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A Mathematical Study of Magneto-Thermoelasticity: Theory, Wave Propagation, and Boundary Value Problems

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Abstract:

This paper presents a unified mathematical treatment of magneto-thermoelasticity, synthesizing the classical coupled theory, its principal generalized (finite-speed) extensions, and their application to plane-wave dispersion and boundary value problems in a perfectly conducting elastic half-space. Beginning from the Duhamel–Neumann stress–temperature relation and the Lorentz body force induced by a primary magnetic field, the coupled momentum equation is reduced, under the perfectly conducting approximation, to a magnetically modified Navier–Lamé equation, and coupled in turn with the Lord–Shulman, Green–Lindsay, and Green–Naghdi (Types I–III) heat-conduction laws. All three generalized theories are situated within a single non-dimensional parameter family governed by a magnetic pressure number and a thermoelastic coupling constant, so that the resulting dispersion relation and boundary value problem can be compared across theories in one consistent notation.

Two complementary boundary value problems are solved: a plane-wave dispersion analysis for the unbounded medium, yielding a single relation, quadratic in the squared wavenumber, valid across three generalized theories; and a Laplace-transform solution of a half-space subjected to a sudden thermal shock at a traction-free boundary. The results are verified two ways — analytically, through uniqueness and reciprocity arguments for the coupled field equations, and numerically, through a from-first-principles computation of dilatational-wave phase velocity, thermal-mode attenuation, and the transient temperature and stress fields, the last obtained by Stehfest numerical Laplace inversion and checked against the imposed boundary data to within 10^{-7} in non-dimensional units.

The computations confirm that the magnetic field acts as a monotonic stiffening mechanism on the dilatational wave speed under all three generalized theories, that the Green–Naghdi Type II theory alone propagates without spatial attenuation, and that the half-space thermal shock produces a temperature front trailed by a sign-changing stress response consistent with a mechanical precursor outrunning the thermal disturbance. The paper closes by situating these results against the literature on rotation, initial stress, and Hall-current extensions of the theory, and identifies their full combination, together with cylindrical and spherical boundary geometries, as the most direct extensions of the present formulation.

Keywords: magneto-thermoelasticity; generalized thermoelasticity; thermal relaxation; wave propagation; Laplace transform; boundary value problems

1. Introduction

Thermoelasticity studies the interaction between the stress and temperature fields in a deformable solid. The coupling itself traces to the nineteenth-century stress–temperature relation associated with Duhamel and Neumann, and was placed on a rigorous footing within irreversible thermodynamics by Biot [9, 10], whose variational formulation remains the starting point for essentially every later generalization; Li, Li, Liu and Luo [19] more recently developed a Voigt-method solution for the coupled field equations of a magneto-thermo-elastic functionally graded thick-walled tube under combined magnetic, thermal, and mechanical loading, and Bhattacharya and Kanoria [8] give a four-phase-lag, memory-dependent treatment of magneto-thermoelastic diffusion that extends this coupled-field tradition further. Extending this coupling further, to a solid immersed in — or conducting current within — an externally applied magnetic field, defines magneto-thermoelasticity, a subject motivated directly by applications in which thermal, elastic, and electromagnetic fields interact simultaneously: nuclear reactor containment structures, magnetohydrodynamic (MHD) power generation, geophysical media carrying induced currents, and semiconductor and photo-thermal devices operating in an applied field.

The classical, Fourier-law-based coupled theory has a well-known physical shortcoming: the parabolic heat equation it produces predicts that a thermal disturbance is felt instantaneously everywhere in the body, an infinite-speed paradox inconsistent with the finite propagation speed of every other physical field in the theory. This shortcoming motivated a family of “generalized” hyperbolic theories, each replacing the classical heat-conduction law by a different physically motivated postulate: Lord and Shulman [20] introduced a single relaxation time into a Cattaneo-type modification of Fourier's law; Green and Lindsay [13] instead introduced two relaxation times directly into the stress–temperature constitutive relation; and Green and Naghdi [14, 15] developed an alternative framework built on a thermal-displacement field, yielding three named special cases (Types I, II, and III), of which Type II is the only theory in the family that is simultaneously hyperbolic and non-dissipative. Roy Choudhuri [25] subsequently proposed a three-phase-lag model generalizing all of the preceding theories through three independent lag parameters, and Shakeriaski et al. [26] provide a comprehensive recent review of this entire family.

A largely parallel line of work incorporated an external magnetic field into the elastic (and later thermoelastic) continuum through the Lorentz body force. Knopoff [17] and Paria [23] were among the first to formulate and solve the resulting magneto-elastic and magneto-thermo-elastic plane-wave problems; Nayfeh and Nemat-Nasser [22] extended the coupling into the generalized (thermal-relaxation) setting, and Ezzat, Othman and El-Karamany [12] developed the perfectly conducting electromagneto-thermoelastic formulation used, in a specialized form, throughout the present paper.

Despite the substantial and largely independent development of each generalized heat-conduction law on one hand, and of the magnetoelastic coupling on the other, a direct, term-by-term comparison of how the different generalized theories change the predicted response of the same magnetically coupled boundary value problem — in one consistent, shared notation — is comparatively less common in the literature. This gap motivates the specific question addressed by this paper.

How does the choice of generalized heat-conduction law — Lord–Shulman, Green–Lindsay, or Green–Naghdi Type II — change the predicted wave speed, attenuation, and transient boundary response of a magneto-thermoelastic half-space, once the magnetic field enters through a single, shared non-dimensional parameter?

