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Continuous Linear Operators on Bicomplex Multi-Normed Spaces

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Abstract

In order to interact with a multi-norm, continuous bicomplex-linear operators must distinguish between ordinary boundedness and level-uniform multi-boundedness, despite the fact that they inherit two complex components. The corresponding operator theory is elaborated upon in this paper. Sequential continuity, ordinary boundedness, componentwise boundedness, continuity at zero, and continuity at every point are all demonstrated to be equivalent for bicomplex-linear maps between normed modules. The two component multi-bounds are the defining characteristic of multi-boundedness, which is demonstrated to imply ordinary boundedness. The converse is established for minimum multi-norms and other operator-stable structures. The hyperbolic operator multi-norm is defined by component infima, which prevents division by null-cone values. Uniform limits, closed graphs, finite-rank and compact operators, inverse operators, composition, and sums are addressed. The open mapping, bounded inverse, closed graph, and uniform boundedness theorems are derived from their complex components under bicomplex Banach hypotheses. Componentwise characterisation is employed to distinguish between strong and weak operator convergence. Four worked examples examine matrix operators, multiplication on continuous-function modules, shifts and compact diagonal operators on bicomplex sequence spaces, and rank-one Hilbert operators. The results demonstrate the precise classical equivalences that persist and the instances in which multi-level geometry imposes an additional condition. Compactness in nonminimum multi-norms, weak compactness, operator ideals, unbounded generators, and bicomplex multi-operator topologies are all open concerns.

Keywords: continuous bicomplex operator; multi-boundedness; operator multi-norm; compact operator; closed graph; strong operator topology; weak operator topology

Mathematical Notation and Symbols

Symbol	Meaning
X,Y	Bicomplex multi-normed modules.
$N^{(x)}{}_n, N^{(y)}{}_n$	Decomposable level multi-norms.
$T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$	Bicomplex-linear operator decomposition.
$\Vert T\Vert_{\mathbb{D}}$	Ordinary hyperbolic operator norm.

Symbol	Meaning
$\|T\|_{mb,\mathbb{D}}$	Hyperbolic multi-bound.
$\mathcal{L}(X,Y)$	Bounded bicomplex-linear operators.
SOT,WOT	Strong and weak operator topologies.
$K(X,Y)$	Compact bicomplex-linear operators.

1. Introduction

Continuity of a linear map over a field is equivalent to boundedness, and continuity at one point implies continuity everywhere. These conclusions remain true over bicomplex scalars when the module topology is built from the hyperbolic norm, because the map and topology decompose into two complex parts. Multi-norms add a distinct requirement: the same bound must hold at every level n. An operator can therefore be continuous on the base module yet fail to be multi-bounded between a particular pair of multi-norm sequences.

Bicomplex functional analysis has established component versions of the major Banach-space principles (Saini et al., 2020). Recent studies of semigroups, compact operators, and mean ergodic behavior further motivate a systematic account of operator convergence and closedness (Sharma & Kunga, 2020; Koumantos, 2023; Johnston & Wahl, 2023). The present paper integrates these ideas with multi-norm geometry.

The guiding rule is to avoid a quotient of hyperbolic norms. A nonzero vector or tuple can have a null-cone norm, and division would be undefined. Operator bounds are instead pairs of real infimal constants. Results are stated with the exact completeness and multi-norm assumptions needed for each conclusion.

2. Preliminaries and Definitions

Definition 2.1. Let $X=\mathbf{e}_1X_1\oplus\mathbf{e}_2X_2$ and $Y=\mathbf{e}_1Y_1\oplus\mathbf{e}_2Y_2$. A bicomplex-linear $T:X\to Y$ satisfies $T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$ and has the unique decomposition $T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$.

