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Modeling and Analysis of Queueing Systems Using Fuzzy Estimators

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Abstract:
Queueing models are investigated through fuzzy estimators in this study. Queueing modeling refers to a mathematical technique used in the analysis of queueing systems. The central challenge in any queueing system is one of balancing between the increased costs due to better service provision and the costs incurred due to customers' wait time. This study proposes a novel fuzzy technique for use in queueing models to overcome the shortcomings of point estimations and associated paradoxes in determining the efficiency of queues when dealing with inconsistent information. Fuzzy estimators are used in considering both the Poisson distribution for the arrival rate and the exponential distribution for service times in queueing systems.

Keywords*:* Fuzzy sets, Queueing systems, Fuzzy estimators, Fuzzy statistics.

1. Introduction

A waiting line, or queue, is one of the most important aspects of operations management and often one of the situations we cannot avoid in our everyday lives.It arises when a system's capacity or service capability is exceeded by the system's demand.The field of telecommunications saw the earliest applications of queueing theory in the early 1900s. Subsequently, several scholars expanded on it, and it now has a wide range of practical uses.

Although fuzzy queueing models have been extensively studied to represent uncertainty in arrival and service parameters, most existing approaches primarily replace crisp arrival and service rates with fuzzy numbers or use fuzzy arithmetic to obtain performance measures. Such approaches are useful when the system parameters themselves are imprecise; however, they do not adequately address the uncertainty that arises when the parameters have to be estimated from unreliable or limited sample observations. In particular, there is a need for an approach that combines statistical estimation with fuzzy representation so that uncertainty in parameter estimation can be directly incorporated into the performance analysis of a queueing system.

The present study addresses this gap by introducing a fuzzy-estimation-based framework for an M/M/S queueing system. Instead of relying only on conventional point estimates of the Poisson arrival rate and exponential service rate, fuzzy estimators are constructed from confidence-interval information and subsequently incorporated into the standard M/M/S performance measures. This provides a systematic way of transferring uncertainty present in observed data into the estimated queueing characteristics. The proposed approach therefore establishes a link between statistical fuzzy estimation and classical multi-server queueing analysis.

The novelty of this study lies in the integration of non-asymptotic fuzzy estimators with the performance analysis of an M/M/S queueing system. The proposed methodology enables the arrival and service parameters to be represented as fuzzy quantities derived from sample information rather than treating them as fixed point estimates. Consequently, the resulting measures, including the probability of an empty system, probability of waiting, expected number of customers in the queue, expected number of customers in the system, expected waiting time, and expected time in the system, are expressed in a fuzzy form. This allows the uncertainty associated with parameter estimation to be retained throughout the queueing analysis rather than being removed at the initial estimation stage.

The main contributions of this study are summarized as follows:

1. **Development of a fuzzy estimation framework:** A systematic procedure is developed for constructing fuzzy estimators of the arrival and service parameters of an M/M/S queueing system using confidence-interval information.
2. **Integration with M/M/S queueing analysis:** The fuzzy estimators are incorporated into the fundamental performance measures of a multi-server queue, providing fuzzy estimates of important system characteristics.
3. **Uncertainty-preserving analysis:** Unlike conventional point-estimation approaches, the proposed framework retains the uncertainty associated with the observed data throughout the queueing analysis.
4. **Derivation of fuzzy performance measures:** Analytical expressions are developed for the fuzzy probability of an empty system, probability that an arriving customer has to wait, expected queue length, expected system size, expected waiting time, and expected time in the system.
5. **Practical applicability:** The proposed framework can be used for queueing environments in which arrival and service observations are uncertain, imprecise, or insufficiently reliable for obtaining meaningful point estimates, such as healthcare service systems, communication networks, transportation facilities, and other service-oriented systems.

Thus, the study contributes to fuzzy queueing theory by providing an estimation-based approach in which uncertainty in sample-derived queueing parameters is explicitly propagated into the performance evaluation of an M/M/S system.

2. Literature Review: -

Fuzzy queueing theory has been developed to analyse queueing systems in which the arrival rate, service rate, waiting time, or other system characteristics are uncertain or imprecise. The classical queueing models generally assume that the system parameters are known precisely, whereas in practical situations such parameters may be uncertain because of limited observations, measurement errors, or variability in the operating environment. Fuzzy set theory provides an appropriate mathematical framework for representing such uncertainty.

Prade (1980) [1] presented an outline of fuzzy or possibilistic models for queueing systems and provided one of the early foundations for incorporating fuzzy concepts into queueing analysis. Li and Lee (1989) [2] developed an analytical framework for fuzzy queues and investigated queueing performance under fuzzy arrival and service parameters. Negi and Lee (1992) [3] further extended fuzzy queueing analysis through analytical and simulation approaches, demonstrating the usefulness of fuzzy representations in queueing systems.

Kao, Li and Chen (1999) [4] proposed a parametric programming approach for analysing fuzzy queues. Their method provided a systematic way of obtaining fuzzy queueing performance measures through mathematical programming. Pardo and Fuente (2008) [5] developed a fuzzy finite-capacity queueing model based on customer satisfaction and fuzzy optimization, thereby extending fuzzy queueing theory to systems with capacity restrictions and optimization considerations.

Klir and Yuan (1995) [6] provided fundamental theoretical concepts related to fuzzy sets and fuzzy logic, which form an important mathematical basis for fuzzy modelling. Nguyen (1978) [7] discussed the extension principle for fuzzy sets, which later became an important tool for transforming fuzzy input parameters into fuzzy output characteristics.

Chrysafis and Papadopoulos (2006) [8] investigated the use of fuzzy estimators in the theoretical pricing of options. Their study demonstrated that statistical information can be transformed into fuzzy estimators for representing uncertainty in estimated parameters. Chrysafis and Papadopoulos (2008) [9] subsequently proposed a model for cost-volume-profit analysis under uncertainty using fuzzy estimators based on confidence intervals. This approach is particularly important because it provides a statistical basis for constructing fuzzy estimators rather than simply assuming fuzzy parameters.

Tsironis, Sfiris and Papadopoulos (2010) [10] investigated fuzzy performance evaluation of workflow stochastic Petri nets using block reduction. Their study demonstrated the application of fuzzy methods to stochastic systems and performance evaluation. Sfiris and Papadopoulos (2012) [11] further examined fuzzy estimators in expert systems and illustrated how statistical information can be represented through fuzzy estimators.

