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A Family of Hailstone Maps on the Primes

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Abstract

We study prime-to-prime iterated maps in which the branch is selected by the residue of the current prime modulo six, with the output obtained by rounding a real or integral target to a nearby prime. Two motivating maps arise from OEIS entries A365048 and A367479. We embed the affine map underlying A365048 in a two-parameter family and organize it with square-root and hybrid variants in a common growth–decay framework. An exact criterion is proved for consecutive parameter values to agree at a given affine-branch state, together with a congruence condition under which this agreement holds unconditionally. We also obtain a necessary imbalance between affine and decay steps in any sufficiently large cycle. Finite computations extend and correct the reported parameter searches, compare stopping-time growth on a common range of starting primes, and exhibit explicit cycles for nearby modified maps. Universal termination is left as conjecture.

Keywords: Integer Sequences, Primes, Discrete Dynamics, Collatz-Type Maps, Prime Gaps.

1. Introduction

A hailstone sequence is generated by iterating a map whose trajectory typically exhibits a mixture of growth and decay steps before entering a terminal state or a periodic cycle. The best-known example is the Collatz map, which sends an integer n to n/2 when n is even and to 3n+1 when n is odd. Whether every positive integer eventually reaches 1 remains open; see Lagarias [9] for a thorough survey.

The maps introduced here act only on primes. Because every prime greater than 3 is congruent to either 1 or 5 modulo 6, this residue provides a natural two-way branch. The present author introduced two such prime-to-prime maps in OEIS entries A365048 and A367479 [11, 12]. The entry A365048 counts how many steps an odd prime takes to reach 3 under an affine target followed by prime rounding. The entry A367479 uses a square-root increment on the growth branch and a square-root decay, counting steps to the terminal prime 2.

These systems are of interest as elementary prime-only dynamical systems whose behavior is simultaneously shaped by congruence arithmetic and by the local distribution of primes. The affine branch resembles an accelerated 3x±1 step, but prime rounding inserts the geometry of prime gaps into every iteration, making the maps suitable both for integer-sequence study and for structural questions about cycles and stopping times.

1.1. Related Work

The broader literature on Collatz-type dynamics informs the context of the present work. Belaga and Mignotte [2] analyzed the cyclic structure of 3x+d systems, giving a framework for how cycle lengths depend on the integer parameter d in maps closely related to the family studied here. Applegate and Lagarias [1] derived lower bounds for the total stopping time of 3x+1 iterates, showing that stopping-time growth reflects the interplay between expansion and contraction rates. Simons and de Weger [13] established theoretical and computational bounds for m-cycles of the 3n+1 problem, combining geometric-mean estimates with machine-assisted cycle exclusion. Choi [4, 5] connected modified Collatz maps on ternary strings to Jacobsthal numbers, and Hercher [8] ruled out nontrivial Collatz cycles of length at most 91 by a careful computer-assisted argument. Bradshaw [3] studied primes arising from an arithmetic differential equation built around the Collatz map. None of these works considers maps restricted to the primes, so the present paper adds a complementary perspective.

1.2. Contributions

The paper makes four main contributions. First, a two-parameter family H(ε, X) is defined, embedding A365048 as the special case H(5, −1) and placing the square-root and hybrid variants in the same framework (Definitions 2.1–2.4, Table 1). Second, Theorem 3.5 gives an exact local criterion for blocks of consecutive parameters X, X+2, …, X+2r to produce identical images at a fixed prime, and Proposition 3.7 identifies a congruence condition under which agreement holds for every prime simultaneously. Third, Proposition 3.14 proves a necessary ratio between affine and decay steps in any cycle lying entirely above a computable threshold, extending the classical geometric-mean argument to the prime-rounding setting. Fourth, the parameter screening of A365048 is corrected and extended: fourteen previously listed parameters fail a deeper screen, and the corrected lists are presented in Section 5.

Throughout, universal termination is not claimed; every finite computation is described as a screen, not a proof. The maps are also distinguished carefully from the classical Collatz map, and no implication in either direction is claimed.

1.3. Paper Organization

Section 2 defines the family of maps. Section 3 establishes their structural properties. Section 4 describes the computational method. Section 5 reports the computational evidence and parameter screens. Section 6 compares stopping-time growth across maps. Section 7 gives explicit cycles for nearby modified maps, and Section 8 lists open problems. Section 9 concludes.

2. Prime Hailstone Maps

For a real number x, let

$$\mathrm{P}^{+}(x) = \min\{q : q \text{ is prime and } q \geq x\}$$

For x ≥ 2, let

$$\mathrm{P}^{-}(x) = \max\{q : q \text{ is prime and } q \leq x\}$$

denote the largest prime q with q ≤ x (the prime floor). The first operator is defined for all real x; the second is used only when its argument is at least 2.