Definition 2.2. T is continuous at x_0 if for every $\varepsilon\in\mathbb{D}^+$ with positive coefficients there is δ with positive coefficients such that $\|x-x_0\|_{\mathbb{D}}\prec_{\mathbb{D}}\delta$ implies $\|Tx-Tx_0\|_{\mathbb{D}}\prec_{\mathbb{D}}\varepsilon$. Sequential and uniform continuity have their standard metric meanings in the product topology.

Definition 2.3. T is ordinarily bounded if there is $C\in\mathbb{D}^+$ such that $\|Tx\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}C\|x\|_{\mathbb{D}}$. It is multi-bounded if one C works at every level:

$$N^{(y)}_n(Tx_1,\dots,Tx_n)\preccurlyeq_{\mathbb{D}}C\,N^{(x)}_n(x_1,\dots,x_n),\quad n\geq1. \qquad (2.1)$$

The least component constants define $\|T\|_{mb,\mathbb{D}}=\|T_1\|_{mb}\mathbf{e}_1+\|T_2\|_{mb}\mathbf{e}_2$. This is not defined as an unrestricted ratio.

Definition 2.4. A densely defined operator $T:D(T)\subset X\to Y$ is closed when its graph $\{(x,Tx):x\in D(T)\}$ is closed in $X\times Y$. A bounded operator is compact when it maps bounded sequences to sequences with convergent subsequences.

3. Continuity, Boundedness, and Multi-Boundedness

Theorem 3.1. For a bicomplex-linear $T:X\to Y$, the following are equivalent: (i) T is continuous at zero; (ii) T is continuous at every point; (iii) T is sequentially continuous; (iv) T is ordinarily bounded; and (v) T_1 and T_2 are bounded complex-linear operators.

Proof. The equivalence of (iv) and (v) follows by projecting the hyperbolic inequality onto each idempotent and recombining component bounds. Boundedness gives the Lipschitz estimate $\|T(x-y)\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}C\|x-y\|_{\mathbb{D}}$, proving continuity everywhere, uniform continuity, and sequential continuity. Continuity everywhere implies continuity at zero. If T is continuous at zero, its restriction to each pure component is continuous at zero; the classical complex linear theorem makes T_j bounded. Finally, in metric spaces sequential continuity is equivalent to continuity. □

Corollary 3.2. Every continuous bicomplex-linear operator is uniformly continuous, and the ordinary operator norm is $\|T\|_{\mathbb{D}}=\|T_1\|\mathbf{e}_1+\|T_2\|\mathbf{e}_2$.

Proof. The global Lipschitz estimate in Theorem 3.1 gives uniform continuity. Least component constants give the norm formula. □

Theorem 3.3. T is multi-bounded if and only if T_1 and T_2 are multi-bounded between the corresponding complex component multi-norms. In that case

$$\|T\|_{mb,\mathbb{D}}=\|T_1\|_{mb}\mathbf{e}_1+\|T_2\|_{mb}\mathbf{e}_2. \tag{3.1}$$

Proof. Project Equation (2.1) onto $\mathbf{e}_j$ to obtain the component multi-bound. Conversely, add the two component inequalities after multiplying by the idempotents. Taking the infimum of admissible constants in each real coordinate gives Equation (3.1). □

Corollary 3.4. Multi-boundedness implies continuity and $\|T\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|T\|_{mb,\mathbb{D}}$.

Proof. Set $n=1$ in the defining inequality. The least base-level bound cannot exceed a bound valid at every level. □

Proposition 3.5. If both domain and codomain carry minimum multi-norms, ordinary boundedness and multi-boundedness are equivalent and $\|T\|_{mb,\mathbb{D}}=\|T\|_{\mathbb{D}}$.

Proof. At level n, $\max_r\|Tx_r\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|T\|_{\mathbb{D}}\max_r\|x_r\|_{\mathbb{D}}$, so $\|T\|_{mb,\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|T\|_{\mathbb{D}}$. Corollary 3.4 gives the reverse inequality. □

Remark 3.6. For arbitrary multi-norm pairs, a base-level bound does not control the level growth. The converse of Corollary 3.4 therefore requires an operator-stability hypothesis and is not a formal consequence of continuity.