Taheri, Falsafain and Mashinchi (2008) [12] studied fuzzy estimation of parameters in statistical models. Their work is closely related to the statistical foundation of fuzzy estimators because it considers the construction of fuzzy representations for uncertain statistical parameters. Rose (2010) [13] provided a comprehensive treatment of fuzzy logic and its engineering applications, highlighting the usefulness of fuzzy approaches for modelling systems involving imprecision.

Srinivasan (2014) [14] investigated a fuzzy queueing model using the DSW algorithm. The study demonstrated how fuzzy queueing parameters can be processed through the DSW method to obtain system performance measures. Rose (2010) [15] also discussed fuzzy logic with engineering applications and provided theoretical and computational foundations for the application of fuzzy methods in engineering systems.

More recent research has continued to extend fuzzy queueing models to multi-server systems. Sujatha, Murthy Akella and Deekshitulu (2017) [16] analysed single-server and multi-server fuzzy queueing models using the α-cut method. Triangular fuzzy numbers were used for the arrival and service rates, and the authors considered Poisson arrivals and Erlang-k service. Different queueing performance measures were obtained in fuzzy form, and the effectiveness of the DSW algorithm was also demonstrated. However, the arrival and service parameters were treated as fuzzy inputs rather than being statistically estimated from sample observations.

Mueen, Ramli and Zaibidi (2017) [17] investigated a parametric nonlinear programming approach for fuzzy queues using hexagonal membership functions. Their approach extended the representation of uncertain queueing parameters beyond conventional triangular and trapezoidal fuzzy numbers. The study demonstrated the usefulness of parametric programming for obtaining queueing performance measures under uncertainty. However, the fuzzy parameters were specified directly and were not constructed from statistical confidence intervals.

Ritha and Vinnarasi (2018) [18] studied priority queueing systems using pentagonal fuzzy numbers. Their work incorporated multiple customer classes into a fuzzy queueing framework and used fuzzy mathematical programming to obtain the corresponding performance measures. The study showed that fuzzy queueing models can be extended to priority systems, although statistical estimation of the fuzzy arrival and service parameters was not considered.
Visalakshi and Suvitha (2018) [19] investigated the performance measures of an FM/FM/1 fuzzy queue using pentagonal fuzzy numbers. The α-cut method was used to obtain the corresponding performance measures, and a numerical example was presented to demonstrate the procedure. The study demonstrated the usefulness of non-triangular fuzzy numbers in queueing analysis; however, the uncertainty associated with estimating the queueing parameters from sample data was not incorporated.
Ramasamy and Susmitha (2018) [20] analysed priority queues with fuzzy parameters using trapezoidal fuzzy numbers. Their work used α-cut and parametric programming approaches to obtain queueing performance measures. The study provided an extension of fuzzy queueing theory to priority systems, but the fuzzy parameters were assumed rather than estimated statistically.
Suvitha and Visalakshi (2019) [21] studied an Erlang-service queueing model with fuzzy parameters. In their model, the inter-arrival and service parameters were represented by octagonal fuzzy numbers, and a numerical example was used to demonstrate the proposed approach. The study illustrates the flexibility of fuzzy membership functions for queueing applications, but it does not provide a confidence-interval-based procedure for constructing fuzzy estimators.
Chen, Liu and Zhang (2020) [22] analysed strategic customer behaviour in fuzzy queueing systems. Their study extended fuzzy queueing analysis to situations in which customers make strategic decisions under uncertainty. The work demonstrates that fuzzy queueing models can incorporate behavioural characteristics in addition to uncertain system parameters. Nevertheless, the main focus was on customer behaviour and strategic decisions rather than on statistical estimation of arrival and service rates.
El-Paoumy (2020) [23] investigated a fuzzy simple queue with balking and feedback customers using symmetric heptagonal fuzzy numbers. The study incorporated customer behaviour into fuzzy queueing analysis and obtained performance measures under uncertain parameters. This work demonstrates the ability of fuzzy queueing models to represent more realistic customer behaviour, although the underlying fuzzy parameters were assumed rather than estimated from sample observations.
Panta, Ghimire, Panthi and Pant (2021) [24] developed an M/M/s/N queueing model with reneging in a fuzzy environment. The model considered Poisson arrivals and exponential service and derived fuzzy queue length, waiting time, response time and the optimal number of servers. A recursive method and numerical illustrations were used to validate the model. Their results showed that fuzzy queueing can provide a more realistic representation than a crisp model when the system parameters are uncertain. However, the study did not construct the fuzzy arrival and service parameters through statistical estimation.
Aarthi and Shanmugasundari (2023) [25] compared single-server fuzzy queueing and intuitionistic fuzzy queueing models with infinite capacity. Their study evaluated the performance measures obtained under both approaches and reported that intuitionistic fuzzy queueing can provide a more flexible representation of uncertainty. The work represents an important recent development in fuzzy queueing theory. However, the study focuses on representing uncertain parameters through fuzzy and intuitionistic fuzzy numbers rather than constructing fuzzy estimators from confidence-interval information.

Vijaya, Srinivasan and Jayasri (2024) [26] analysed a limited-capacity FM/FM/1 queue using trapezoidal fuzzy numbers and the LR method. Their results were compared with the DSW algorithm, and a numerical example demonstrated the applicability of the proposed method. The study contributes to the development of computational techniques for fuzzy queueing systems; however, it does not explicitly address statistical uncertainty arising during the estimation of the queueing parameters.

Our work addresses this fascinating field of study by developing estimators for the parameters of a queueing system's probability distributions using fuzzy statistics. Fuzzy estimators are integrated into queueing systems with parallel channels of service (M/M/S queueing models, (Fig. 1). We demonstrate that the suggested analysis approach can be executed with ease and efficiency using our created formulas, in addition to yielding a more accurate estimation of the performance metrics.

The objective of queuing analysis is to strike a balance between the cost of maintaining units (i.e., customers, cars, phone calls, etc.) waiting cost (WC) and the cost of providing a level of service capacity (SC) (Fig. 2). The cost of service capacity rises with it, either continuously or, more often, in steps, depending on the service level (see Fig. 2). However, waiting costs (WC) tend to decrease as capacity increases due to a decrease in the number of units waiting and the length of time they wait. The trade-off between those two costs is then represented on the graph by a total cost (TC) curve. Quoting analysis's ultimate objective is to determine the service capacity level that will minimize overall costs.