Definition 2.1 (Affine prime hailstone map). Let ε ∈ {1, 5} and let X be odd. Set $H_{\varepsilon,X}(2) = 2$ and $H_{\varepsilon,X}(3) = 3$. For a prime p ≥ 5, define

$$H_{\varepsilon,X}(p) = \begin{cases} \mathrm{P}^{+}\left(\dfrac{3p+X}{2}\right), & p \equiv \varepsilon \pmod 6, \\[2ex] \mathrm{P}^{-}\left(\dfrac{p+1}{2}\right), & p \not\equiv \varepsilon \pmod 6. \end{cases}$$

The first branch is called the affine branch and the second the halving branch. The affine branch is expanding whenever p + X ≥ 2.

The convention $H_{\varepsilon,X}(3) = 3$ matches the stopping rule in A365048. The fixed value at 2 is irrelevant when starting from an odd prime but makes the map defined on the full set of primes.

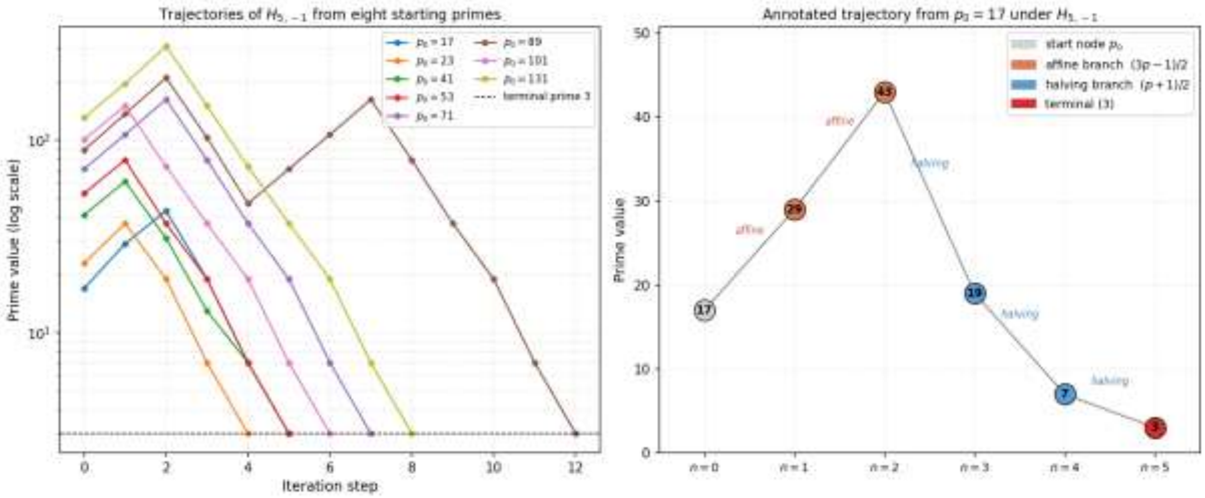
Example 2.2. The map $H_{5,-1}$ underlies A365048. Starting from the prime 17 gives the trajectory

$17,\ 29,\ 43,\ 19,\ 7,\ 3.$

The stopping time to 3 is therefore 5, as recorded in A365048. Figure 1 illustrates this trajectory (right panel) and compares it with several other converging trajectories (left panel).

Remark 2.3. The affine target is $(3p + X)/2$. For $X = 1$ this is the standard accelerated $3p + 1$ target before prime rounding; for $X = -1$ it is the accelerated $3p - 1$ target used in A365048. Prime rounding is essential because the target itself need not be prime.

Figure 1. Left: Trajectories of $H_{5,-1}$ from eight starting primes on a log scale, all converging to the terminal prime 3. Right: Annotated trajectory from $p_0 = 17$ (Example 2.2). Orange nodes were reached via an affine step; blue nodes via a halving step; the red terminal node is 3.



Definition 2.4 (Square-root and hybrid maps). Set $H_{\mathrm{sqrt}}(2) = 2$. For a prime $p \geq 3$, define

$$H_{\mathrm{sqrt}}(p) = \begin{cases} \mathrm{P}^{+}\big(p + \lceil\sqrt{p}\rceil\big), & p \equiv 5 \pmod 6, \\ \mathrm{P}^{-}\big(\lceil\sqrt{p}\rceil\big), & p \not\equiv 5 \pmod 6. \end{cases}$$

The hybrid map H_{hyb} uses the same root-increment branch as H_{sqrt}, but replaces the root-decay branch with the halving branch

$$H_{\mathrm{hyb}}(p) = \mathrm{P}^{-}\left(\frac{p+1}{2}\right) \qquad \text{when } p \not\equiv 5 \pmod 6$$

with $H_{\mathrm{hyb}}(2) = 2$.

