 3.5. Every bicomplex multi-norm satisfies

$$\max_r\|x_r\|\mathbb{D}\preccurlyeq\mathbb{D}N_n(x_1,\dots,x_n)\preccurlyeq\mathbb{D}\Sigma_r\|x_r\|\mathbb{D}. \qquad (3.3)$$

Proof. Zero all but one coordinate and use permutation and zero stability for the lower estimate. Express a tuple as the sum of its one-coordinate tuples and use the triangle inequality for the upper estimate. □

4. Topology, Convergence, and Completeness

Definition 4.1. For $\varepsilon=\varepsilon_1\mathbf{e}_1+\varepsilon_2\mathbf{e}_2$ with $\varepsilon_1,\varepsilon_2>0$, the hyperbolic ball is $B\mathbb{D}(x;\varepsilon)=\{y:\|x-y\|\mathbb{D}\prec\mathbb{D}\varepsilon\}$. At level n, replace $\|\cdot\|\mathbb{D}$ by N_n.

Theorem 4.2. The hyperbolic-ball topology on X is exactly the product topology of $X_1\times X_2$. It is metrizable, for example by

$$d(x,y)=\max\{\|x_1-y_1\|_1,\ \|x_2-y_2\|_2\}. \qquad (4.1)$$

Proof. A hyperbolic ball is $B_1(x_1;\varepsilon_1)\times B_2(x_2;\varepsilon_2)$, so such balls form the standard product base. Conversely, every product of component balls is a hyperbolic ball. The maximum metric generates the same base. □

Proposition 4.3. For a decomposable N_n, tuple convergence and Cauchy behavior are equivalent to the corresponding component convergence and Cauchy behavior.

Proof. The inequality $N_n(x^{(m)}-x)\prec\mathbb{D}\varepsilon$ is exactly the two strict component inequalities. Quantifying over $\varepsilon_1,\varepsilon_2$ proves convergence; replacing x by $x^{(q)}$ proves the Cauchy statement. □

Theorem 4.4. Completeness. X is a bicomplex Banach module if and only if X_1 and X_2 are complex Banach spaces. If this holds, every fixed power X^n is complete under every decomposable N_n.
Proof. A Cauchy sequence in X is a pair of component Cauchy sequences, so component completeness gives a recombined limit. The converse follows by embedding pure component sequences. For fixed n, Proposition 3.5 makes N_n,j equivalent to a finite product norm on X_j^n, which is complete when X_j is Banach. □
Definition 4.5. Two decomposable multi-norms N and M are uniformly equivalent if there exist invertible $c,C\in\mathbb{D}^+$, independent of n, such that $cN_n\preccurlyeq_{\mathbb{D}}M_n\preccurlyeq_{\mathbb{D}}CN_n$ for all n.
Proposition 4.6. Uniform equivalence holds exactly when both component multi-norm sequences are uniformly equivalent. Fixed-level equivalence only requires component equivalence at that n.
Proof. Projection onto $\mathbf{e}_j$ gives necessity. Conversely, combine the component constants as $c=c_1\mathbf{e}_1+c_2\mathbf{e}_2$ and $C=C_1\mathbf{e}_1+C_2\mathbf{e}_2$. Positivity of every coefficient makes these constants invertible. □

5. Subspaces, Products, and Quotients

Proposition 5.1. Every bicomplex submodule $M\subset X$ inherits a multi-norm by restriction. It is complete if and only if M is closed in X and X is complete.
Proof. Restriction preserves every axiom. A closed subspace of each Banach component is complete. Conversely, a complete subspace of a metric space is closed; apply this to the product topology. □
Theorem 5.2. Product construction. If X and Y have decomposable multi-norms N and M, define

$$P_n((x_1,y_1),\dots,(x_n,y_n))=\max_{\mathbb{D}}\{N_n(x_1,\dots,x_n),M_n(y_1,\dots,y_n)\}. \tag{5.1}$$