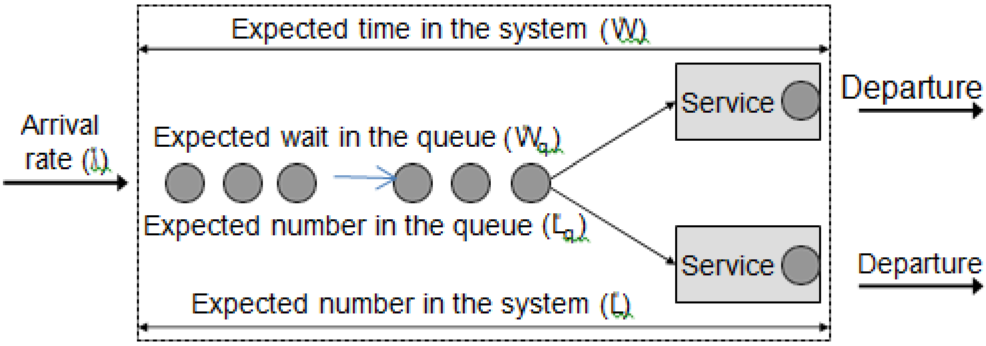

Figure1.Aparallel channels of service queuing system (M/M/S).

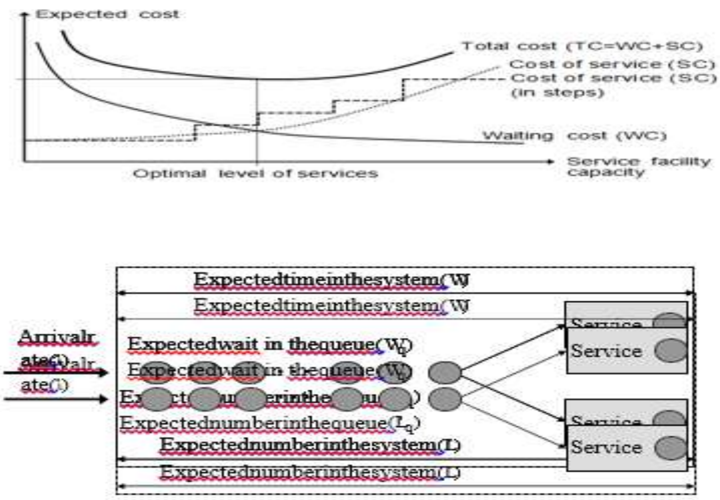

Figure 2. The whole cost of a queuing system, including the cost of waiting and services.

3. PRELIMINARIES

Throughout the study, the following fuzzy set theory concepts and notations have been employed. On the real field R, we assume closed and bounded (compact) fuzzy numbers. The critical sets define the α-cuts of a fuzzy set $\tilde{A}$

$$\alpha_{\tilde{A}} = \{x \in R : \tilde{A}(x) \geq \alpha\}, \alpha \in (0,1] \qquad (2.1)$$

On the other hand, its support $0_{\tilde{A}}$ is the closureof the union of all the α-cuts in the topology of $\tilde{A}$:

$$0_{\tilde{A}} = \overline{\cup_{\alpha\in(0,1]}\alpha\tilde{A}} = \overline{x : \tilde{A}(x) > 0\}} \qquad (2.2)$$

It is established that the fuzzy set $\tilde{A}$ is determined by the α-cuts of equation (2.1). If $\tilde{A}$ is a normal fuzzy set, convex fuzzy, upper semi-continuous conditions fuzzy set with a compact support, then it is a fuzzy number [6]. The notation we use to represent a fuzzy number is as follows:

$$\alpha_{\tilde{A}} = [A_1(\alpha), A_2(\alpha)] \qquad (2.3)$$

Where $A_1(\alpha)$, $A_2(\alpha)$ can be regarded as function on interval (0,1] [7]. Then:

(i) $A_l(\alpha)$ is increasing and left continuous,

(ii) $A_r(\alpha)$ is decreasing and left continuous,

(iii)$A_l(\alpha) \leq A_r(\alpha)$

We apply the following α-cuts operations to realize fuzzy arithmetic operations. Let $\widetilde{M}$ and $\widetilde{N}$ be two fuzzy numbers, then

(i) Addition: $(\widetilde{M} \oplus \widetilde{N})^\alpha = [m_l^\alpha + n_l^\alpha, m_r^\alpha + n_r^\alpha]$.

(ii) Subtraction: $(\widetilde{M} - \widetilde{N})^\alpha = (\widetilde{M} \oplus (-\widetilde{N}))^\alpha = [m_l^\alpha - n_r^\alpha, m_r^\alpha + n_l^\alpha]$,

Where $-N^\alpha = [-n_r^\alpha, -n_l^\alpha]$.

(iii)Multiplication: $(\widetilde{M} \otimes \widetilde{N})^\alpha = [m_l^\alpha n_l^\alpha, m_r^\alpha n_r^\alpha]$.

(iv)Division: $(\widetilde{M}/\widetilde{N})^\alpha = (\widetilde{M} \otimes \widetilde{N}^{-1})^\alpha = [\frac{m_l^\alpha}{n_r^\alpha}, \frac{m_r^\alpha}{n_l^\alpha}]$.

When $M > 0, N > 0$, with $n_l^\alpha > 0$ (where $-N^{-1} = \left[\frac{1}{n_r^\alpha}, \frac{1}{n_l^\alpha}\right]$).

In the above relations $m_l^\alpha = m_l(\alpha), m_r^\alpha = m_r(\alpha), n_l^\alpha = n_l(\alpha), n_r^\alpha = n_r(\alpha)$.

Thefollowingnotationsareusedtodescribethecharacteristicsoftheproposedqueueingsystem:

$\tilde{\lambda}$: fuzzy meanarrivals rate ofPoisson distribution withparameter,

$\frac{1}{\tilde{\mu}}$: fuzzy mean assistance the exponential distribution's time period with parameter,

$\tilde{P}_0$: fuzzy likelihood that the system is empty,

$\tilde{P}_n$: fuzzy likelihood of n in the system items,

$\tilde{P}_w$: fuzzyprobability that an itementering the system and waiting in line,

$\tilde{L}_q$: imprecise estimate of the quantity of items in the queue,

$\tilde{L}$: The anticipated quantity of things in the system,

$\widetilde{W}_q$: Imprecise estimates of how long an item will take to go through the queue,

$\widetilde{W}$: Fuzzy estimate of how long an item will stay in the system.