The map H_{sqrt} underlies A367479. Table 1 organizes all four combinations of growth and decay branches; it also clarifies the distinction between target values and jump sizes, which becomes important in the stopping-time comparison of Section 6.

Table 1. Four Combinations of an Upper Branch and a Decay Branch. A365048 Is $H_{5,-1}$, and A367479 Is H_{sqrt}.

	halving target (p+1)/2	**root target ⌈√p⌉**
affine target (3p+X)/2	H(5,X) [and H(1,Y)]	not investigated
root-incr. target p+⌈√p⌉	H_{hyb}	H_{sqrt} [= A367479]

For a terminating trajectory, write $\tau_3(p; H_{\varepsilon,X})$ for the number of iterations to reach 3, and write $\tau_2(p; F)$ for the number to reach 2 under F = H_{sqrt} or H_{hyb}.

3. Basic Properties

Lemma 3.1 (Well-definedness and an explicit rounding bound). The maps in Definitions 2.1 and 2.4 are well defined. Let t = (3p + X)/2. If t > 396738, then

$$0 \le \mathrm{P}^{+}(t) - t \le \frac{t}{25\log^2 t}$$

Similarly, if u = p + ⌈√p⌉ > 396738, then

$$0 \le \mathrm{P}^{+}(u) - u \le \frac{u}{25\log^2 u}$$

Proof. The decay targets satisfy (p + 1)/2 ≥ 3 for p ≥ 5 and ⌈√p⌉ ≥ 2 for p ≥ 3, so the corresponding intervals contain a prime. The existence of $P^+(x)$ follows from the infinitude of the primes. For x > 396738, Dusart [7] proved that the interval [x, x + x/(25 log² x)] contains a prime; applying this to x = t and x = u gives both bounds. □

Remark 3.2. The explicit form in Lemma 3.1 matters here. A bound stated as O(p/log²p) would be valid only after fixing X and taking p large enough; it is not uniform over all odd X.

Lemma 3.3 (Strict decay). Every halving or root-decay step strictly decreases the current prime. More precisely,

$$\mathrm{P}^{-}\left(\frac{p+1}{2}\right) < p \quad (p > 1), \qquad \mathrm{P}^{-}(\lceil\sqrt{p}\rceil) < p \quad (p \ge 3)$$

Proof. The prime output does not exceed its target, and (p + 1)/2 < p for p > 1 and ⌈√p⌉ < p for p ≥ 3. □

Lemma 3.4 (When the upper branch expands). For p ≥ 5, the affine branch of $H_{\varepsilon,X}$ strictly increases p whenever p + X ≥ 2; in particular it is always increasing when X ≥ −3. The root-increment branch of H_{sqrt} and H_{hyb} is strictly increasing for every prime p ≥ 3.

Proof. If p + X ≥ 2 then (3p + X)/2 ≥ p + 1, so its prime ceiling exceeds p. The root-increment target is at least p + 2 for every prime p ≥ 3. □

The next theorem explains the bands of agreeing parameter values visible in the empirical lists of Section 5.

Theorem 3.5 (Agreement of a block of parameters). Let $p \equiv \varepsilon \pmod 6$ be prime, let X be odd, and set $m = (3p + X)/2$. For an integer $r \geq 1$, the values

$H_{\varepsilon,X}(p),\ H_{\varepsilon,X+2}(p),\ \ldots,\ H_{\varepsilon,X+2r}(p)$

all agree if and only if none of m, m+1, …, m+r−1 is prime. When this condition holds, their common value is $P^{+}(m + r)$.

Proof. The successive affine targets are m, m+1, …, m+r. If none of m, m+1, …, m+r−1 is prime, then no prime lies between any of the first r targets and m+r. Hence all r+1 targets have the same prime ceiling, namely $P^{+}(m+r)$. Conversely, if m+j is prime for the smallest $j \in \{0, \ldots, r-1\}$, then targets m through m+j round to m+j, whereas m+j+1 rounds to a prime strictly larger than m+j; the block therefore lacks a common image. □

Corollary 3.6 (Adjacent-parameter criterion). With the notation of Theorem 3.5,

$$H_{\varepsilon,X}(p) = H_{\varepsilon,X+2}(p) \quad \Leftrightarrow \quad \frac{3p+X}{2} \text{ is not prime}$$

Proposition 3.7 (An unconditional case of agreement). Let $\varepsilon \in \{1, 5\}$. If X is odd, $X \geq -3$, and $3 \mid X$, then $H(\varepsilon,X) = H(\varepsilon,X+2)$ as maps on the primes.