Then P is a multi-norm on X×Y, complete when X and Y are complete.
Proof. All axioms hold componentwise for the maximum of two nonnegative real multi-norms. For the triangle inequality, $\max(a+c,b+d)\leq\max(a,b)+\max(c,d)$. Appending zero and repetition give equality because both constituent multi-norms do. Completeness is product completeness. □
Theorem 5.3. Minimum quotient multi-norm. Let M be a closed submodule of X and suppose X carries the minimum multi-norm. On X/M define

$$Q_n(x_1+M,\dots,x_n+M)=\max_r \inf_{m\in M}\|x_r+m\|_{\mathbb{D}}. \tag{5.2}$$

Then Q is the minimum multi-norm induced by the quotient hyperbolic norm and X/M is complete when X is complete.
Proof. The quotient norm is componentwise the standard complex quotient norm and is definite because M is closed. Equation (5.2) is simply the coordinate maximum of that norm, so permutation, scalar domination, zero stability, and repetition are immediate. Component Banach quotient theorems give completeness. □
Remark 5.4. For an arbitrary multi-norm, taking an infimum over independent representatives at each level need not transparently preserve repetition. A general quotient theory requires additional hypotheses or a specifically chosen quotient multi-norm; the minimum case avoids this ambiguity.
Example 5.5. Let $X=\mathscr{B}\mathbb{C}^2$ with the component Euclidean minimum multi-norm and $M=\{(z,0):z\in\mathscr{B}\mathbb{C}\}$. Then X/M is isomorphic to $\mathscr{B}\mathbb{C}$ through $(z,w)+M\mapsto w$, and Q_n is the coordinate maximum of the hyperbolic moduli of the second coordinates.
Proof. The map is well defined, surjective, and has kernel M. Component Euclidean distance to M removes the first coordinate and leaves $|w|_{\mathbb{D}}$. The quotient multi-norm follows from Equation (5.2). □

6. Continuous Dual Structures

Definition 6.1. The continuous bicomplex dual X' consists of bounded bicomplex-linear maps $f:X\to\mathscr{B}\mathbb{C}$, with $\|f\|_{\mathbb{D}}=\|f_1\|\mathbf{e}_1+\|f_2\|\mathbf{e}_2$.

Theorem 6.2. Dual decomposition. There is an isometric module isomorphism

$$X'=(X_1')\mathbf{e}_1\oplus(X_2')\mathbf{e}_2,\quad f(x)=f_1(x_1)\mathbf{e}_1+f_2(x_2)\mathbf{e}_2. \qquad (6.1)$$

Proof. Bicomplex linearity makes f preserve idempotent components. Boundedness is the pair of complex bounds, yielding $f_j\in X_j'$. Conversely, a pair of bounded complex functionals defines a bounded bicomplex functional. The operator norm coefficients are exactly the component dual norms. □

Corollary 6.3. Hahn–Banach extension for a bicomplex submodule reduces to independent complex Hahn–Banach extensions. Norm preservation holds componentwise.

Proof. Decompose the submodule and functional, extend f_1 and f_2 by the complex Hahn–Banach theorem, and recombine. This is consistent with established bicomplex Hahn–Banach results (Luna-Elizarrarás et al., 2014). □

Remark 6.4. A multi-norm on X does not determine a unique dual multi-norm. One may place the minimum multi-norm on X', use a multi-dual construction inherited from the components, or impose operator-space structure. The choice must be stated rather than treated as canonical.

7. Finite-Dimensional Characterization

Theorem 7.1. If $\dim_{\mathbb{C}}X_1$ and $\dim_{\mathbb{C}}X_2$ are finite, every hyperbolic norm on X is equivalent to every other, every submodule is closed, and every decomposable multi-norm is fixed-level equivalent to the minimum multi-norm.

Proof. All norms on a finite-dimensional complex vector space are equivalent and every subspace is closed. Apply these facts to each component. At level n, X_j^n remains finite dimensional, so N_{n},j is equivalent to the coordinate-maximum product norm. Recombine the two estimates. □

Example 7.2. The pure submodule $\mathbf{e}_1\mathbb{C}^m\subset\mathscr{B}\mathbb{C}^m$ has nonzero vectors with norm values $a\mathbf{e}_1$ in the null cone. It is nevertheless a complete normed bicomplex module in its inherited topology.

Proof. Definiteness requires only that $a\mathbf{e}_1=0$ imply $a=0$, not that $a\mathbf{e}_1$ be invertible. The submodule is isomorphic to the finite-dimensional complex space $\mathbb{C}^m$ and hence complete. This example rules out any definition demanding strictly positive coefficients for every nonzero vector. □

7.3 Boundedness, Compactness, and Separation

Definition 7.3. A subset $A\subset X$ is hyperbolically bounded when there exists $C=C_1\mathbf{e}_1+C_2\mathbf{e}_2\in\mathbb{D}^+$ such that $\|x\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}C$ for all $x\in A$. It is relatively compact when its closure is compact in the product topology.

