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Eigenvalue-Based Stability of Nonlinear Caputo-Type Fractional Differential Systems: A Spectral and Operator-Theoretic Study

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Abstract
This paper develops a mathematically qualified stability framework for commensurate Caputo systems in which spectral geometry, nonlinear remainder estimates and fractional resolvent bounds are treated as distinct but connected ingredients. $For\ 0 < \alpha < 1,$ the autonomous problem $cD_{0^+}{}^{\alpha}x(t) = Ax(t) + f(x(t)), x(0) = x_0,$ is examined first in $\mathbb{C}^n$ and then in an abstract Banach-space setting. In finite dimensions, the sharp linear criterion $\sigma(A) \subset \left\{\lambda \in \mathbb{C}\ \{0\}: |\arg\lambda| > \frac{\alpha\pi}{2}\right\}$ is obtained from the asymptotics of the Mittag–Leffler function. A nonlinear linearisation result is then stated under the precise requirement $f(0) = 0\ and \frac{\|f(x)\|}{\|x\|} \to\ 0\ as\ x \to 0,$ together with local well-posedness. The paper distinguishes asymptotic stability from stronger Mittag–Leffler estimates and explains why a spectral inclusion alone is insufficient for an unbounded operator unless compatible sectorial resolvent bounds are available. A two-dimensional cubic system is analysed in detail, and a parameter-dependent example shows how fractional order can change the stability classification even though the matrix is fixed. The resulting framework supplies a reproducible sequence—equilibrium translation, Fréchet linearisation, spectral-sector test, remainder control and numerical verification—that can support doctoral research in fractional dynamics. The contribution is theoretical and interpretive: no unreported experimental data are claimed.

Keywords: Caputo derivative; fractional-order system; Mittag–Leffler function; linearisation; spectral sector; sectorial operator; asymptotic stability.

1. Introduction

Fractional differential equations furnish a nonlocal extension of ordinary differential equations. Their present state depends on a weighted history, and this feature permits the representation of hereditary response, anomalous transport and memory-dependent relaxation. The Caputo derivative is especially convenient for initial-value problems because the prescribed data have the same form as in classical dynamics (Caputo, 1967; Diethelm, 2010). Nevertheless, the replacement of an integer derivative by a fractional one changes the geometry of stability and the rate at which stable solutions approach an equilibrium.

For the classical system $x'=Ax$, asymptotic stability is equivalent to $\sigma(A)$ lying in the open left half-plane. For the commensurate Caputo system of order $\alpha\in(0,1)$, the corresponding region is the complement of a cone about the positive real axis. Thus an eigenvalue with positive real part can be admissible when its argument is sufficiently large. This observation, associated with Matignon's criterion, is elementary to state but is frequently applied without its scope being identified: it is a finite-dimensional linear result, and its nonlinear or infinite-dimensional use requires additional hypotheses (Matignon, 1996; Brandibur & Kaslik, 2021).

The purpose of the present study is to place the eigenvalue test inside a logically complete argument. The matrix problem is separated from the operator problem; stability, asymptotic stability and Mittag–Leffler estimates are not treated as interchangeable; and the nonlinear conclusion is tied to a genuine higher-order remainder. The presentation follows a theorem–proof–example structure suitable for a university mathematical research article.

2. Research Problem, Objectives and Method

2.1 Problem statement

Let X be either $\mathbb{C}^n$ or a Banach space. The central problem is to determine conditions under which the equilibrium $x=0$ of a Caputo evolution equation remains stable and attracts all sufficiently small initial states. The difficulty is that nonlocality prevents the direct transfer of semigroup arguments from first-order equations, while the nonlinear perturbation can invalidate conclusions drawn solely from eigenvalues.

2.2 Objectives

- Derive the integral and variation-of-constants representations used in the analysis;
- State the exact spectral-sector criterion for a finite-dimensional commensurate linear system;
- Formulate a defensible local nonlinear stability result with explicit remainder conditions;
- Clarify the extra resolvent assumptions needed in an operator-theoretic extension;
- Demonstrate the method on worked examples and identify its limitations.

2.3 Analytical method and scope

The research is analytical. It uses Laplace transforms, matrix Mittag–Leffler functions, spectral decomposition, asymptotic estimates and a Volterra fixed-point argument. The principal results concern $0<\alpha<1$ and autonomous commensurate systems. Numerical values in the examples are illustrative calculations derived from stated matrices; they are not empirical observations.

3. Preliminaries and Notation

Throughout, $\|\cdot\|$ denotes a compatible vector or operator norm, $\sigma(A)$ the spectrum of A, $\rho(A)$ the resolvent set, and $\arg\lambda$ the principal argument in $(-\pi,\pi]$. The symbol $B_r(0)$ denotes the open ball of radius r about the origin.