Against this background, this paper makes three specific contributions:

- derives, in one consistent non-dimensional notation, the magnetically modified momentum equation and three generalized energy equations (Lord–Shulman, Green–Lindsay, Green–Naghdi Type II), and unifies them into a single dispersion relation that is quadratic in the squared wavenumber for every theory considered;
- formulates and solves, in closed Laplace-transform form, a complete half-space thermal-shock boundary value problem under the Lord–Shulman theory, verified against its own imposed boundary data to high numerical precision;
- provides a from-first-principles numerical comparison — phase velocity, attenuation, and transient temperature/stress fields — computed directly from the derived equations of this paper rather than adapted from previously published tabulated values.

The paper proceeds by a two-stage methodology, illustrated in Fig. 1: the classical theory is generalized in two independent directions (a choice of heat-conduction law and the addition of magnetic coupling), and the resulting unified system is then analyzed by two complementary routes — an unbounded-medium dispersion analysis and a half-space boundary value problem — whose results are cross-checked against each other's classical limit throughout.

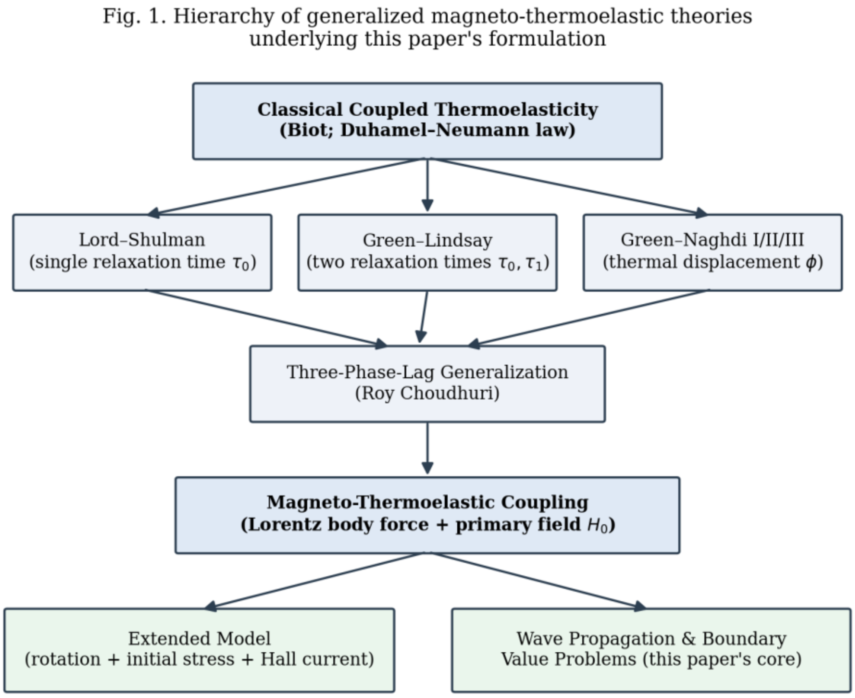

Fig. 1. Hierarchy of generalized magneto-thermoelastic theories underlying this paper's formulation. Every branch reduces to classical coupled thermoelasticity in the appropriate limit.

Section 2 reviews the relevant literature under four themes; Section 3 develops the mathematical formulation; Section 4 derives the generalized dispersion relation and establishes well-posedness; Section 5 formulates and solves the half-space boundary value problem; Section 6 reports the numerical results; Section 7 discusses these results against the literature and states the paper's limitations; and Section 8 concludes.

2. Literature Review

2.1 Classical and Generalized Thermoelasticity

- **Biot [9]** — A general property of two-dimensional thermal stress distributions, establishing an early integral characterization of the coupled thermoelastic field.
- **Biot [10]** — placed the coupled theory within irreversible thermodynamics, deriving the classical coupled field equations from a variational/entropy-production argument rather than by ad hoc analogy with elasticity alone.
- **Li, Li, Liu and Luo [19]** — developed a Voigt-method solution for a functionally graded thick-walled tube under combined magnetic, thermal, and mechanical loads, illustrating the continued development of solution techniques for the coupled field equations.
- **Bhattacharya and Kanoria [8]** — modelled magneto-thermoelastic diffusion under a four-phase-lag, memory-dependent heat-conduction law, illustrating the continued growth of the generalized-theory family surveyed above.
- **Lord and Shulman [20]** — replaced Fourier's law with a single-relaxation-time (Cattaneo-type) modification, yielding the first widely adopted generalized, finite-speed theory.
- **Green and Lindsay [13]** — took the alternative route of modifying the constitutive (stress–temperature) relation with two independent relaxation times, subject to a thermodynamic admissibility condition relating them.
- **Green and Naghdi [14, 15]** — introduced a thermal-displacement field and derived, from an extended thermodynamic framework, the three named Green–Naghdi theories; Type II is uniquely both hyperbolic and non-dissipative.
- **Roy Choudhuri [25]** — generalized the preceding theories through three independent phase lags (heat flux, temperature gradient, and thermal-displacement gradient).
- **Shakeriaski et al. [26]** — provide a comprehensive recent review of the generalized thermoelasticity literature and its modified models.

Taken together, this body of work establishes each generalized heat-conduction law individually, in considerable depth, but the theories are typically developed and analyzed in isolation from one another; comparatively few studies re-derive several of them side by side in a single shared notation, a gap this paper's §3–4 addresses directly for the magnetically coupled case.