Proposition 7.4. A is bounded, totally bounded, relatively compact, or compact if and only if both component projections A_1 and A_2 have the corresponding property in X_1 and X_2.

Proof. Boundedness is immediate from the two norm coefficients. Product metric spaces are totally bounded exactly when both projections are totally bounded, and compactness of a subset of a finite product is equivalent to compactness of its closed projections together with closedness in the product. For A itself, sequence or net characterizations yield the same paired criterion. □

Theorem 7.5. Finite-dimensional Heine–Borel theorem. If X_1 and X_2 are finite dimensional, a subset $A\subset X$ is compact exactly when it is closed and hyperbolically bounded.

Proof. Closed bounded subsets of each finite-dimensional complex component are compact. The product of the two component closures is compact, and a closed subset A of it is compact. The converse holds in every normed product space because compact sets are closed and bounded. □

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Proposition 7.6. All standard real scalar metrics built monotonically from the two component norms generate the same topology. In particular,

$$d_\infty(x,y)=\max(\|x_1-y_1\|,\|x_2-y_2\|), \qquad (7.1)$$

$$d_2(x,y)=\sqrt{(\|x_1-y_1\|^2+\|x_2-y_2\|^2)}$$

are equivalent with $d_\infty \le d_2 \le \sqrt{2}d_\infty$.

Proof. The displayed real inequalities are elementary. Hence the two metrics have the same convergent sequences and open sets. They metrize the topology but do not replace the $\mathbb{D}^+$-valued homogeneous norm in module identities. □

Theorem 7.7. Separation by the bicomplex dual. If $x \notin M$ for a closed submodule M of a normed bicomplex module, there exists $f \in X'$ such that $f|M=0$ and $f(x)\neq 0$.

Proof. At least one component x_j does not belong to the closed component subspace M_j. The complex Hahn–Banach separation theorem supplies f_j vanishing on M_j but not at x_j. Set the other component functional to zero and recombine. The resulting value may lie in the null cone, but it is nonzero and separates x from M. □

Corollary 7.8. The weak topology $\sigma(X,X')$ is the product of the weak topologies $\sigma(X_1,X_1')$ and $\sigma(X_2,X_2')$. Weak convergence is therefore componentwise.

Proof. Every bicomplex functional is a pair of component functionals, and pure functionals test each component independently. □

Proposition 7.9. A bicomplex-linear map $T:X\to Y$ is compact if and only if its component maps are compact. When X is finite dimensional, every continuous T is compact.

Proof. A bounded sequence is a pair of bounded component sequences. Successive subsequence selection proves sufficiency; pure sequences prove necessity. In finite dimensions, bounded sets have compact closure by Theorem 7.5, so continuous images are relatively compact. □

Example 7.10. Let $X=\mathscr{B}\mathbb{C}^2$ and $M=\mathbf{e}_1\mathrm{span}\{(1,0)\}\oplus\mathbf{e}_2\mathrm{span}\{(0,1)\}$. Then M is closed, and the quotient rank vector is (1,1). The quotient minimum norm is computed by deleting the constrained first coordinate in component one and the constrained second coordinate in component two.

Proof. The component quotient spaces are $\mathbb{C}^2/\mathrm{span}\{(1,0)\}$ and $\mathbb{C}^2/\mathrm{span}\{(0,1)\}$, each one dimensional. Orthogonal representatives are (0,x_{1,2}) in the first component and (x_{2,1},0) in the second. Their norms give the quotient coefficients. □

Proposition 7.11. Let $T:X\to Y$ be continuous and $M\subset X$ a closed submodule with $T(M)=\{0\}$. The induced map $\tilde{T}:X/M\to Y$ is continuous and $\|\tilde{T}\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|T\|_{\mathbb{D}}$. If ran T is closed and T is surjective onto ran T, then X/ker T is topologically isomorphic to ran T.

Proof. For any representative x+m, Tx is unchanged. Taking the infimum of $\|x+m\|$ over $m\in M$ gives the quotient estimate. When X and ran T are Banach, the induced bijection is bounded and has bounded inverse by the component open mapping theorem. □

These structural results clarify how topology enters the algebraic product. Compactness and weak convergence are componentwise, but a chosen real metric is only an auxiliary device. Separation functionals can be pure idempotent and therefore have null-cone norm or values. Such functionals remain mathematically effective; requiring invertibility would destroy the dual’s ability to detect one-channel differences.