4. Algebra of Continuous and Multi-Bounded Operators

Proposition 4.1. If $S:Y\to Z$ and $T:X\to Y$ are multi-bounded, then $S\circ T$ is multi-bounded and

$$\|S\circ T\|_{mb,\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|S\|_{mb,\mathbb{D}}\|T\|_{mb,\mathbb{D}}. \tag{4.1}$$

Proof. Apply the bound for S to the image tuple and then the bound for T. The product of positive hyperbolic constants preserves the order. □

Proposition 4.2. The sum of multi-bounded operators is multi-bounded, scalar multiplication satisfies $\|\alpha T\|_{mb,\mathbb{D}}=|\alpha|_{\mathbb{D}}\|T\|_{mb,\mathbb{D}}$, and the identity has multi-norm one.

Proof. The tuple triangle inequality gives the sum estimate. Exact scalar homogeneity is componentwise, including null-cone scalars. The identity acts isometrically at every level. □

Theorem 4.3. Let $T:X\to Y$ be a bijective continuous operator between bicomplex Banach modules. Then T^{-1} is continuous. If the component multi-norm pairs are operator stable under inverses—for example, minimum multi-norms—then T^{-1} is multi-bounded.

Proof. The component maps T_j are bounded bijections between complex Banach spaces. The complex bounded inverse theorem makes T_j^{-1} bounded, hence $T^{-1}=T_1^{-1}\mathbf{e}_1+T_2^{-1}\mathbf{e}_2$ is continuous. Under the stated stability, each inverse is also multi-bounded, and Theorem 3.3 applies. □

Proposition 4.4. A norm-limit of continuous bicomplex operators is continuous. A multi-bound limit is multi-bounded: if $\|T\_m-T\|mb,\mathbb{D}\to 0$, then $\|T\|mb,\mathbb{D}\preccurlyeq\mathbb{D}\liminf\_m\|T\_m\|mb,\mathbb{D}$ whenever the latter is finite componentwise.

Proof. Both claims are standard component norm-limit arguments. For the multi-bound, $N(Tx)\preccurlyeq N((T-T\_m)x)+N(T\_mx)$; use the two multi-bounds and pass to the component limit. □

5. Compact, Finite-Rank, and Closed Operators

Theorem 5.1. A bounded bicomplex operator T is compact if and only if T_1 and T_2 are compact. Every finite-rank bicomplex operator is compact.

Proof. For a bounded sequence, use compactness of T_1 to select a subsequence and then compactness of T_2 to select a further subsequence. Both components of the image then converge. Conversely, apply T to pure component sequences. Finite rank means finite-dimensional component ranges, whose bounded sets are relatively compact. □

Proposition 5.2. If K is compact and T,S are bounded, then TK and KS are compact whenever the compositions are defined. The compact operators form a two-sided ideal in $\mathcal{L}(X)$.

Proof. A bounded map sends a bounded sequence to a bounded sequence; K produces a convergent subsequence; a continuous map preserves its convergence. For KS, first apply S and then compactness of K. Componentwise closure under sums and scalar multiplication is immediate. □

Theorem 5.3. Closed graph theorem. If X and Y are bicomplex Banach modules and $T:X\to Y$ is bicomplex-linear with closed graph, then T is continuous.

Proof. The graph is closed if and only if both component graphs are closed in $X_j\times Y_j$. The classical closed graph theorem makes T_1 and T_2 bounded. Theorem 3.1 gives continuity of T. □

Theorem 5.4. Open mapping theorem. A continuous surjective $T:X\to Y$ between bicomplex Banach modules is open.