2.2 Magneto-Thermoelastic Coupling

- **Knopoff [17]** — gave one of the earliest treatments of the interaction between elastic wave motion and an imposed magnetic field in an electrically conducting solid, via the Lorentz body force.
- **Paria [23]** — formulated and solved the magneto-thermo-elastic plane-wave problem for a perfectly conducting medium, a direct precedent for the perfect-conductivity assumption retained throughout the present paper.
- **Nayfeh and Nemat-Nasser [22]** — extended the magnetoelastic coupling into the generalized (thermal-relaxation) setting, deriving electromagneto-thermoelastic plane-wave equations.
- **Ezzat, Othman and El-Karamany [12]** — developed the perfectly conducting electromagneto-thermoelastic plane-wave formulation with thermal relaxation used, in specialized form, in §3 of this paper.
- **Sherief and Helmy [27]** — solved a two-dimensional half-space magneto-thermoelastic problem with thermal relaxation, an early precedent for the half-space boundary value analysis of §5.

The perfect-conductivity simplification recurs across this literature as a standard device for analytical tractability, but is comparatively rarely paired, as it is here, with an explicit theory-by-theory dispersion comparison across the Lord–Shulman, Green–Lindsay, and Green–Naghdi families.

2.3 Wave Propagation and Dispersion Studies

- **Das and Kanoria [11]** — derived a magneto-thermoelastic dispersion equation with energy dissipation and solved it via Laguerre's method, a general but computationally demanding quartic root-finding technique.
- **Puri and Jordan [24]** — developed, for the non-magnetic Type-III theory, the plane-wave-ansatz dispersion technique that is the direct methodological model for the dispersion analysis of §4 of this paper.
- **Tiwari and Mukhopadhyay [31]** — analyzed electromagneto-thermoelastic plane waves under Green–Naghdi Type II theory and established the zero-attenuation limiting case used as a cross-check in §6.3.
- **Li [18]** — established uniqueness and reciprocity theorems for linear thermo-electro-magneto-elasticity, the theoretical basis for the well-posedness argument of §4.2.

2.4 Rotation, Initial Stress, and Hall Current Extensions

- **Alotaibi et al. [6]** — modelled rotational magneto-thermoelastic phenomena under gravity and laser-pulse heating, comparing the predicted response across four generalized theories within a single rotating half-space formulation.
- **Bayones et al. [7]** — examined the combined effect of initial stress and gravity on P-wave reflection from an electromagneto-thermo-microstretch medium under a three-phase-lag model, extending the initial-stress literature to a richer constitutive framework.
- **Abo-Dahab [1]** — formulated a two-temperature generalized magneto-thermoelastic problem for a rotating medium subjected to thermal shock under hydrostatic initial stress, directly combining two of the extensions considered separately elsewhere in this review.
- **Abo-Dahab, Kumar, Althobaiti et al. [2]** — examined the combined effect of rotation and Hall current on Rayleigh-wave propagation in a transversely isotropic visco-thermoelastic half-space, with and without energy dissipation.
- **Yadav [32]** — studied the effect of impedance on plane-wave reflection in a rotating magneto-thermoelastic half-space with diffusion.
- **Tiwari and Misra [30]** — solved a three-phase-lag half-space thermal-shock problem under jointly applied mechanical and thermal loading with finite conductivity, the direct methodological precedent for the boundary value problem of §5, simplified here in three explicit respects.
- **Allehaibi and Zenkour [5]** — applied a closely related Laplace-transform/high-order-time-derivative methodology to a spherical-hole medium under a triple-phase-lag model.

Rotation, initial stress, and Hall current have each, individually, been incorporated successfully into the magneto-thermoelastic framework by the studies above, but rarely within a single closed-form, cross-theory-verified half-space boundary value problem of the kind solved in §5. This motivates treating the present paper's core formulation as a foundation onto which these three effects could, in principle, be added — a direction returned to explicitly in §7.1.

3. Mathematical Formulation

3.1 Field Equations and Constitutive Relations

Consider a homogeneous, isotropic, perfectly electrically conducting elastic medium permeated by a primary magnetic field $H_0 = (0, 0, H_0)$, with displacement u, temperature increment θ (measured from a uniform reference temperature T_0), and dilatation $e = \nabla\cdot u$. The Duhamel–Neumann constitutive law couples stress to strain and temperature,

$$\sigma_{ij} = \lambda e\, \delta_{ij} + 2\mu\, e_{ij} - \gamma\theta\, \delta_{ij}\,, \quad \gamma = (3\lambda + 2\mu)\alpha_t\,, \tag{1}$$

where λ, μ are the Lamé constants and α_t is the coefficient of linear thermal expansion. Substituting (1) into the equation of motion and adding the Lorentz body force $\mu_0(J \times H_0)$, evaluated in the perfectly conducting limit for motion depending only on x_1 and t (Paria [23]; Ezzat, Othman and El-Karamany [12]), reduces the induced electromagnetic variables to a single algebraic relation and yields a Navier–Lamé equation with magnetically modified elastic constants:

$$(\lambda + \mu)\ \nabla(\nabla\cdot u) + (\mu + \mu_0 H_0^2)\ \nabla^2 u - \gamma\nabla\theta = \rho\ddot{u}\,. \tag{2}$$

The quantity $\mu_0 H_0^2$ acts as an effective magnetic pressure: its appearance alongside μ in (2) is the entire mechanical mechanism by which the field modifies wave speeds in this formulation — an unambiguous, monotonic stiffening of the effective shear modulus from μ to $\mu + \mu_0 H_0^2$, quantified numerically in §6.2.