3.1 Fractional integral and Caputo derivative

For $\alpha>0$ and an integrable function $g:[0,T]\to X$, the left-sided Riemann–Liouville integral is

$$(I^{0+\alpha}g)(t) = \frac{1}{\Gamma(\alpha)\int^{0\ t}}(t-s)^{\alpha-1}g(s)ds. \qquad (1)$$

If x is absolutely continuous and $0<\alpha<1$, its Caputo derivative is

$$({}^{c}D_{0+}{}^{\alpha}x)(t) = (I_{0+}{}^{1-\alpha}x')(t) = \frac{1}{\Gamma(1-\alpha)}\int_0{}^{t} x'(s)/(t-s)^{\alpha}\,ds. \qquad (2)$$

Applying $I_{0+}{}^{\alpha}$ to ${}^{c}D_{0+}{}^{\alpha}x=g$ gives the equivalent Volterra equation

$$x(t) = x^0 + \frac{1}{\Gamma(\alpha)\int^{0}{}_{t}}(t-s)^{\alpha-1}g(s)ds. \quad (3)$$

3.2 Mittag–Leffler functions

For α>0 and $\beta \in \mathbb{C}$, the one- and two-parameter Mittag–Leffler functions are entire functions defined by

$$\frac{\frac{E^{\alpha}(z)=\Sigma_{k}{}^{=0\infty}z^{k}}{\Gamma(\alpha k+1)}, \quad E^{\alpha},{}^{\beta}(z)=\Sigma_{k}{}^{=0\infty}z^{k}}{\Gamma(\alpha k+\beta)}. \quad (4)$$

The identity $E_1(z)=e^z$ explains their role as fractional analogues of the exponential. In a sector separated from $|\arg z| \le \frac{\alpha\pi}{2}, E^{\alpha}(z)$ decays algebraically; on an unstable ray it contains a growing exponential contribution. These sectorial asymptotics generate the stability boundary (Gorenflo et al., 2014).

3.3 Stability concepts

The equilibrium is stable if, for every ε>0, sufficiently small $\|x_0\|$ implies $\|x(t;x^0)\| < \varepsilon$ for all t≥0. It is locally asymptotically stable if it is stable and $x(t;x^0) \to 0$ for all sufficiently small initial states. A Mittag–Leffler estimate is a quantitative statement of the form

$$\|x(t;x_0)\| \le M\|x_0\|\, E^{\alpha}(-ct^{\alpha}), \quad M, c > 0. \quad (5)$$

Such a bound implies attraction but is stronger than the bare definition of asymptotic stability. For non-normal matrices, polynomial factors and constants may enter the estimate; consequently, one should not infer (5) from eigenvalue locations without checking the relevant matrix-function bound.

4. Linear Spectral Stability

Consider $\mathrm{c}D_{0+}{}^{\alpha}x(t) = Ax(t), x(0) = x_0, where\ A \in \mathbb{C}^{n\mathrm{x}n}$. Laplace transformation yields $X(s) = s^{\alpha-1}(s^{\alpha}I - A)^{-1}x^0$. Inverting the transform gives

$$x(t) = E^{\alpha}(At^{\alpha})x_0. \quad (6)$$

Theorem 1 (Matignon spectral criterion)

Let $0 < \alpha < 1\ and\ A \in \mathbb{C}^{n\mathrm{x}n}$. The zero solution of (6) is asymptotically stable if and only if every $\lambda \in \sigma(A)$ satisfies

$$|\arg(\lambda)| > \frac{\alpha\pi}{2}. \quad (7)$$

Proof. Put $A = PJP^{-1}$in Jordan form. Each Jordan block contributes terms obtained from derivatives of $E^{\alpha}(\lambda t^{\alpha})$ and hence polynomial factors multiplied by sectorially decaying Mittag–Leffler terms. If (7) holds strictly, the large-argument expansion in the stable sector makes every block tend to zero. Conversely, when an eigenvalue lies inside the closed instability sector, the corresponding scalar mode fails to decay; on the boundary it is not asymptotically stable. Similarity preserves convergence, proving the result.

The strict inequality is essential. The point λ=0 is excluded because

$E^{\alpha}(0) = 1. When\ \alpha \to 1^{-}$, (7) approaches $|\arg\lambda| > \frac{\pi}{2}$, which is precisely Re λ<0. When α decreases, the forbidden cone narrows; however, the temporal decay of an admissible negative-real mode becomes algebraic rather than exponential.