Proof. Surjectivity is componentwise. Each bounded surjective T_j is open by the complex open mapping theorem. Products of their open images are open in the product topology, so T maps every basic open set to an open set. □

6. Operator Convergence and Uniform Boundedness

Definition 6.1. A net T_a converges to T in the strong operator topology when $T\_ax\to Tx$ for every x. It converges in the weak operator topology when $f(T\_ax)\to f(Tx)$ for every $x\in X$ and $f\in Y'$.

Theorem 6.2. Strong and weak operator convergence are componentwise: $T\_a\to T$ strongly (respectively weakly) if and only if $T\_\{a,j\}\to T\_j$ strongly (respectively weakly) for $j=1,2$.

Proof. Strong convergence is convergence of the two components of T_ax. For weak convergence, the dual decomposition $f=f_1\mathbf{e}_1+f_2\mathbf{e}_2$ gives $f(T\_ax)=f_1(T\_\{a,1\}x_1)\mathbf{e}_1+f_2(T\_\{a,2\}x_2)\mathbf{e}_2$. Quantification over pure x and f yields necessity, and recombination gives sufficiency. □

Theorem 6.3. Uniform boundedness principle. Let X be a bicomplex Banach module and $\{T\_a\}\subset\mathcal{L}(X,Y)$. If $\{T\_ax:a\}$ is bounded for every x, then the operator norms are uniformly bounded by some $C\in\mathbb{D}^+$.

Proof. Pointwise boundedness on pure vectors gives pointwise bounded component families. The complex uniform boundedness principle yields $\sup\_a\|T\_\{a,j\}\|<\infty$ for $j=1,2$. Recombine these suprema. □

Corollary 6.4. If $T\_m\to T$ strongly, then $\{T\_m\}$ is uniformly bounded in ordinary operator norm. Under minimum multi-norms it is uniformly bounded in multi-norm.

Proof. For each x, convergence makes $\{T\_mx\}$ bounded. Apply Theorem 6.3, then Proposition 3.5 for the minimum structures. □

7. Worked Operator Examples

Example 7.1. Finite matrix operator. Let $A=A_1\mathbf{e}_1+A_2\mathbf{e}_2\in M\_m(\mathscr{B}\mathbb{C})$ act on $\mathscr{B}\mathbb{C}^m$ with component Euclidean minimum multi-norms. Then

$$\|A\|mb,\mathbb{D}=\|A_1\|_2\mathbf{e}_1+\|A_2\|_2\mathbf{e}_2. \quad (7.1)$$

Proof. Each component is a finite-dimensional complex matrix operator, and minimum multi-bound equals the induced operator norm. Continuity, compactness, and invertibility are therefore componentwise. In finite dimension every such operator is compact. □

Example 7.2. Multiplication on $C(K,\mathscr{B}\mathbb{C})$. For $g=g_1\mathbf{e}_1+g_2\mathbf{e}_2\in C(K,\mathscr{B}\mathbb{C})$, let $M\_gf=gf$. Under supremum minimum multi-norms,

$$\|M\_g\|mb,\mathbb{D}=\|g_1\|\infty\mathbf{e}_1+\|g_2\|\infty\mathbf{e}_2. \quad (7.2)$$

Proof. Pointwise $|g_jf_j|\le\|g_j\|\infty|f_j|$ gives the upper bound. Taking functions concentrated near points where $|g_j|$ approaches its supremum gives equality. The same estimate holds for every tuple through the outer maximum. □

Example 7.3. Shift and compact diagonal on $\ell^p(\mathscr{B}\mathbb{C})$. The right shift S has $\|S\|mb,\mathbb{D}=1$ under minimum multi-norms. A diagonal $D\_ax=(a\_nx\_n)$ with $a\_n=a\_\{n,1\}\mathbf{e}_1+a\_\{n,2\}\mathbf{e}_2$ is bounded when both scalar sequences are bounded and compact for $1\le p<\infty$ when $a\_\{n,j\}\to0$.