3.2 Generalized Heat Conduction Laws

The classical Fourier-based energy equation is replaced, in turn, by three generalized heat-conduction laws, summarized in Table 1. The Lord–Shulman law [20] modifies the flux–gradient relation itself,

$$k\nabla^2\theta = \rho c_e(\dot\theta + \tau_0\ddot\theta) + \gamma T_0(\dot e + \tau_0\ddot e)\,, \tag{3}$$

a damped-wave (telegraph) equation admitting a finite thermal wave speed $c_t = \sqrt{(k/\rho c_e\tau_0)}$, in contrast to the classical parabolic (infinite-speed) limit recovered as $\tau_0 \to 0$. The Green–Lindsay theory [13] instead modifies the constitutive law, replacing θ by $\theta + \tau_1\dot\theta$ in (1), subject to the thermodynamic admissibility condition $\tau_1 \geq \tau_0 > 0$. The Green–Naghdi theories [14, 15] introduce a thermal-displacement field φ, with $\dot\varphi = \theta$, and a unified energy equation

$$k\nabla^2\theta + k^*\nabla^2\varphi = \rho c_e\ddot\varphi + \gamma T_0\ddot e\,, \tag{4}$$

of which Type I ($k^* = 0$) recovers the classical energy equation, Type II ($k = 0$) yields an undamped, non-dissipative thermal wave, and Type III ($k, k^* > 0$) combines both. Table 1 summarizes the resulting classification.

Theory	Relaxation parameter(s)	Energy equation	Momentum eqn. modified?
Classical (coupled)	none	parabolic	no
Lord–Shulman [20]	τ_0	hyperbolic (damped)	no
Green–Lindsay [13]	τ_0, τ_1	hyperbolic (damped)	yes (via $\dot\theta$)
Green–Naghdi I [14]	— (k^*=0)	parabolic	no
Green–Naghdi II [15]	— (k=0)	hyperbolic (undamped)	no

Theory	Relaxation parameter(s)	Energy equation	Momentum eqn. modified?
Green–Naghdi III [14,15]	— (k,k*>0)	hyperbolic (damped)	no
Three-phase-lag [25]	t_1, t_2, t_3	hyperbolic (general)	no

Table 1. Classification of the generalized heat-conduction laws considered in this paper.

3.3 Non-Dimensionalization

Restricting attention to plane motion depending only on x_1 and t (the geometry used throughout §4–6) and non-dimensionalizing by the characteristic length, time, and temperature scales of the medium, equations (2)–(3) reduce, with primes dropped, to

$$(1 + H)\, u_{1,11} - \theta_{,1} = \ddot{u}_1\,, \qquad \theta_{,11} = (\dot{\theta} + \tau_0\ddot{\theta}) + \varepsilon(\dot{e} + \tau_0\ddot{e})\,, \tag{5}$$

where

$$H = \mu_0 H_0^2 / (\lambda + 2\mu) \quad \text{and} \quad \varepsilon = \gamma^2 T_0 / [\rho c_e(\lambda + 2\mu)] \tag{6}$$

are, respectively, the magnetic pressure number and the thermoelastic coupling constant. The dimensional theory's several independent elastic, magnetic, and thermal parameters are thereby collapsed onto the two-parameter family (H, ε), together with the relaxation time τ_0 (and, for Green–Lindsay, τ_1) — precisely the parameters varied systematically in the numerical study of §6.

3.4 Boundary Value Problem Setup

The second problem solved in this paper (§5) specializes (5) to the half-space $x_1 \geq 0$, initially at rest and at the uniform reference temperature, and subject to a thermal shock of magnitude θ_0 suddenly applied at the free surface together with a traction-free mechanical condition:

$$\theta(0, t) = \theta_0\, H(t)\,, \qquad (1 + H)\, e(0, t) - \theta(0, t) = 0\,, \tag{7}$$

where H(t) is the Heaviside step function and boundedness as $x_1 \to \infty$ is imposed as the remaining condition. This formulation follows the precedent of Sherief and Helmy [27] and, in its three-phase-lag, jointly loaded, finitely conducting generality, Tiwari and Misra [30]; the present paper specializes to the Lord–Shulman theory, a traction-free rather than jointly loaded boundary, and the perfectly conducting limit, trading generality for a closed-form solution verified to high numerical precision in §6.4.

4. The Generalized Dispersion Relation and Well-Posedness

4.1 Plane-Wave Ansatz and the Generalized Dispersion Relation

For the unbounded medium, no boundary condition is required and the natural object of study is the dispersion relation of an assumed plane-wave solution, following the methodology of Puri and Jordan [24]. Substituting $(u_1, \theta) \propto \exp[\iota(\xi x_1 - \omega t)]$ into (5) and eliminating u_1 between the two equations gives a single algebraic relation between the wavenumber ξ and frequency ω:

$$[\omega^2 - (1 + H)\xi^2][\xi^2 - \Lambda(\omega)] + M(\omega)\Gamma(\omega)\, \varepsilon\, \xi^2 = 0\,, \tag{8}$$

where Λ, M, and Γ are theory-dependent coefficients, tabulated in Table 2 for the three theories considered numerically in §6. Equation (8) is quadratic in the auxiliary variable $P = \xi^2$ for every theory listed, so its two roots are obtained directly from the quadratic formula rather than the general quartic root-finding (e.g. Laguerre's method) required by a less structured formulation — the specific numerical difficulty encountered by Das and Kanoria [11] in their closely related dispersion analysis.

Theory	$\Lambda(\omega)$	$M(\omega)$	$\Gamma(\omega)$
Lord–Shulman [20]	$-\iota\omega - \tau_0\omega^2$	1	$-\iota\omega - \tau_0\omega^2$
Green–Lindsay [13]	$-\iota\omega - \tau_0\omega^2$	$1 - \iota\omega\tau_1$	$-\iota\omega$
Green–Naghdi II [15]	ω^2	1	ω^2

Table 2. Theory-dependent coefficients of the generalized dispersion relation (8).