4. FUZZY ESTIMATION OF QUEUEING ARRIVALS AND SERVICE

Confidence interval estimate is expanded upon by fuzzy estimation. To implement fuzzy statistics in queues, the authors introduce their non-Asymptotic Fuzzy Estimators (NAFE). Current research is being done on their usage in a variety of applied statistic disciplines [8,], [9], [10], and [11]. Fig. 3 shows a form of a typical b fuzzy estimator.

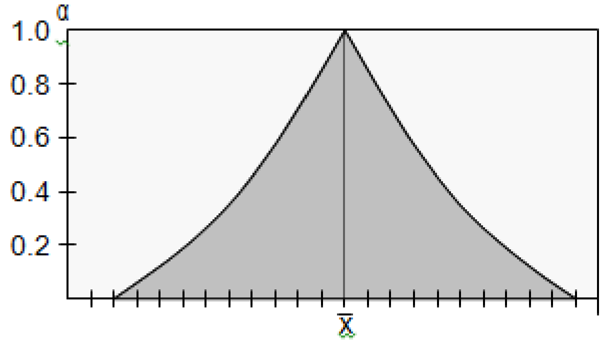

Fig.3. A typical fuzzy estimator.

Take a random sample of size n items from a distribution with unknown parameter θ, and denoted by X_1, X_2,..., X_n.Let $[\theta_1(\alpha), \theta_2(\alpha)]$ be the (1-α)100% confidence intervals for θ. This is the key proposition. Using any continuous, onto, monotonically growing function

$$h_y(\alpha):(0,1] \to \left[\frac{y}{2}, 0.5\right], \quad y \in (0,1) \tag{3.1}$$

then the following stacked intervals yield a fuzzy estimator, $\tilde{\theta}_y$:

$$\alpha_{\tilde{\theta}_y} = \left[\theta_1\left(2h_y(\alpha)\right), \theta_2\left(2h_y(\alpha)\right)\right], \ \alpha \in (0,1] \tag{3.2}$$

The arrival and service times in the suggested fuzzy queueing model are distributed exponentially since both arrivals and services are Poisson processes. Therefore, the union of all α-cuts of a fuzzy estimator may be used to build the Poisson parameter λ and the exponential parameter μ as fuzzy numbers. To find these α-cuts, we employ a collection of observations.

Let's say we have an exponential distribution Exponential (λ) and the random sample (x_1, x_2,...x_n) of observations. It is possible to build a fuzzy estimate for the exponential distribution's 1/λ [12].

We demonstrate the construction of a non-asymptotic fuzzy estimator for 1/λ in the following statement. Let x_1, x_2,..., x_n are sample values that the sample has assumed,andthat X_1,X_2,..., X_n is a random sample. Let $y \in [0,1)$ as well. Then, $(1/\tilde{\lambda})_y$ is a fuzzy number, whose α-cuts are the closed intervals, and the exact (1- y)100% confidence interval for 1/λ serves as its support.

$$\alpha\left(\frac{1}{\tilde{\lambda}}\right)_y = \left[\frac{1}{\hat{\lambda}}\frac{2\sum_{i=1}^{n} x_i}{X^2_{2n,1-h_y(\alpha)}}, \frac{1}{\hat{\lambda}}\frac{2\sum_{i=1}^{n} x_i}{X^2_{2n,h_y(\alpha)}}\right], \alpha \in (0,1] \quad (3.3)$$

where $\hat{\lambda} = 1/\bar{x}$is the fuzzy estimate with the highest likelihood., $X^2_{2n}$is the fuzzy estimate with the highest likelihood.$X^2_{2n,h(\alpha)} = F^{-1}\left(1 - h_y(\alpha)\right)$, $h_Y(\alpha) = \left(\frac{1}{2} - \frac{Y}{2}\right)\alpha + \frac{Y}{2}$and the cumulative distribution is indicated by F. The distribution's median is denoted by M.

5. FUZZY QUEUEING MODEL APPROACH: FROM CRISP TO FUZZY MEASUREMENTS

It is necessary to ascertain the values for a collection of distinctive measures that characterize the system's performance. Since these numbers are never totally accurate, we aim to offer membership grades to a range of potential real values rather than actual values.

We cannot estimate the performance of the queueing system based on sample means or proceed with point estimation of the parameters of distributions when the available data are unreliable, since we would propagate the errors of the unreliable data.

The benefits of both point and fuzzy procedures are combined in fuzzy estimating, which is a combined probability and fuzzy methodology. To generate fuzzy estimations for the average interarrival time and average service time, one only has to provide a collection of observations for interarrival and service times.

These closed intervals determine the fuzzy estimators $\bar{\lambda}, \bar{\mu}$, which are based on the Poisson and exponential distributions, respectively.

$$^{\alpha}\tilde{\lambda}_Y = \left[\lambda_l^{\alpha}, \lambda_r^{\alpha}\right] = \left[\hat{\lambda}\frac{X^2_{2y,h_y(\alpha)}}{2n}, \hat{\lambda}\frac{X^2_{2y+2,1-h_y(\alpha)}}{2n}\right], \alpha \in (0,1] \tag{4.1}$$

Where $y = \sum_{i=1}^{n} x_i \neq 0$ and $^{\alpha}\hat{\lambda}_y = \left[0, \frac{X^2_{2,1-h_Y(\alpha)}}{2n}\right]$ when y = 0

$$^{\alpha}\tilde{\mu}_Y = \left[\mu_l^{\alpha}, \mu_r^{\alpha}\right] = \frac{1}{\left[\left(\frac{1}{\mu}\right)_l^{\alpha}, \left(\frac{1}{\mu}\right)_r^{\alpha}\right]} =$$

(4.2)

$$\left[\hat{\mu}\frac{X^2_{2n,h_Y(\alpha)}}{2\sum_{i=1}^{n} x_i}, \hat{\mu}\frac{X^2_{2n,1-h_Y(\alpha)}}{2\sum_{i=1}^{n} x_i}\right], \alpha \in (0,1]$$

Where $X^2_{2n,h(\alpha)} = F^{-1}(1-h(\alpha))$ and the cumulative distribution is indicated by F. Based on equation (4.1) and equation (4.2), we prove the fuzzy characteristics of the proposed queueing system.