Proof. Write $m = (3p + X)/2$. Since $2m = 3p + X \equiv X \pmod 3$ and 2 is invertible modulo 3, we have $m \equiv 2X \pmod 3$. If $3 \mid X$ then $3 \mid m$; if also $X \geq -3$ and $p \geq 5$ then $m \geq (15-3)/2 = 6$, so m is composite. Corollary 3.6 then yields $H(\varepsilon,X)(p) = H(\varepsilon,X+2)(p)$ for every prime $p \equiv \varepsilon \pmod 6$. The decay branch is independent of X, so the two maps agree everywhere. □

Remark 3.8. Proposition 3.7 accounts for 45 of the 53 adjacent pairs (X, X+2) in the list of Section 5.1, and 11 of the 13 adjacent pairs in the corrected list of Section 5.2; in each case these are exactly the pairs whose smaller member is divisible by 3. The remaining adjacent pairs are addressed only by the heuristic of Remark 3.11.

Proposition 3.9 (Congruence obstructions to disagreement). Let $p \equiv \varepsilon \pmod 6$ be prime, let X be odd, and put $m = (3p + X)/2 > 3$. If $H(\varepsilon,X)(p) \neq H(\varepsilon,X+2)(p)$, then $3 \nmid X$ and $4 \nmid (p - X)$.

Proof. By Corollary 3.6, disagreement forces m to be prime. If $3 \mid X$ then $3 \mid m$, contradicting primality when $m > 3$. If $p \equiv X \pmod 4$, then $3p + X \equiv 4X \equiv 0 \pmod 4$, so m is even and greater than 3; again a contradiction. □

Corollary 3.10 (Orbitwise agreement). Fix $p_0 \geq 5$. Suppose that along the $H_{\varepsilon,X}$ trajectory from p_0, every affine-branch state p satisfies $(3p + X)/2 \notin \mathbb{P}$. Then the trajectories of $H_{\varepsilon,X}$ and $H(\varepsilon,X+2)$ from p_0 are identical until their first terminal value or repeated state.

Proof. Apply Corollary 3.6 inductively at every affine-branch state; the decay branch does not depend on X. □

Remark 3.11 (Prime-density heuristic after congruence sieving). Corollary 3.6 says that adjacent parameters disagree at an affine state exactly when m = (3p + X)/2 is prime. Proposition 3.9 shows that m is automatically composite when 3 | X or p ≡ X (mod 4). When neither condition holds, m is odd and coprime to 3, so by a standard prime-density estimate the crude probability of disagreement at a single affine state is roughly 3/log m. This is used only as qualitative motivation; it does not predict the global band counts.

Proposition 3.12 (Branch composition of cycles). Assume X ≥ −3. Every nontrivial cycle of $H_{\varepsilon,X}$ outside the terminal fixed points uses at least one affine step and at least one halving step. The analogous statement holds for H_{sqrt} and H_{hyb}.

Proof. A cycle consisting only of decay steps is impossible by Lemma 3.3. A cycle consisting only of affine steps is impossible by Lemma 3.4 under the stated condition. □

Remark 3.13 (Scope of Proposition 3.12 for X < −3). The argument requires X ≥ −3 to guarantee that the affine branch always expands. For X < −3, the affine target (3p + X)/2 is strictly below p when p < −X and equals p when p = −X. The equality case can itself create a fixed point; for example, H(5,−5)(5) = $P^+(5)$ = 5. For X = −9, the affine-state prime p = 5 satisfies p < 9, and one checks directly that H(5,−9)(5) = $P^+((15−9)/2) = P^+(3)$ = 3, a terminal value. Thus, for a general X < −3, every affine-state prime p ≤ −X must be checked individually before the branch-composition conclusion of Proposition 3.12 can be extended to that parameter. The finite computations of Section 5 provide such checks only for the parameter values and starting-prime ranges explicitly reported there.

Proposition 3.14 (A necessary growth–decay ratio in large cycles). Fix odd X and 0 < δ < 1/2. There exists P = P(X, δ) with the following property. Let C be a cycle of $H_{\varepsilon,X}$ all of whose elements exceed P. If g affine steps and d halving steps occur in one traversal of C, then

$$\frac{d}{g} \le \frac{\log(3/2+\delta)}{-\log(1/2+\delta)}$$

For every η > 0 there is a threshold $P_\eta = P_\eta(X)$ such that every such cycle satisfies

$$\frac{d}{g} \le \log_2(3/2) + \eta = 0.5849625\ldots$$

Proof. For an affine step with target t = (3p + X)/2 > 396738, Lemma 3.1 gives

$$\frac{H_{\varepsilon,X}(p)}{p} \le \left(\frac{3}{2}+\frac{X}{2p}\right)\left(1+\frac{1}{25\log^2 t}\right)$$