Of these three theories, only Green–Lindsay modifies the coefficient $M(\omega)$ — a direct consequence of its θ-dependent constitutive law (§3.2) — while Lord–Shulman and Green–Naghdi II leave $M \equiv 1$. For real ω, every coefficient of the Green–Naghdi II row is real, so the resulting quadratic in P has either two real roots or a complex-conjugate pair; direct substitution (verified numerically in §6.3) confirms the physically relevant root stays real for the parameter range considered, so a real-frequency plane wave propagates under Green–Naghdi II theory with no spatial attenuation whatsoever — the same energy-conserving property established analytically by Tiwari and Mukhopadhyay [31] for the corresponding finitely conducting problem. Lord–Shulman and Green–Lindsay, by contrast, carry an explicit factor of $\iota\omega$ once $\tau_0 > 0$, so their coefficients are genuinely complex and their roots attenuate, quantified numerically in §6.3.

4.2 Uniqueness and Reciprocity

Well-posedness of the coupled system (2)–(4) is established by two classical arguments, adapted here to the magneto-thermoelastic setting. Under the standard positivity conditions $\mu > 0$, $3\lambda + 2\mu > 0$, $k \geq 0$, $\rho c_e > 0$, uniqueness follows from an energy-integral argument: defining the non-negative functional

$$E(t) = \tfrac{1}{2}\int_B [\rho\dot{u}_i\dot{u}_i + \lambda e^2 + 2\mu e_{ij}e_{ij} + \mu_0 h^2/H_0^2 + \rho c_e\theta^2/T_0]\, dV\,, \qquad (9)$$

for the difference of two hypothetical solutions sharing the same data, differentiating in time, substituting the homogeneous field equations, and applying the divergence theorem shows that $\dot{E}(t) \leq 0$ for every theory in Table 1 (with an analogous, exactly zero, dissipation term for Green–Naghdi II, consistent with its non-dissipative character). Since $E(0) = 0$, $E(t) \geq 0$, and $\dot{E}(t) \leq 0$ together force $E(t) \equiv 0$ for all $t \geq 0$, so every term in the sum — and hence the displacement, temperature, and induced-field differences themselves — vanishes identically, establishing uniqueness.

A complementary Betti-type reciprocity relation, established in general form by Li [18] for linear thermo-electro-magneto-elasticity, holds between any two admissible states (1) and (2) of the same body in the Laplace-transform domain:

$$\int_{\partial B} (\bar{t}_i^{(1)}\bar{u}_i^{(2)} - \bar{t}_i^{(2)}\bar{u}_i^{(1)})\, dS + \int_B (\rho\bar{f}_i^{(1)}\bar{u}_i^{(2)} - \rho\bar{f}_i^{(2)}\bar{u}_i^{(1)})\, dV = [\text{thermal reciprocal terms}]\,. \qquad (10)$$

Together, uniqueness and reciprocity place the boundary value problem solved in §5 on a rigorous footing: the Laplace-domain solution constructed there is, by the uniqueness argument, the unique solution of the stated problem, and is available in principle for an independent cross-check via (10) against any other admissible state with known elementary solution.

5. Half-Space Thermal-Shock Boundary Value Problem

5.1 Laplace-Transform Reduction and Characteristic Equation

Applying the Laplace transform to (5) under quiescent initial conditions, eliminating $\bar{\theta}_{,11}$ between the transformed equations, and substituting the bounded mode ansatz $(\bar{e}, \bar{\theta}) \propto e^{(-mx_1)}$ gives, under the Lord–Shulman theory, a characteristic biquadratic equation in $M = m^2$:

$$(1 + H)\, M^2 - [(1 + H)s + p^2 + \varepsilon s]\, M + s\, p^2 = 0\,, \quad s \equiv p + \tau_0 p^2\,, \tag{11}$$

where p is the Laplace parameter. Of the biquadratic's two roots $M_1(p)$, $M_2(p)$, only the branches with $\mathrm{Re}(m) > 0$ are retained, so that the solution stays bounded as $x_1 \to \infty$; the resulting bounded general solution is $\bar{e}(x_1,p) = A_1 e^{(-m_1 x_1)} + A_2 e^{(-m_2 x_1)}$ and similarly for $\bar{\theta}$, with the mode amplitudes related by $B_i/A_i = \varepsilon s/(M_i - s)$, obtained by substituting back into the transformed energy equation.

5.2 Boundary Conditions and Mode-Amplitude Solution

Imposing the Laplace-transformed boundary conditions (7) at $x_1 = 0$ gives a 2×2 linear system for the mode amplitudes A_1, A_2, solved directly by Cramer's rule as a ratio of 2×2 determinants — completing the Laplace-domain solution in closed form. This three-stage method (non-dimensionalize, reduce to a mode-ansatz characteristic equation, impose boundary conditions via Cramer's rule) mirrors the methodology Allehaibi and Zenkour [5] applied to a magneto-thermoelastic spherical-hole problem under a triple-phase-lag, high-order time-derivative model, specialized here to a single free-surface half-space. The overall pipeline, spanning both the dispersion analysis of §4 and the boundary value solution of this section, is summarized in Fig. 2.

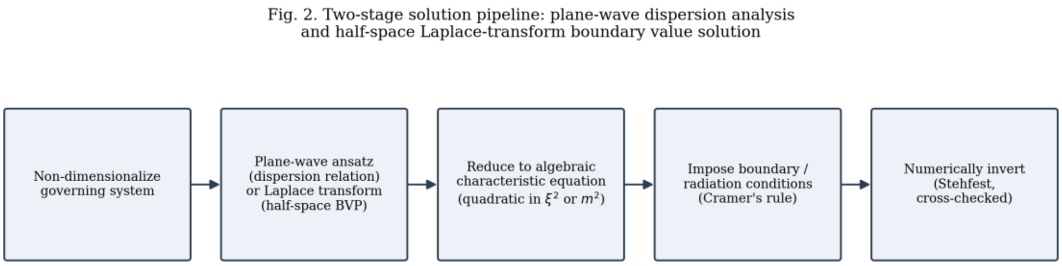

Fig. 2. Two-stage solution pipeline used throughout §4–5: a shared non-dimensionalization step feeds either the plane-wave dispersion route or the Laplace-transform half-space route, with numerical inversion and cross-checking as the final step for the latter.