5.1. THE FUZZY PROBABILITY OF THE QUEUEING SYSTEM BEING EMPTY

The fuzzy number ${}^{\alpha}\tilde{P}_0 = \left[P_{0I}^{\alpha}, P_{0r}^{\alpha}\right]$ that represents the likelihood the line-up . The following formula indicates when the system is empty:

$$P_{0I}^{\alpha} = \frac{1}{\sum_{n=0}^{s-1}\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s + \left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \frac{s\mu_I^{\alpha}}{\lfloor s(s\mu_r^{\alpha} - \lambda_I^{\alpha})}}$$

(4.3)

$$P_{0r}^{\alpha} = \frac{1}{\sum_{n=0}^{s-1}\frac{\lambda_r^{a}}{\lfloor n\mu_I^{\alpha}}} + \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)\frac{s\mu_r^{\alpha}}{\lfloor s\left[s\mu_I^{\alpha} - \lambda_r^{\alpha}\right]}$$

where ${}^{\alpha}\lambda, {}^{\alpha}\mu$ are the α- cuts of the fuzzy estimated parameters for the poisson and the exponential distributions and s is the number of servers.

Proof.

$$ {}^{\alpha}P_0 = \frac{1}{\sum_{n=0}^{s-1}\frac{\left({}^{\alpha}\lambda / {}^{\alpha}\mu\right)^n}{\lfloor n} + \frac{\left({}^{\alpha}\lambda / {}^{\alpha}\mu\right)^s}{\lfloor s}\frac{\left(s\,{}^{\alpha}\mu\right)}{\left(s\,{}^{\alpha}\mu - {}^{\alpha}\lambda\right)}} =$$

$$= \frac{1}{\sum_{n=0}^{s-1}\frac{\left[\lambda_I^{\alpha}, \lambda_r^{\alpha}\right]\frac{1}{\left[\mu_I^{\alpha}, \mu_r^{\alpha}\right]}}{\lfloor n} + \frac{\left[\frac{[\lambda_I^{\alpha}, \lambda_r^{\alpha}]}{[\mu_I^{\alpha}, \mu_r^{\alpha}]}\right]^s}{\lfloor s}\frac{[s\mu_I^{\alpha}, s\mu_r^{\alpha}]}{[s\mu_I^{\alpha}, s\mu_r^{\alpha}] - [\lambda_I^{\alpha}, \lambda_r^{\alpha}]}} =$$

$$= \frac{1}{\sum_{n=0}^{s-1}\frac{\left[\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}, \frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right]}{\lfloor n} + \frac{\left[\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}, \frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right]^s}{\lfloor s}\frac{\left[s\mu_I^{\alpha}, s\mu_r^{\alpha}\right]}{\left[s\mu_I^{\alpha} - \lambda_r^{\alpha}, s\mu_r^{\alpha} - \lambda_I^{\alpha}\right]}} =$$

$$=\frac{1}{\sum_{n=0}^{s-1}\frac{\left[\frac{\lambda_I^\alpha}{\mu_r^\alpha},\frac{\lambda_r^\alpha}{\mu_I^\alpha}\right]}{\lfloor n}+\frac{\left[s\mu_I^\alpha\left(\frac{\lambda_I^\alpha}{\mu_r^\alpha}\right)^s,s\mu_r^\alpha\left(\frac{\lambda_r^\alpha}{\mu_I^\alpha}\right)^s\right]}{\lfloor s\left[s\mu_I^\alpha-\lambda_r^\alpha,s\mu_r^\alpha-\lambda_I^\alpha\right]}}=$$

$$=\frac{1}{\sum_{n=0}^{s-1}\frac{\left[\frac{\lambda_I^\alpha}{\mu_r^\alpha},\frac{\lambda_r^\alpha}{\mu_I^\alpha}\right]}{\lfloor n}+\left[\frac{s\mu_I^\alpha\left(\frac{\lambda_I^\alpha}{\mu_r^\alpha}\right)^s}{\lfloor s\left(s\mu_r^\alpha-\lambda_I^\alpha\right)},\frac{s\mu_r^\alpha\left(\frac{\lambda_r^\alpha}{\mu_I^\alpha}\right)^s}{\lfloor s\left[s\mu_I^\alpha-\lambda_r^\alpha\right]}\right]}$$

Hence

$${}^\alpha P_0=\left[P_{0I}^\alpha,P_{0r}^\alpha\right]=\left[\frac{1}{\sum_{n=0}^{s-1}\frac{\lambda_I^\alpha}{\lfloor n\mu_r^\alpha}+\left(\frac{\lambda_I^\alpha}{\mu_r^\alpha}\right)^s\frac{s\mu_I^\alpha}{\lfloor s\left(s\mu_r^\alpha-\lambda_I^\alpha\right)}},\frac{1}{\sum_{n=0}^{s-1}\frac{\lambda_r^\alpha}{\lfloor n\mu_I^\alpha}+\left(\frac{\lambda_r^\alpha}{\mu_I^\alpha}\right)^s\frac{s\mu_r^\alpha}{\lfloor s\left[s\mu_I^\alpha-\lambda_r^\alpha\right]}}\right]$$

5.2. FUZZY PROBABILITY AN ITEM ENTERING AT THE SYSTEM WILL WAIT IN QUEUE

The fuzzy probability ${}^\alpha P_w=\left[P_{wI}^\alpha,P_{wr}^\alpha\right]$ an item entering the system and waiting in line is given by equation (4.4)

$$P_{wI}^\alpha=P_{0I}^\alpha\frac{1}{\lfloor s}\left(\frac{\lambda_I^\alpha}{\mu_r^\alpha}\right)^s\frac{s\mu_I^\alpha}{s\mu_r^\alpha-s_I^\alpha},$$

(4.4)

$$P_{wr}^\alpha=P_{0r}\frac{1}{\lfloor s}\left(\frac{\lambda_r^\alpha}{\mu_I^\alpha}\right)^s\frac{s\mu_r^\alpha}{s\mu_I^\alpha-\lambda_r^\alpha}$$