For fixed X, the right-hand side is at most 3/2 + δ once p is large enough. Similarly,

$$\frac{P^-((p+1)/2)}{p} \le \frac{p+1}{2p} \le \frac{1}{2}+\delta$$

for large p. Choose P so that both bounds hold for p > P. Since the product of the step ratios around any cycle equals 1,

$$1 \le \left(\frac{3}{2}+\delta\right)^g \left(\frac{1}{2}+\delta\right)^d$$

and taking logarithms yields the stated inequality. □

Remark 3.15. Proposition 3.14 is the prime-rounding analogue of the classical geometric-mean argument for the 3x+1 map; compare Lagarias [9] and Simons and de Weger [13]. The result is necessary but not sufficient: it constrains the branch composition of any cycle lying entirely above the threshold P, but it does not by itself exclude such cycles and gives no conclusion for cycles that pass through smaller primes.

4. Computational Method

Each iteration requires either $P^+(x)$ or $P^-(x)$. For the forward search $P^+(x)$, Lemma 3.1 gives an explicit upper bound on the distance to the next prime when $x > 396738$. No corresponding backward-gap bound is needed for the computations reported here: the implementation uses SymPy's adjacent-prime routines directly. Primality testing and prime searching are therefore described operationally rather than by a uniform asymptotic complexity bound; see Crandall and Pomerance [6] for background on computational primality testing.

All computations used exact Python integer arithmetic. Primality tests and adjacent-prime searches were supplied by SymPy's isprime, nextprime, and prevprime functions (Meurer et al. [10]). The original computations used Python 3.13.5 with SymPy 1.14.0; the independent rescreens of Section 5 used Python 3.12.3 with SymPy 1.14.0. The repeated-state cycle certificates are exact once the visited states have been identified; the computational primality-testing assumptions are those of the cited SymPy implementation.

For a starting prime p, cycle detection works as follows. Each visited prime and its first index are stored. Iteration stops if the designated terminal prime is reached. If a stored prime reappears, the algorithm returns the preperiod length and the repeated-state cycle. A step-count limit is used only as a safety guard; it is never interpreted as evidence of nontermination. Every reported cycle is therefore accompanied by a repeated-state certificate, and every statement called exhaustive means that every starting prime in the stated finite range was tested. No aggregate CPU time is reported, because the original searches were carried out in several interactive sessions without a unified timing log.

5. Computational Evidence

Before reporting the computations, it is worth being precise about what the finite checks establish. A map passes a screen if every starting prime in a specified finite test set reaches the designated terminal prime. A parameter fails a screen if at least one tested prime enters a cycle that avoids the terminal prime. No finite screen, however deep, constitutes a proof of universal termination; the language in this section is chosen accordingly.

For $\varepsilon \in \{1, 5\}$, $T > 0$, and $N \geq 1$, let

$$S_\varepsilon(T,N) = \left\{ Z : \begin{array}{l} Z \text{ is odd, } |Z| \leq T, \text{ and each of the first } N \\ \text{odd-prime starts reaches 3 under } H_{\varepsilon,Z} \end{array} \right\}$$

This notation is used only when every parameter in the stated interval was tested under the same value of N. Since a deeper screen can only remove parameters,

$$S_\varepsilon(T,N) \supseteq S_\varepsilon(T,N') \qquad \text{for } N \leq N'$$

every finite-screen set is therefore an upper bound for the set of universally terminating parameters.

Conjecture 5.1. For every prime $p \geq 5$, the orbit of p under $H_{5,-1}$ eventually reaches 3.

Conjecture 5.2. For every prime p, the orbit of p under H_{sqrt} eventually reaches 2.

Conjecture 5.3. For every prime p, the orbit of p under H_{hyb} eventually reaches 2.

Conjecture 5.1 was checked exhaustively for the first four million odd primes, up to 67867979, with no exception found and largest stopping time 261. An earlier author-reported check in the OEIS comments claimed verification up to 2041000859 [11]; this figure was not independently re-examined for this paper, and a spot check of 200 primes spread across that range found no exceptions.

Conjecture 5.2 was checked for the first 300000 primes, up to 4256233, with maximum stopping time 32. Conjecture 5.3 was checked for the first 20000 odd primes, up to 224743, with maximum stopping time 51.

Remark 5.4. The three verification depths are far from uniform: 4000000 starts for $H_{5,-1}$, 300000 for H_{sqrt}, and only 20000 for H_{hyb}. Conjecture 5.3 therefore rests on considerably weaker evidence than the other two; extending the exhaustive check for H_{hyb} to at least 300000 starting primes is a priority for follow-up work.

Table 2 summarizes these checks.