The quotient statement also separates algebra from analysis. Algebraically, X/ker T is always isomorphic to ran T. Topologically, the inverse is continuous only under closed-range and completeness hypotheses. This distinction is essential for operator equations and for any attempt to define bicomplex Banach multi-module exact sequences.

7.12 Reflexivity and Weak Compactness

Definition 7.12. A bicomplex Banach module X is reflexive when the canonical map $J:X\to X''$ is onto, where both duals are continuous bicomplex duals.

Theorem 7.13. X is reflexive if and only if X_1 and X_2 are reflexive complex Banach spaces.

Proof. The continuous dual decomposition gives $X'=X_1'\mathbf{e}_1\oplus X_2'\mathbf{e}_2$ and hence $X''=X_1''\mathbf{e}_1\oplus X_2''\mathbf{e}_2$. The canonical map is $J=J_1\mathbf{e}_1+J_2\mathbf{e}_2$. It is surjective exactly when both component canonical maps are surjective. □

Corollary 7.14. The closed hyperbolic unit ball of X is weakly compact if and only if X is reflexive.

Proof. The unit ball is the product of the two component unit balls under the weak product topology. By the complex Kakutani characterization, each component ball is weakly compact exactly when the component space is reflexive. Finite products preserve compactness. □

Proposition 7.15. A bounded sequence has a weakly convergent subnet in X whenever both component spaces are reflexive. In separable reflexive components, bounded sequences have weakly convergent subsequences.

Proof. Weak compactness of the component balls gives convergent subnets; use product compactness. Under the standard metrizability conditions on bounded weak sets supplied by separability of appropriate dual structures, the component subsequences can be selected successively and recombined. □

Reflexivity provides a clean example of a property that is completely componentwise yet analytically consequential. It supports weak compactness arguments in variational problems and operator equations. However, a multi-norm can impose additional family geometry not detected by the base-space bidual, so “multi-reflexivity” would require a separate definition involving multi-duals or operator ideals.

8. Applications and Interpretive Analysis

The product topology is compatible with continuous bicomplex models, which are characterised by the existence of two complex channels with distinct tolerances. An evaluation bound $|f(z)|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}C(z)\|f\|_{\mathbb{D}}$ is a pair of complex evaluation bounds in analytic function spaces. In sequence spaces, coordinate convergence and completeness are also reduced to paired classical statements.

Constraints and identifiability are significantly influenced by subspace and quotient constructions. Admissible states can be represented by a closed submodule, while the quotient identifies vectors that differ by an unobservable component.. The family norm is rigorously controlled by the minimal quotient multi-norm. Product multi-norms represent coupled systems without imposing a total order between channel magnitudes.

Optimisation conditions and weak topologies are componentized through dual decomposition. $f(x)=0$ is a bicomplex linear constraint that consists of two complex constraints. A family of functionals separates points precisely when both component families do. These facts are essential for operator equations, reproducing kernels, and bicomplex learning models; however, their practical utility is contingent upon the application's provision of meaningful channel coupling.

9. Open Problems and Limitations

The current framework provides coverage for multi-norms that are idempotent-decomposable. Nondecomposable gauges may encode cross-channel interaction; however, they necessitate a novel positivity and axiom analysis. Quotient multi-norms that exceed the minimum construction are not automatic, and a canonical dual multi-norm is not available in the absence of an additional category or tensor structure.

The following are open problems: reflexivity criteria, tensor products, bicomplex Banach lattices, C*-modules, interpolation, multi-duals, and uniform equivalence. Additionally, weak and weak-star compactness are included. The two coefficients should be retained in computational methods, and division by null-cone norms should be avoided. The broad testing ground for these questions is suggested by connections with recent bicomplex Lebesgue and sequence spaces (Değirmen & Sağır Duyar, 2022; Toksoy & Sağır Duyar, 2024).

10. In summary

A bicomplex multi-norm is correctly understood as a $\mathbb{D}^+$-valued structure that is associated with the hyperbolic modulus. The axioms are generated by idempotent component multi-norms, which are uniquely recovered under decomposability. The product topology is generated by hyperbolic spheres. The exact component criteria for convergence, completeness, equivalence, subspace structure, products, quotients in the minimum case, and continuous duals are all met. Null-cone norms may be present in pure vectors without compromising definiteness. The topology can be metrized by real scalarizations; however, they do not serve as a substitute for bicomplex homogeneity. The bicomplex Banach multi-module theory that emerges is structurally transparent and reveals the precise tasks that remain for research in weak-topology, tensors, duals, and quotients.

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