Proof.

$${}^\alpha P_w={}^\alpha P_0\frac{1}{\lfloor s}\left(\frac{{}^\alpha\lambda}{{}^\alpha\bar{\mu}}\right)^s\left(\frac{s^\alpha\mu}{s^\alpha\mu-{}^\alpha\bar{\lambda}}\right)=$$

$$^{\alpha}P_0 \frac{1}{\lfloor s} \left[\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} \right)^s, \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^s \right] \left[\frac{\left[s\mu_I^{\alpha}, s\mu_r^{\alpha} \right]}{s\mu_I^{\alpha} - \lambda_r^{\alpha}, s\mu_r^{\alpha} - \lambda_I^{\alpha}} \right] =$$

$$=^{\alpha} P_0 \frac{1}{\lfloor s} \left[\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} \right)^s \frac{s\mu_I^{\alpha}}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha} \right)}, \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^s \frac{s\mu_r^{\alpha}}{\left[s\mu_I^{\alpha} - \lambda_r^{\alpha} \right]} \right]$$

Hence

$$^{\alpha}P_w = \left[P_{wI}^{\alpha}, \mathrm{P}_{wr}^{\alpha} \right] = \left[P_{wI}^{\alpha} = P_{0I}^{\alpha} \frac{1}{\lfloor s} \left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} \right)^s \frac{s\mu_I^{\alpha}}{s\mu_r^{\alpha} - s_I^{\alpha}}, P_{wr}^{\alpha} = P_{0r} \frac{1}{\lfloor s} \frac{s\mu_r^{\alpha}}{s\mu_I^{\alpha} - \lambda_r^{\alpha}} \right]$$

5.3. FUZZY PROBABILITY OF N ITEMS IN THE SYSTEM

This fuzzy probability $^{\alpha}P_n = \left[P_{nI}^{\alpha}, P_{nr}^{\alpha} \right]$ is given by the formula below :

$$P_{nI}^{\alpha} = \begin{cases} P_{0I}^{\alpha} \frac{1}{\lfloor n} \left(\frac{\lambda_I^{\varepsilon}}{\mu_r^{\alpha}} \right)^n, (n=1,2,....,s) \\ P_{0I}^{\alpha} \frac{1}{\lfloor s s^{n-s}} \left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} \right)^n, (n=s,s+1,....) \end{cases}$$

(4.5)

$$P_{nr}^{\alpha} = \begin{cases} P_{0r}^{\alpha} \frac{1}{\lfloor n} \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^n, (n=1,2,....,s) \\ P_{0r}^{\alpha} \frac{1}{\lfloor s s^{n-s}} \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^n, (n=s,s+1,....) \end{cases}$$

Proof.

$$^{\alpha}P_n = \begin{cases} ^{\alpha}P_0 \frac{\left(\lambda^{\alpha} / \mu^{\alpha} \right)^n}{\lfloor n} =^{\alpha} P_0 \left[\frac{1}{\lfloor n} \left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} \right)^n, \frac{1}{\lfloor n} \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^n \right], (n=1,2,....,s) \\ ^{\alpha}P_0 \frac{\left(\lambda^{\alpha} / \mu^{\alpha} \right)^n}{\lfloor s s^{n-s}} =^{\alpha} P_0 \left[\frac{1}{\lfloor s s^{n-s}} \left(\frac{\lambda_i^{\alpha}}{\mu_r^{\alpha}} \right)^n, \frac{1}{\lfloor s s^{n-s}} \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^n \right], (n=s,s+1,s+2,....) \end{cases}$$

Hence

$$^{\alpha}P_n = \left[P_{nI}^{\alpha}, P_{nr}^{\alpha} \right] = \left[\begin{cases} P_{0I}^{\alpha} \frac{1}{\lfloor n} \left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} \right)^n, (n=1,2,....,\mathrm{s}) \\ P_{0I}^{\alpha} \frac{1}{\lfloor s s^{n-s}} \left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} \right)^n, (n=s,s+1,....) \end{cases}, \begin{cases} P_{0r}^{\alpha} \frac{1}{\lfloor n} \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^n, (n=1,2,...,s) \\ P_{0r}^{\alpha} \frac{1}{\lfloor s s^{n-s}} \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right)^n, (n=s,s+1,....) \end{cases} \right]$$

5.4.THE EXPECTED FUZZY SIZE OF ITEMS IN THEQUEUE

The expected fuzzy size of items in the queue $\left({}^{\alpha}L_q = \left[L_{qI}^{\alpha}, L_{qr}^{\alpha}\right]\right)$ is given by the formula:

$$L_{qI}^{\alpha} = P_{0I}^{\alpha} \frac{1}{(s-1)!} \frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2}$$

(4.6)

$$L_{qr}^{\alpha} = P_{0r}^{\alpha} \frac{1}{(s-1)!} \frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2}$$

Proof.

$${}^{\alpha}L_q = \left[{}^{\alpha}P_0 \cdot \frac{\left(\lambda^{\alpha}/\mu^{\alpha}\right)^s \lambda^{\alpha}\mu^{\alpha}}{\left((s-1)!\right)\left(s\mu^{\alpha} - \lambda^{\alpha}\right)^2}\right] = {}^{\alpha}P_0 \frac{\left[\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s, \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s\right].\left(\lambda_I^{\alpha}\mu_I^{\alpha}, \lambda_r^{\alpha}\mu_r^{\alpha}\right)}{(s-1)!\left[\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)\right]^2} =$$

$$= {}^{\alpha}P_0 \frac{\left[\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right), \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)\right]}{(s-1)!\left[\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2, \left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2\right]} =$$

$$= {}^{\alpha}P_0 \left[\frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right)}{(s-1)!\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2}, \frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)}{(s-1)!\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2}\right]$$

Hence

$$ {}^{\alpha}L_q = \left[L_{qI}^{\alpha}, L_{qr}^{\alpha}\right] = \left[P_{0I}^{\alpha} \frac{1}{\lfloor\underline{(s-1)}} \frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2}, P_{0r}^{\alpha} \frac{1}{\lfloor\underline{(s-1)}} \frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2} \right] $$