Table 2. Exhaustive Finite Verification of the Three Named Maps.

map	largest start	starts tested	exceptions	max. stopping time
$\mathbf{H_{5,-1}}$	67867979	4000000 odd primes	0	261 to 3
$\mathbf{H_{sqrt}}$	4256233	300000 primes	0	32 to 2
$\mathbf{H_{hyb}}$	224743	20000 odd primes	0	51 to 2

5.1. The Family H(5,X)

Earlier computations linked to A365048 screened odd parameters up to 29951, retaining only those for which every one of the first one million odd prime starts reaches 3. The present work reproduced the reported values in the interval $1999 \le X \le 3001$ (together with nearby failing parameters), then extended the search to $29951 < X < 32000$. This produced eleven further candidates, one of which was subsequently eliminated.

The parameter $X = 30007$ yields a repeated-state cycle certificate. Starting from the prime 227219, the trajectory under H(5,30007) has a preperiod of length 25 followed by a cycle of length 65; the smallest cycle element is 12209207 and the largest orbit value is 730351333. Since 227219 is the 20197th odd prime, it lies within the range of the original one-million-start screen. The retention of $X = 30007$ in the earlier list is therefore inconsistent with the stated criterion, indicating either a transcription error or a difference in the enumeration of starting primes used in the original computation. The present paper treats the earlier lists as historical records; every parameter displayed below was subsequently rechecked on the first 200000 odd-prime starts.

The remaining ten extension candidates are

30397, 30447, 30449, 30589, 30747, 30749, 30753, 30755, 31089, 31091.

Every parameter in the list below survived the 200000-start recheck with no exception found; the largest observed stopping time was 734:

−9, −3, −1, 3, 5, 15, 17, 33, 35, 55, 75, 77, 87, 89, 103, 109, 201, 203, 219, 221, 309, 311, 529, 531, 533, 595, 1123, 1141, 1317, 1319, 1321, 1857, 1859, 2023, 2167, 2325, 2327, 2509, 2679, 2681, 2685, 2687, 2727, 2729, 2731, 2899, 3301, 3321, 3323, 3471, 3473, 3597, 3599, 3727, 3741, 3743, 4023, 4025, 4255, 4269, 4271, 4861, 4863, 4865, 5647, 6151, 6397, 6513, 6515, 7393, 8551, 8553, 8555, 8893, 8973, 8975, 8977, 8995, 9127, 9163, 9367, 9991, 10053, 10055, 10353, 10355, 11133, 11135, 11509, 13507, 13599, 13601, 14371, 15061, 15955, 19269, 19271, 19353, 19355, 19357, 20139, 20141, 20157, 20159, 21043, 21129, 21131, 21307, 23229, 23231, 24135, 24137, 24139, 24889, 25489, 25495, 26901, 26903, 27591, 27593, 28041, 28043, 28111, 29949, 29951, 30397, 30447, 30449, 30589, 30747, 30749, 30753, 30755, 31089, 31091.

Within the previously searched parameter windows, omitted values had been eliminated by detected cycles, but those eliminations were not all performed at the same screening depth. Consequently, the displayed survivor list must not be identified with S5(32000, 200000) unless every odd parameter in the full symmetric window $|X| \leq 32000$ is screened uniformly at depth 200000. What is established here is narrower: every parameter displayed above survived a 200000-start recheck, while the omitted parameters were eliminated in the earlier searches at their reported depths. This distinction prevents an unsupported set inclusion or equality claim.

Conjecture 5.5. For every X in the list above, the orbit of every prime $p \geq 5$ under H(5,X) eventually reaches 3.

Let $N_5(T)$ denote the number of odd X with $|X| \leq T$ for which H(5,X) terminates universally at 3.

Conjecture 5.6. The function $N_5(T)$ is unbounded as $T \to \infty$.

The corrected list contains 135 parameters, of which 122 are positive and below 30000. Grouping into maximal bands where successive gaps are at most 6 gives 76 bands: 32 singletons, 33 pairs, 8 triples, 2 quadruples, and 1 quintuple. Theorem 3.5 accounts for local agreement of consecutive parameter maps at a fixed prime, but a global theory of band counts must also track how disagreements at early states propagate through the dynamics; Remark 3.11 offers only qualitative guidance at present.

5.2. The Family H(1,Y)

The companion family H(1,Y) swaps the residue classes so that the affine branch acts on primes $p \equiv 1 \pmod 6$. In the earlier OEIS comments the decay condition was written in a form that left one residue class ambiguous; throughout this paper the decay branch is taken to be the complementary case $p \equiv 5 \pmod 6$, as in Definition 2.1.