5. Nonlinear Linearisation Principle

Consider the autonomous equation

$$\mathrm{c}D^{0+\alpha}x(t) = Ax(t) + f\big(x(t)\big), \quad x(0) = x^0, \quad (8)$$

where $f{:}B_{\mathrm{r}}(0) \to \mathbb{C}^{n}$ is locally Lipschitz, $f(0) = 0$ and $f(x) = o(\|x\|)$ as x→0. A mild solution satisfies the fractional variation-of-constants formula

$x(t) = E^{\alpha}(At^{\alpha})x^0 + \int^0{}^t (t-s)^{\alpha-1}E^{\alpha},{}^{\alpha}(A(t-s)^{\alpha})f\big(x(s)\big)ds.$ (9)

Theorem 2 (local asymptotic stability)

Assume $0 < \alpha < 1, A \in \mathbb{C}^{n\mathrm{x}n}, \sigma(A)$ satisfies (7), and f is locally Lipschitz with f(0)=0 and $\lim_r \to 0 \sup_0 < \|x\| \leq_r \|f(x)\|/\|x\| = 0$. Then the zero equilibrium of (8) is locally asymptotically stable.

Proof. The spectral condition supplies integrable bounds for the kernel $K(t) = t^{\alpha-1}E^{\alpha},{}^{\alpha}(At^{\alpha})$ and decay of $E^{\alpha}(At^{\alpha})$. Choose r so that the local Lipschitz modulus of f on B_r is small relative to the convolution bound of K. On the complete space of bounded continuous trajectories remaining in B_r, the right-hand side of (9) defines a contraction for sufficiently small x_0. The resulting solution is bounded. Splitting the convolution into an early-time and a late-time part, using decay of the linear term, integrability of K and $f(x) = o(\|x\|)$, shows $x(t) \to 0$. Continuous dependence provides Lyapunov stability.

Remark 1 (scope of the theorem)

Eigenvalues alone do not prove global stability. The theorem is local and requires local existence, uniqueness and a higher-order remainder. If the Jacobian has a boundary eigenvalue, if f is not suitably regular, or if solutions can leave every bounded set, a Lyapunov, invariant-set or centre-manifold argument is required.

6. Operator-Theoretic Extension

Let X be a complex Banach space and A:D(A)⊂X→X a closed densely defined operator. In this setting σ(A) can contain continuous spectrum, and spectral mapping need not provide the finite-dimensional decay estimate by itself. A suitable hypothesis is that A is sectorial of angle θ with $\theta < \pi - \frac{\alpha\pi}{2}$ (under the chosen sign convention) and that its resolvent satisfies an estimate of the form

$\|(\lambda I - A)^{-1}\| \leq C/|\lambda|$ for λ outside a sector containing σ(A). (10)

Under compatible sectoriality assumptions, the solution operators may be represented by a Dunford contour integral

$S^{\alpha}(t) = \left(\frac{1}{2\pi i}\right)\int \Gamma\, e^{\mathrm{l}t}\,\lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}d\lambda,$ (11)

or, equivalently, through subordination when A generates an appropriate C_0-semigroup. Decay then follows from resolvent-family estimates, not merely from the set-theoretic inclusion

$\sigma(A) \subset \left\{ |\arg\lambda| > \frac{\alpha\pi}{2} \right\}$. For a semilinear problem, local Lipschitz continuity and a small remainder are combined with the bound on S^{α} and its convolution kernel. This distinction prevents the common but invalid practice of transferring the matrix criterion verbatim to every unbounded operator (Bazhlekova, 2001; Prüss, 1993).

7. Dependence on Fractional Order

For a fixed nonzero eigenvalue λ with principal angle $\theta = |\arg\lambda|$, condition (7) is equivalent to $\alpha < \frac{2\theta}{\pi}$. Hence the critical fractional order is

$\alpha crit = \min\left\{1, 2\, min\lambda \in \frac{\sigma(A)|\arg\lambda|}{\pi}\right\}.$ (12)

If $0 < \alpha < \alpha crit$, all modes satisfy the sector test; if α>αcrit, at least one fails. This geometric fact does not mean that decreasing α always produces faster convergence. $For\ \lambda < 0, E^{\alpha}(\lambda t^{\alpha})$ typically has an algebraic tail proportional to $t^{-\alpha}$, so a smaller order can strengthen memory and slow long-time relaxation.

Eigenvalue λ	|arg λ|	Condition on α	Interpretation
−2	π	0<α<1	Stable for every admissible order
$-1+i$	$\frac{3\pi}{4}$	0<α<1	Stable for every admissible order
$1+2i$	$arctan(2) \approx 1.107$	α<0.705	Order-dependent classification
1	0	No α>0	Unstable positive-real mode

8. Worked Examples

8.1 Cubic triangular system

Consider

$$cD_{0^+}{}^{\alpha}x_1 = -2x_1 + x_2 - x_1{}^3, \quad cD_{0^+}{}^{\alpha}x_2 = -3x_2 - x_1x_2{}^2, \quad 0 < \alpha < 1. \tag{13}$$

At x*=0, the Jacobian and spectrum are

$$A = [-2 \ \ 1\,;\, 0 \ \ -3\,], \quad \sigma(A) = \{-2, -3\}. \tag{14}$$

Both eigenvalues have principal argument π; consequently (7) holds for every $\alpha \in (0,1)$. The remainder $f(x) = (-x_1{}^3, -x_1x_2{}^2)^{\mathrm{T}}$ satisfies $\|f(x)\| \le C\|x\|^3$ in a neighbourhood of zero, so $\frac{\|f(x)\|}{\|x\|} \le C\|x\|^2 \to 0$. Theorem 2 therefore proves local asymptotic stability for all admissible orders. This conclusion is stronger than a visual trajectory plot because it covers every sufficiently small initial condition and does not depend on a chosen time step.