5.5.THE EXPECTED FUZZY SIZE OF ITEMS IN THE QUEUE SYSTEM

The expected fuzzy size of items in the system (items in the queue and being served), $\left({}^{\alpha}L = \left[L_I^{\alpha}, L_r^{\alpha}\right]\right)$ is given by the formula:

$$ L_I^{\alpha} = P_{0I}^{\alpha} \frac{1}{\lfloor\underline{(s-1)}} \frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2} + \frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}} $$

(4.7)

$$ L_r^{\alpha} = P_{0r}^{\alpha} \frac{1}{\lfloor\underline{(s-1)}} \frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2} + \frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} $$

Proof.

$$ {}^{\alpha}L = \left[L_q^{\alpha} + \frac{\lambda^{\alpha}}{\mu^{\alpha}}\right] = \left[P_{0I}^{\alpha}, P_{0r}^{\alpha}\right] \frac{1}{\lfloor\underline{(s-1)}} \left[\frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2}, \frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2} \right] + \left[\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}, \frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right] $$

Hence

$$ {}^{\alpha}L = \left[L_I^{\alpha}, L_r^{\alpha}\right] = \left[P_{0I}^{\alpha} \frac{1}{\lfloor\underline{(s-1)}} \frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^s \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2} + \frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}, P_{0r}^{\alpha} \frac{1}{\lfloor\underline{(s-1)}} \frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^s \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2} + \frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}} \right] $$

5.6.THE EXPECTED FUZZY TIME AN ITEM SPENDS WAITING IN QUEUE

The expected fuzzy time ${}^{\alpha}W_q = \left[W_{qI}^{q}, W_{qr}^{\alpha}\right]$ an item spends waiting in queue is given by the following equation (4.8):

$$W_{qI}^{\alpha} = P_{0I}^{r} \frac{1}{\lfloor(s-1)} \left[\frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^{s} \left(\lambda_I^{\alpha}\mu_I^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^{2} \lambda_I^{\alpha}} \right]$$

(4.8)

$$W_{qr}^{\alpha} = P_{0r}^{\alpha} \frac{1}{\lfloor(s-1)} \left[\frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^{s} \left(\lambda_r^{\alpha}\mu_r^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^{2} \lambda_r^{\alpha}} \right]$$

Proof.

$${}^{\alpha}W_q = \frac{{}^{\alpha}L_q}{{}^{\alpha}\lambda} = \frac{\left[{}^{\alpha}P_0 \cdot \frac{\left({}^{\alpha}\lambda / {}^{\alpha}\mu\right)^{s} \cdot {}^{\alpha}\lambda \cdot {}^{\alpha}\mu}{\lfloor(s-1) \cdot \left(s\,{}^{\alpha}\mu - {}^{\alpha}\lambda\right)^{2}} \right]}{\lambda^{\alpha}} =$$

$$= \left[P_{0I}^{\alpha}, P_{0r}^{\alpha}\right] \frac{\left[\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^{s}, \left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^{s}\right] . \left(\lambda_I^{\alpha}\mu_I^{\alpha}, \lambda_r^{\alpha}\mu_r^{\alpha}\right)}{\lfloor(s-1) \left[\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^{2}, \left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^{2}\right] \left(\lambda_I^{\alpha}, \lambda_r^{\alpha}\right)} =$$

$$= \frac{\left[P_{0I}^{\alpha}, P_{0r}^{\alpha}\right]}{\lfloor (s-1)} \left[\frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^{s} \left(\lambda_I^{\alpha} \mu_I^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2 \lambda_I^{\alpha}}, \frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^{s} \left(\lambda_r^{\alpha} \mu_r^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2 \lambda_r^{\alpha}} \right]$$

Hence

$$^{\alpha}W_q = \left[W_{qI}^{\alpha}, W_{qr}^{\alpha}\right] = \left[P_{0I}^{r} \frac{1}{\lfloor (s-1)} \left[\frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right) \left(\lambda_I^{\alpha} \mu_I^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2 \lambda_I^{\alpha}} \right], P_{0r}^{\alpha} \frac{1}{\lfloor (s-1)} \left[\frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^{s} \left(\lambda_r^{\alpha} \mu_r^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2 \lambda_r^{\alpha}} \right] \right]$$

5.7. THE EXPECTED FUZZY TIME AN ITEM SPENDS IN THE SYSTEM

Finally, the expected fuzzy time $^{\alpha}W = \left[W_I^{\alpha}, W_r^{\alpha}\right]$ a thing stays in the system is given by the equation (4.9):

$$W_I^{\alpha} = P_{0I}^{\alpha} \frac{1}{\lfloor (s-1)} \left[\frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^{s} \left(\lambda_I^{\alpha} \mu_I^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2 \lambda_I^{\alpha}} \right] + \frac{1}{\mu_r^{\alpha}}$$

(4.9)

$$W_r^{\alpha} = P_{0r}^{\alpha} \frac{1}{\lfloor (s-1)} \left[\frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^{s} \left(\lambda_r^{\alpha} \mu_r^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2 \lambda_r^{\alpha}} \right] + \frac{1}{\mu_I^{\alpha}}$$

Proof.

$$^{\alpha}W = {}^{\alpha}W_q + \frac{1}{^{\alpha}\mu} = \left[W_{qI}^{\alpha}, W_{qr}^{\alpha}\right] + \left[\frac{1}{\mu_r^{\alpha}}, \frac{1}{\mu_I^{\alpha}}\right]$$

Hence

$$^{\alpha}W = \left[W_I^{\alpha}, W_r^{\alpha}\right] = \left[\begin{array}{l} P_{0I}^{\alpha} \frac{1}{\lfloor (s-1)} \left[\frac{\left(\frac{\lambda_I^{\alpha}}{\mu_r^{\alpha}}\right)^{s} \left(\lambda_I^{\alpha} \mu_I^{\alpha}\right)}{\left(s\mu_I^{\alpha} - \lambda_r^{\alpha}\right)^2 \lambda_I^{\alpha}} \right] + \frac{1}{\mu_r^{\alpha}}, \\ P_{0r}^{\alpha} \frac{1}{\lfloor (s-1)} \left[\frac{\left(\frac{\lambda_r^{\alpha}}{\mu_I^{\alpha}}\right)^{s} \left(\lambda_r^{\alpha} \mu_r^{\alpha}\right)}{\left(s\mu_r^{\alpha} - \lambda_I^{\alpha}\right)^2 \lambda_r^{\alpha}} \right] + \frac{1}{\mu_I^{\alpha}} \end{array} \right]$$

6. Conclusion

Fuzzy estimators are used in a new approach to quantify the fuzzy M/M/S queueing system performance. The suggested approach might be applied when studying or developing a queuing model has been done with uncertain data. Therefore, when the parameters of the distributions of a queueing system cannot be point estimated from uncertain sample data, we eliminate the disadvantages of point estimation or confidence interval estimation.