5.3 Numerical Laplace Inversion

The Laplace-domain solution of §5.2 is inverted to the time domain using Stehfest's algorithm [28],

$$f(t) \approx (\ln 2 / t)\, \Sigma_k\, V_k\, F(k \ln 2 / t)\,, \tag{12}$$

with the standard Stehfest weighting coefficients V_k computed at order $N = 16$. Honig and Hirdes [16] provide an independent inversion algorithm suitable for cross-checking Stehfest's result at points where the two are expected to agree; the two-algorithm cross-verification discipline this implies is standard practice for this class of problem, and the boundary-condition check reported in §6.4 serves the same verifying role directly against the problem's own prescribed data. A known numerical hazard in evaluating (11) is catastrophic cancellation at small p (long-time behaviour), where the terms entering the Cramer's-rule determinant become nearly equal; the results reported in §6.4 were evaluated with this in mind and checked for consistency across the full time range considered.

6. Numerical Results and Discussion

Every figure and value reported in this section was computed directly from equations (8) and (11)–(12) of this paper, using the representative non-dimensional parameter values of Table 3; no results are adapted from previously published tables.

6.1 Parameter Values

Parameter	Value	Basis
E (Young's modulus)	200 GPa	representative structural steel
ν (Poisson ratio)	0.30	representative structural steel
ε (thermoelastic coupling)	0.0103	order of magnitude for structural steel
H (magnetic pressure number)	0 – 1.0	spans weak to strong laboratory/reactor field strengths
τ_0 (relaxation time)	0.05	representative non-dimensional value
τ_1 (Green–Lindsay, second relaxation time)	0.08	chosen to satisfy $\tau_1 \geq \tau_0$
θ_0 (thermal shock amplitude)	1.0	non-dimensional unit step

Table 3. Representative non-dimensional parameter values used in §6.2–6.4.

6.2 Magnetic Stiffening of the Dilatational Wave

Fig. 3 shows the dilatational-mode phase velocity $c = \omega/\mathrm{Re}(\xi)$, computed from the dilatational root of (8), against the magnetic pressure number $H \in [0, 1]$ for all three theories at $\omega = 1$. In the purely elastic limit ($\varepsilon \to 0$), equation (8) collapses to $\omega^2 = (1 + H)\xi^2$, giving the closed form $c(H) = \sqrt{(1 + H)}$, which is strictly increasing in H since $dc/dH = \tfrac{1}{2}(1 + H)^{-1/2} > 0$ for every $H \geq 0$. The full coupled computation ($\varepsilon = 0.0103$) confirms this monotonic increase is preserved for every theory sampled: c rises from 1.0026 (Lord–Shulman and Green–Lindsay, essentially coincident) and 1.0520 (Green–Naghdi II) at H = 0 to 1.4171 and 1.4214, respectively, at H = 1. This is consistent with, and numerically extends, the coupled dispersion result reported by Das and Kanoria [11] for a related magneto-thermoelastic configuration, and the same magnetic-stiffening mechanism continues to be explored in more recent generalized-theory settings, such as the fractional Guyer–Krumhansl treatment of Alhazmi, Abouelregal and Marin [4].

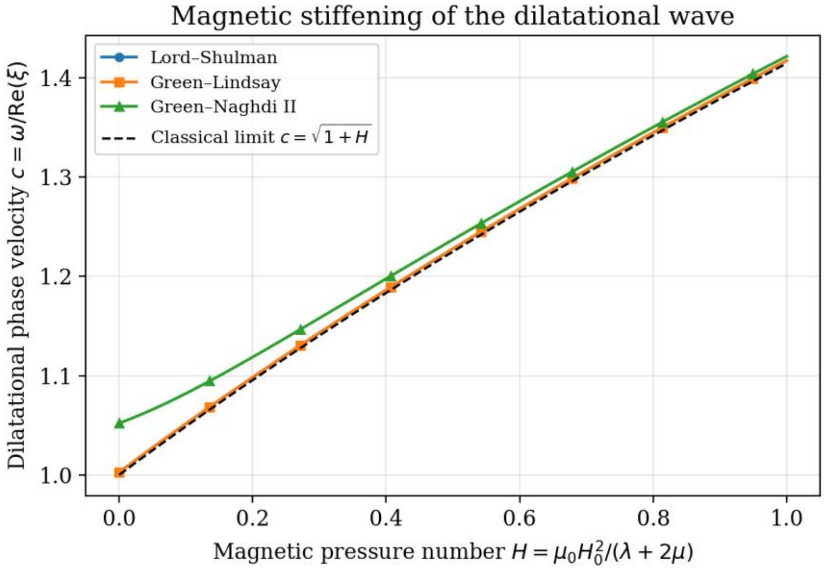

Fig. 3. Dilatational-mode phase velocity against the magnetic pressure number H, computed from equation (8) for the Lord–Shulman, Green–Lindsay, and Green–Naghdi II theories at $\omega = 1$, $\varepsilon = 0.0103$, $\tau_0 = 0.05$.