The original parameter screen for $|Y| \leq 30000$ used 400 starting primes per parameter, and a two-stage extension to $Y \leq 50000$ used 600 starts for screening and 20000 starts for confirmation, yielding the additional survivor Y = 31321. All reported values were subsequently rescreened at 200000 starts.

Thirteen of the reported values fail the deeper screen:

8673, 8675, 10243, 10831, 11413, 13665, 13667, 13683, 13685, 25407, 25409, 29013, 29015.

In each case the trajectory enters a cycle that avoids 3. Table 3 records the exact certificates. All thirteen first fail beyond the 400th odd prime, which is consistent with their survival under the original shallower screen.

Table 3. Failure Certificates for the Thirteen Y-Values Eliminated by the 200000-Start Rescreen. Each Row Exhibits a Starting Prime Whose Orbit Is Eventually Periodic with a Cycle Avoiding 3.

Y	first failing p	preperiod	cycle len.	least cycle elt.	largest orbit
8673	130687	5	8	150991	799679
8675	130687	5	8	150991	799679
10243	128431	2	8	197767	791111
10831	106963	10	8	265621	701147
11413	145477	1	16	223759	1179173
13665	107581	17	8	301183	2111597
13667	107581	17	8	301183	2111597
13683	154723	1	8	238939	1265273

13685	154723	1	8	238939	1265273
25407	560491	1	8	560503	2242157
25409	560491	1	8	560503	2242157
29013	497509	1	8	577831	3043217
29015	497509	1	8	577831	3043217

Ten of the thirteen values form five adjacent pairs: (8673, 8675), (13665, 13667), (13683, 13685), (25407, 25409), and (29013, 29015). In each pair the smaller value is divisible by 3, so Proposition 3.7 shows the two maps are identical on all primes; their shared failure certificate is a consistency check on the computation.

The corrected candidate list $S_1(30000, 200000)$ is:

−5, 1, 115, 121, 175, 181, 187, 331, 363, 365, 367, 475, 877, 963, 965, 1255, 1257, 1259, 1273, 2425, 2449, 3351, 3353, 5265, 5267, 6757, 7113, 7115, 8281, 9183, 9185, 10305, 10307, 11247, 11249, 12559, 14391, 14393, 16951, 19063, 20797, 21697, 24367, 25005, 25007, 28015.

This gives 46 values, of which 45 are positive and below 30000. The extension value Y = 31321 also passes the 200000-start screen and remains an additional finite-screen survivor. The largest stopping time among all surviving parameters at this depth was 426.

Conjecture 5.7. For every Y in the corrected list above, and for Y = 31321, the orbit of every prime $p \geq 5$ under H(1,Y) eventually reaches 3.

6. Stopping-Time Growth

Table 4 compares $H_{5,-1}$ and H_{sqrt} on the common range of all primes up to 4256233, the upper limit of the exhaustive H_{sqrt} check. For $H_{5,-1}$ the starting prime 2 is omitted and stopping is at 3; for H_{sqrt} stopping is at 2.

Table 4. Mean Stopping Times by Number of Digits on the Common Range p ≤ 4256233. The Seven-Digit Row Is Truncated at 4256233. The Column N Gives the Count of Primes with the Stated Digit Count; Mean Stopping Time Is Reported for H5,−1 (Stopping at 3) and Hsqrt (Stopping at 2).

digits	**N (H(5,−1))**	**mean τ3**	**N (Hsqrt)**	**mean τ2**
1	3	1.0	4	2.8
2	21	5.4	21	4.8
3	143	13.0	143	8.0
4	1061	20.5	1061	6.9
5	8363	30.6	8363	8.9
6	68906	42.4	68906	10.6
7	221502	51.3	221502	8.4

Remark 6.1 (Stopping-time heuristic). For H5,−1, suppose that along a terminating family of trajectories the logarithmic drift is uniformly negative in the sense that $d - \log 2(3/2)g \geq c(g+d)$ for some constant $c > 0$, where g and d are the numbers of affine and halving steps. Since an affine step changes log2 p by roughly log2(3/2) and a halving step changes it by roughly −1, this condition suggests O(log p) total steps. For Hsqrt, a root-decay step approximately halves log2 p, while the root-increment branch

changes p by only about √p; under a comparable assumption that decay steps occur with a nonvanishing frequency, an O(log log p) scale is suggested. Neither estimate is unconditional. The data in Table 4 are consistent with this qualitative picture: the mean for $H_{5,-1}$ grows steadily across digit classes, while the mean for Hsqrt remains in a comparatively narrow band. The irregularity of the Hsqrt means cautions against fitting a precise asymptotic formula from these data alone.

7. Nearby Maps with Explicit Cycles

The examples in this section show that small changes to a branch can introduce nonterminal cycles. Exact certificates are given rather than step-count bounds.