8.2 Order-induced change of classification

Let $A = [1 \ -2;\ 2\ 1]$. Its eigenvalues are 1±2i, and $|\arg(1 \pm 2i)| = \arctan(2) \approx 1.10\ 7$ radians. Formula (12) gives αcrit≈0.705. Therefore, the Caputo system is asymptotically stable for α=0.6 but not for α=0.8. In contrast, the integer-order system is unstable because both eigenvalues have positive real part. This example isolates the angular mechanism behind fractional stability.

9. Verification Framework for Analytical and Numerical Studies

A reproducible stability investigation should use the following sequence.

Stage	Required mathematical evidence
Equilibrium identification	Solve $Ax + f(x) = 0$ and state the equilibrium under study.
Translation and linearisation	Set $y = x - x*$ and compute the Fréchet derivative $J = A + Df(x*)$.
Spectral computation	Report eigenvalues, algebraic multiplicities and principal arguments; assess non-normality or Jordan structure.
Sector test	Compare$\min|\arg\lambda|\ with \frac{\alpha\pi}{2}$ and treat equality as a boundary case.
Remainder estimate	Prove $f(x*+y) - f(x*) - Df(x*)y = o(\|y\|)$, uniformly where required.
Well-posedness	State continuity/Lipschitz and domain assumptions guaranteeing a unique local solution.

Stage	Required mathematical evidence
Quantitative bound	Use matrix Mittag–Leffler or resolvent-kernel estimates; do not claim exponential decay by analogy.
Numerical corroboration	Declare the discretisation, step size, memory treatment, convergence study and residual; distinguish numerical evidence from proof.

For computational work, a predictor–corrector or convolution-quadrature method may be used, but discrete stability is a separate property. The continuous sector condition does not automatically guarantee that an arbitrary time discretisation reproduces the correct decay. Step refinement and a comparison with a known linear Mittag–Leffler solution are therefore essential (Diethelm et al., 2002; Lubich, 1986).

10. Discussion and Contribution

The analysis establishes three levels of reasoning. At the first level, the spectrum of a finite matrix determines the asymptotic behaviour of the linear commensurate system through a sharp angular condition. At the second level, a nonlinear equilibrium inherits local stability only when the nonlinear remainder is small relative to the state and the Volterra problem is well posed. At the third level, an infinite-dimensional extension requires sectorial resolvent estimates that control a fractional solution family.

The principal conceptual contribution is the separation of these levels. This separation makes the framework safer to apply: a computed eigenvalue plot is recognized as decisive only for the stated finite-dimensional linear problem; the nonlinear theorem is invoked after verifying a remainder condition; and the operator case is treated through resolvent theory. The two examples further show that the fractional order affects both the geometry of admissible eigenvalues and the temporal character of decay.

The study also clarifies a terminological point. ‘Mittag–Leffler stability’ is useful when an explicit estimate such as (5) has been established; it should not be used as a decorative synonym for asymptotic stability. In applications, reporting constants, domains of attraction and the norm in which the estimate holds makes the claim testable.

11. Limitations and Future Research

- The sharp criterion (7) is stated for finite-dimensional, autonomous, commensurate systems with $0 < \alpha < 1$.
- Boundary eigenvalues, defective spectral configurations and non-Lipschitz nonlinearities require separate analysis.
- For unbounded operators, sectorial resolvent estimates and domain compatibility cannot be replaced by a plot of point eigenvalues.
- The examples are analytical demonstrations; extensive numerical experiments and physical validation lie outside the present paper.

12. Conclusion

A rigorous eigenvalue-based analysis of nonlinear Caputo systems must do more than compute a Jacobian. For a finite-dimensional commensurate linear equation, the strict sector condition $|\arg \lambda| > \frac{\alpha\pi}{2}$

is necessary and sufficient for asymptotic stability. For a nonlinear equation, the conclusion transfers locally when the remainder is genuinely higher order and the fractional integral equation is well posed. In a Banach space, spectral location must be accompanied by sectorial resolvent control. These distinctions produce a coherent mathematical workflow and prevent overstatement. The proposed framework is therefore suitable as a theoretical foundation for subsequent numerical, control-theoretic and applied investigations of memory-dependent systems.

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