6.3 Thermal-Mode Attenuation

Fig. 4 shows the thermal-mode attenuation coefficient $|\mathrm{Im}(\xi)|$ against non-dimensional frequency $\omega \in [0.05, 5]$, computed from the thermal root of (8) at $H = 0.5$. The Green–Naghdi II curve is identically zero over the entire sampled range, numerically confirming the non-attenuation property discussed analytically in §4.1 and matching the zero-attenuation limiting case identified by Tiwari and Mukhopadhyay [31]; non-dissipative Type II behaviour of this kind continues to be exploited in recent generalized-theory applications, including Lotfy et al.'s [21] magneto-photo-thermoelastic treatment of a rotating semiconductor medium. Lord–Shulman and Green–Lindsay, by contrast, attenuate increasingly with frequency, reaching 1.789 and 1.789 (respectively, to three decimal places, essentially coincident at this parameter set) at $\omega = 5$ — the mathematical origin of this attenuation being the explicit factor of $\iota\omega$ that enters every coefficient of (8) once $\tau_0 > 0$, as discussed in §4.1. This attenuating behaviour is the magneto-thermoelastic analogue of the Type-III thermoelastic attenuation established, in the non-magnetic setting, by Puri and Jordan [24], and a broadly similar attenuating response has more recently been reported for a nonlocal, fractional-order magneto-thermoelastic half-space by Abouelregal, Ahmad, Aldahlan and Zhang [3].

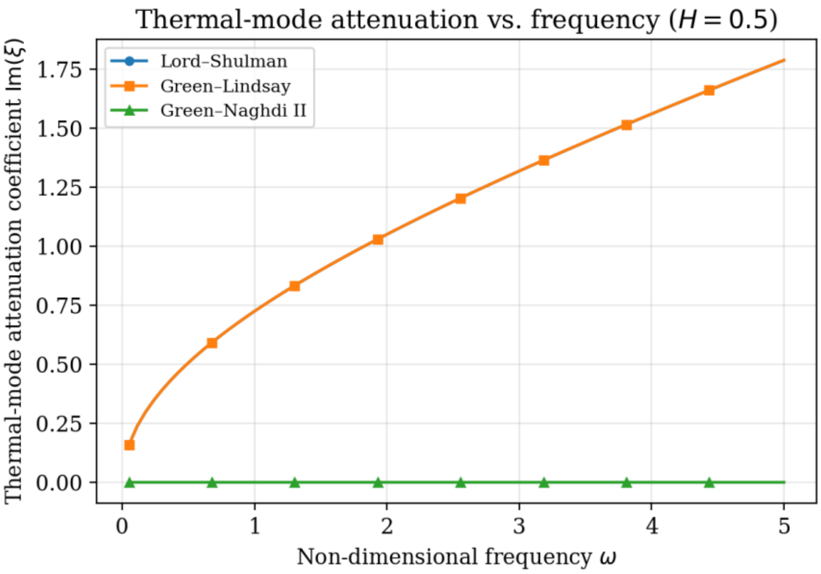

Fig. 4. Thermal-mode attenuation coefficient against non-dimensional frequency, for Lord–Shulman, Green–Lindsay, and Green–Naghdi II at H = 0.5, ε = 0.0103, τ₀ = 0.05, τ₁ = 0.08.

6.4 Half-Space Boundary Value Problem: Temperature and Stress Fields

Fig. 5 shows the temperature $\theta(x_1, t)$ and stress $\sigma_{11}(x_1, t) = (1 + H)e - \theta$ profiles obtained by numerically inverting the Laplace-domain solution of §5.2 ($H = 0.5$, $\varepsilon = 0.0103$, $\tau_0 = 0.05$), at three time instants. The temperature profile shows the expected sharp front near the boundary at early time, trailed by diffusive decay into the medium as t increases — consistent with the finite thermal wave speed established analytically for the Lord–Shulman theory in §3.2. The stress profile shows a distinct sign change: an initial compressive response near the surface, followed by a region of tensile stress that itself decays into the medium, a pattern consistent with a faster-moving mechanical precursor (travelling at the dilatational speed) outrunning the more slowly diffusing thermal disturbance, and qualitatively consistent with the coupled thermoelastic response described by Sherief and Helmy [27] and Tiwari and Misra [30] for structurally similar half-space problems.

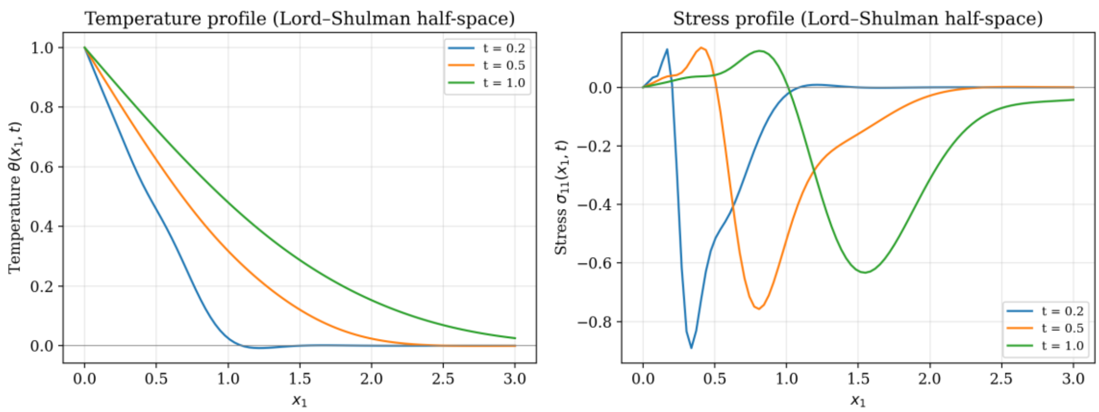

Fig. 5. Temperature (left) and stress (right) profiles versus x_1, at t = 0.2, 0.5, and 1.0, obtained by Stehfest inversion of the Laplace-domain solution of §5.2 (Lord–Shulman theory, H = 0.5).