7. REFERENCES

1. Prade, H. M. (1980). An outline of fuzzy or possibilistic models for queuing systems. In P. P. Wang & S. K. Chang (Eds.), *Fuzzy Sets*. Plenum Press, New York.
2. Li, R. J., & Lee, E. (1989). Analysis of fuzzy queues. *Computers & Mathematics with Applications, 17*(7), 1143–1147.
3. Negi, D. S., & Lee, E. S. (1992). Analysis and simulation of fuzzy queues. *Fuzzy Sets and Systems, 46*(3), 321–330.
4. Kao, C., Li, C. C., & Chen, S. P. (1999). Parametric programming to the analysis of fuzzy queues. *Fuzzy Sets and Systems*, *107*(1), 93–100.
5. Pardo, M. J., & Fuente, D. (2008). Design of a fuzzy finite capacity queuing model based on the degree of customer satisfaction: Analysis and fuzzy optimization. *Fuzzy Sets and Systems*, *159*(24), 3313–3332.
6. Klir, G. J., & Yuan, B. (1995). *Fuzzy sets and fuzzy logic: Theory and applications*. Prentice Hall, Upper Saddle River, NJ.
7. Nguyen, H. T. (1978). A note on the extension principle for fuzzy sets. *Journal of Mathematical Analysis and Applications*, *64*(2), 369–380.
8. Chrysafis, K. A., & Papadopoulos, B. K. (2006). On theoretical pricing of options with fuzzy estimators. *Journal of Computational and Applied Mathematics*, *223*(2), 552–566.
9. Chrysafis, K. A., & Papadopoulos, B. K. (2008). Cost-volume-profit analysis under uncertainty: A model with fuzzy estimators based on confidence intervals. *International Journal of Production Research*, *47*(21), 5977–5999.
10. Tsironis, L., Sfiris, D. S., & Papadopoulos, B. K. (2010). Fuzzy performance evaluation of workflow stochastic Petri nets by means of block reduction. *IEEE Transactions on Systems, Man, and Cybernetics—Part A: Systems and Humans*, *40*(2), 352–362.
11. Sfiris, D. S., & Papadopoulos, B. K. (2012). Fuzzy estimators in expert systems. *Applied Mathematical Sciences*, *6*(35), 1695–1718.
12. Taheri, S., Falsafain, A., &Mashinchi, M. (2008). Fuzzy estimation of parameters in statistical models. *Engineering and Technology*, (38), 318–324.
13. Rose, T. J. (2010). *Fuzzy logic with engineering applications*. John Wiley & Sons.
14. Srinivasan, R. (2014). Fuzzy queuing model using DSW algorithm. *International Journal of Advanced Research in Mathematics and Applications*, *1*(1), 57–62.
15. Sujatha, N., Murthy Akella, V. S. N., &Deekshitulu, G. V. S. R. (2017). Analysis of multiple server fuzzy queueing model using α-cuts. *International Journal of Mechanical Engineering and Technology*, *8*(10), 35–41.
16. Mueen, Z., Ramli, R., &Zaibidi, N. Z. (2017). Parametric nonlinear programming approach with fuzzy queues using hexagonal membership functions. *Journal of Computational and Theoretical*

Nanoscience, 14(10), 4979–4985.
17. Ritha, W., & Vinnarasi, S. J. (2018). Analysis of priority queues with pentagon fuzzy number. *International Journal of Engineering Technologies and Management Research, 5*(4), 90–100.
18. Visalakshi, V., & Suvitha, V. (2018). Performance measure of fuzzy queue using pentagonal fuzzy numbers. *Journal of Physics: Conference Series, 1000*, 012015.
19. Ramasamy, V., & Susmitha, R. (2018). Evaluation of performance measures of priority queues with fuzzy parameters using α-cut approach. *ARPN Journal of Engineering and Applied Sciences*, *13*(7), 2636–2641.
20. Suvitha, V., & Visalakshi, V. (2019). Erlang service queueing model with fuzzy parameters. *AIP Conference Proceedings*, *2112*, 020130. https://doi.org/10.1063/1.5112315.
21. Chen, G., Liu, Z., & Zhang, J. (2020). Analysis of strategic customer behavior in fuzzy queueing systems. *Journal of Industrial and Management Optimization, 16*(1), 371–386. https://doi.org/10.3934/jimo.2018157.
22. El-Paoumy, M. S. (2020). Performance measure of fuzzy simple queue with balking and feedback customers using symmetric heptagonal fuzzy numbers. *International Journal of Computer Applications*, *177*(47), 28–33.
23. Panta, A. P., Ghimire, R. P., Panthi, D., & Pant, S. R. (2021). Optimization of M/M/s/N queueing model with reneging in a fuzzy environment. *American Journal of Operations Research, 11*, 121–140. https://doi.org/10.4236/ajor.2021.113008.
24. Aarthi, S., & Shanmugasundari, M. (2023). Comparison of single server queuing performance measures using fuzzy queuing models and intuitionistic fuzzy queuing models with infinite capacity. *Journal of Intelligent & Fuzzy Systems, 44*(3), 4733–4746. https://doi.org/10.3233/JIFS-221367.
25. Vijaya, S., Srinivasan, N., & Jayasri, D. (2024). Analyzing performance measure of fuzzy queueing problem with trapezoidal fuzzy numbers by LR method. *Journal of Propulsion Technology, 45*(1).
26. Chrysafis, K. A., Sfiris, D. S., & Papadopoulos, B. K. (2014). Non-asymptotic fuzzy estimators based on confidence intervals. *Information Sciences, 279*, 446–459. https://doi.org/10.1016/j.ins.2014.03.131.