7.1. Changing X = −1 to X = 1

For $H_{5,1}$, the prime 449 has preperiod

449, 677, 1019, 1531, 761, 1151, 1733

and then enters a cycle of length 84 beginning at 2609. Direct iteration confirms $H_{5,1}^{84}(2609) = 2609$, with all intermediate states distinct, so this trajectory never reaches 3.

7.2. Changing the Halving Target

Let G be obtained from $H_{5,-1}$ by replacing the halving target $(p + 1)/2$ with $(p + 3)/2$, while retaining $G(3) = 3$.

Proposition 7.1. The map G has the nonterminal cycles

(5, 7) and (11, 17, 29, 43, 23, 37, 19).

Proof. Direct substitution gives $5 \mapsto 7 \mapsto 5$. For the second cycle, $11 \mapsto 17 \mapsto 29 \mapsto 43 \mapsto 23 \mapsto 37 \mapsto 19 \mapsto 11$. Each arrow follows immediately from Definition 2.1 with the modified halving target. □

Among the first 2000 odd prime starts, the only eventual cycles of G found were the fixed point (3), the two-cycle (5, 7), and the seven-cycle of Proposition 7.1, with observed basin sizes 1, 3, and 1996, respectively. This is a finite observation; the classification of all cycles of G remains open.

7.3. A Modified Square-Root Decay

Replace the decay target in H_{sqrt} by $\lfloor\sqrt{(p+1)}\rfloor + 1$, leaving the root-increment branch unchanged. Under this modification, 3 is a fixed point because the new decay target at $p = 3$ is 3. In the first 1000 odd prime starts, every trajectory converged to this fixed point. As before, this is a finite observation, not a complete classification.

These three examples illustrate that termination is sensitive to the precise choice of branch targets. In particular, the examples demonstrate sensitivity to modifications of both growth and decay branches; they do not support a general ranking of one branch type as intrinsically more fragile than another.

8. Open Problems

1. Characterize the odd parameters X for which H(5,X) terminates at 3 from every prime, and the parameters Y for which H(1,Y) does so. Determine the growth rate of the counting functions N5(T) and N1(T), where N1(T) counts the odd Y with $|Y| \le T$ for which H(1,Y) terminates universally at 3.
2. Explain the band structure of the parameter lists. Theorem 3.5 gives an exact local agreement criterion, but a global account must track how disagreements at early states affect later targets in the orbit.
3. Investigate the fourth entry in Table 1: the combination of an affine upper branch with a square-root decay target.
4. Prove or refute Conjectures 5.1–5.3, without using the unresolved classical Collatz conjecture.

5. Strengthen Proposition 3.14. Can one exclude cycles with all elements above a specified threshold, or bound the small primes that every cycle must contain?
6. Classify the eventual cycles of the modified maps in Section 7. In particular, are the three cycles found for G the only ones?
7. Quantify how often the affine targets (3p + X)/2 generated by the dynamics are prime. Standard prime-density heuristics do not directly apply because the states p are themselves generated by the map under study.
8. Determine which parameters in the lists of Section 5 survive further deepening. The rescreens reported there already remove fourteen parameters from earlier lists, so stability under arbitrarily deep screens is open even within the computed windows.

9. Conclusion

This paper organized two OEIS sequences, A365048 and A367479, inside a common two-parameter family of prime-to-prime hailstone maps, together with a hybrid variant. Theorem 3.5 and its corollaries give an exact, checkable criterion for when neighboring parameters act identically on a given prime, and Proposition 3.7 isolates a congruence condition under which that agreement is unconditional; together these results help explain the banded structure visible in the parameter screens. Proposition 3.14 transports the classical geometric-mean argument for the 3x+1 map into the prime-rounding setting, giving a necessary bound on the ratio of affine to decay steps in any sufficiently large cycle. On the computational side, a 200000-start recheck of the retained candidates corrected earlier parameter lists, removing fourteen previously reported parameters that fail at greater depths and producing repeated-state cycle certificates for the failures. The examples of Section 7 show that the observed termination behavior is sensitive to small changes in the branch targets.

No claim of universal termination is made for any of the three named maps; each is left as an explicit conjecture, consistent with how the finite computations are described throughout. The open problems of Section 8, particularly the band-structure question and the possibility of strengthening the cycle-ratio bound into an effective exclusion result, set the main directions for follow-up work.

10. Data and Code Availability

The reference implementation, machine-readable screening results, and failure certificates for the eliminated parameters are available from the author upon reasonable request. The software versions and testing conventions used for the reported computations are stated in Section 4. A persistent public repository record may be added after publication. The finite-screen parameter lists and their testing conventions will also be submitted separately to the OEIS where appropriate.

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