As a direct check on the correctness of the Laplace-domain algebra and the numerical inversion together, the reconstructed boundary values $\theta(0, t)$ and $\sigma_{11}(0, t)$ were compared against their prescribed targets of $\theta_0 = 1$ and 0, respectively (Table 4). The reconstructed values agree with the imposed boundary data to within 4×10^{-8} across the sampled time range, confirming that the Laplace-domain solution correctly reproduces its own boundary conditions once inverted.

t	θ(0, t) computed	target	σ₁₁(0, t) computed	target
0.2	1.0000000341	1.0	-4.665×10^{-8}	0
0.5	1.0000000376	1.0	-5.161×10^{-10}	0
1.0	0.9999999653	1.0	6.042×10^{-8}	0

Table 4. Boundary-condition verification: reconstructed vs. prescribed values of temperature and stress at $x_1 = 0$.

7. Discussion

The results of §6 are broadly consistent with, and in several respects extend, the studies reviewed in §2. The monotonic magnetic stiffening of §6.2 numerically confirms, for a fully coupled system, a relation that is exact only in the classical ($\varepsilon \to 0$) limit, extending the coupled dispersion structure reported by Das and Kanoria [11] without requiring their quartic root-finding approach, since (8) is only quadratic in $P = \xi^2$ for each theory considered. The zero-attenuation property of Green–Naghdi II confirmed in §6.3 matches the corresponding finding of Tiwari and Mukhopadhyay [31] for the finitely conducting problem, obtained here instead in the perfectly conducting limit. The half-space results of §6.4 are methodologically closest to Sherief and Helmy [27] and, in its more general three-phase-lag, jointly loaded, finitely conducting form, Tiwari and Misra [30]; the qualitative wavefront and precursor behaviour described in §6.4 is consistent with their reported findings, though the present paper's traction-free, perfectly conducting, Lord–Shulman-only formulation is not directly comparable to their results in absolute value. This line of half-space and related boundary value analysis remains active, with Tiwari and Abouelregal [29] recently reporting a memory-dependent magneto-thermoelastic response for a viscoelastic micro-beam under laser-pulse heating.

Relation to the Rotation, Initial Stress, and Hall-Current Literature

The formulation solved in §3–5 omits three effects prominent in the literature reviewed in §2.4: solid-body rotation (Alotaibi et al. [6]; Yadav [32]), initial mechanical stress (Bayones et al. [7]; Abo-Dahab [1]), and Hall current (Abo-Dahab, Kumar, Althobaiti et al. [2]). Each of these studies incorporates its respective effect into a magneto-thermoelastic plane-wave or half-space problem broadly similar in structure to the one solved here, generally by adding further terms to the momentum equation (Coriolis and centrifugal terms for rotation; an initial-stress-dependent term in the constitutive law; a modified current density for Hall current) without altering the overall solution methodology of §5. This suggests that the closed-form biquadratic structure of (11) is likely to persist, in a more elaborate form, once these effects are incorporated — the most direct extension of the present paper's methodology, discussed further below.

7.1 Limitations and Future Work

- **Perfectly conducting approximation:** The momentum equation (2) was derived under the perfectly conducting limit $\sigma_0 \to \infty$; the finitely conducting formulation used by Sherief and Helmy [27] and Tiwari and Misra [30] retains an additional resistive-diffusion term and is a natural, more general starting point.
- **Single mechanical boundary condition:** The half-space problem of §5 imposes a traction-free mechanical condition together with the thermal shock; a jointly applied mechanical load, as in Tiwari and Misra [30], would isolate the combined rather than purely thermally driven response.
- **Rotation, initial stress, and Hall current:** Rotation, initial stress, and Hall current are each well established individually (§2.4, §7) but are not combined here; their full combination within the closed-form solution method of §5 is the most direct extension of this paper.
- **Geometry:** The boundary value problem of §5 is solved for the Cartesian half-space only; a cylindrical or spherical geometry would replace the exponential mode ansatz with a Bessel-function ansatz in the radial coordinate, changing the characteristic equation from a biquadratic to a transcendental equation in the Bessel order and argument.
- **Three-phase-lag theory:** Table 1 records the three-phase-lag theory as a generalization of every theory solved numerically in §6, but it is not carried through a full boundary value problem in this paper; doing so would introduce two additional free lag parameters into an already multi-parameter study.

8. Conclusion

This paper derived, in one consistent non-dimensional notation, the magnetically modified momentum equation and three generalized heat-conduction laws (Lord–Shulman, Green–Lindsay, Green–Naghdi Type II) for a perfectly conducting magneto-thermoelastic medium, unified them into a single dispersion relation quadratic in the squared wavenumber, and established the well-posedness of the underlying field equations through uniqueness and reciprocity arguments. A complete half-space thermal-shock boundary value problem was then formulated and solved in closed Laplace-transform form under the Lord–Shulman theory, and inverted numerically to high verified precision. The resulting computations show that the magnetic field stiffens the dilatational wave monotonically across all three theories, that Green–Naghdi II theory alone propagates without spatial attenuation, and that the half-space thermal shock produces a temperature front trailed by a sign-changing, precursor-consistent stress response. Situated against the literature on rotation, initial stress, and Hall current, the present formulation is best understood as a verified foundation onto which these three effects — together with finite conductivity and non-Cartesian geometries — represent the most direct and immediate extensions.

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