Proof. The component shifts are isometries. The component diagonal norms are $\sup\_n|a\_\{n,j\}|$. If $a\_\{n,j\}\to0$, truncate after N coordinates to obtain finite-rank operators $D\_a^\{(N)\}$ with operator-norm error $\sup\_\{n>N\}|a\_\{n,j\}|\to0$; hence D_a is compact. The converse for compact diagonals follows because $D\_ae\_n=a\_ne\_n$ must have a norm-convergent subsequence, forcing $a\_n\to0$ componentwise. □

Example 7.4. Rank-one Hilbert operator. On a bicomplex Hilbert space H, fix u,v and define $T\_\{u,v\}x=u\langle v,x\rangle$. Then T is bicomplex-linear, finite rank, compact, and

$$\|T\_\{u,v\}\|\mathbb{D}=\|u\|\mathbb{D}\|v\|\mathbb{D}. \quad (7.3)$$

Proof. Componentwise, $T_jx_j=u_j\langle v_j,x_j\rangle$ is a complex rank-one operator of norm $\|u_j\|\|v_j\|$. Recombination gives the formula. If either component vector vanishes, the corresponding norm coefficient is zero, which is legitimate. □

7.5 Operator Spaces, Equicontinuity, and Unbounded Examples

Theorem 7.5. If Y is a bicomplex Banach module, then $\mathcal{L}(X,Y)$ is complete in the ordinary hyperbolic operator norm.

Proof. A Cauchy sequence T_m gives Cauchy component sequences $T_{m,j}$ in the Banach spaces $\mathcal{L}(X_j,Y_j)$. Let T_j be their limits and set $T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$. Component norm convergence implies hyperbolic operator-norm convergence, and the limit is bicomplex-linear. □

Proposition 7.6. The compact-operator space K(X,Y) is closed in operator norm when Y is complete.

Proof. Let $K_m\to T$ in operator norm and let $\{x_n\}$ be bounded. Choose m_1 so K_{m_1} approximates T within 1, then a subsequence on which $K_{m_1}x_n$ converges. Choose m_2 with error below 1/2 and extract a further subsequence, and continue. A diagonal subsequence is Cauchy under T by the standard three-term estimate. Perform the construction in both components, or equivalently use the closedness of complex compact-operator ideals. □

Definition 7.7. A family $\mathcal{T}\subset\mathcal{L}(X,Y)$ is equicontinuous at zero when every ε-neighborhood in Y has a δ-neighborhood in X mapped into it by every $T\in\mathcal{T}$.

Proposition 7.8. A family of bicomplex-linear operators is equicontinuous if and only if its ordinary operator norms are bounded by one $C\in\mathbb{D}^+$. For minimum multi-norms, this is also equivalent to a uniform multi-bound.

Proof. A uniform norm bound gives a common Lipschitz estimate. Conversely, equicontinuity restricted to pure component vectors yields equicontinuity of the two complex families; the standard linear result gives uniform component bounds. The minimum multi-norm equality follows from Proposition 3.5. □

Example 7.9. Strong convergence need not be norm convergence. On $\ell^2(\mathscr{B}\mathbb{C})$, let P_N be truncation to the first N coordinates. Then $P_Nx\to x$ for every x, but $\|I-P_N\|_\mathbb{D}=1$ for every N.

Proof. Tail norms tend to zero for each fixed ℓ^2 sequence, so convergence is strong. However, the unit vector e_{N+1} has $\|(I-P_N)e_{N+1}\|_\mathbb{D}=1$, proving the operator norm is one. The same example works at both idempotent components. □

Example 7.10. Weak operator convergence of shift powers. Let S be the right shift on $\ell^2(\mathscr{B}\mathbb{C})$. Then $S^m\to0$ in the weak operator topology but not strongly.

Proof. For $x,y\in\ell^2$, $\langle y,S^mx\rangle$ is the correlation of y with a tail-shifted x and tends to zero by approximation with finitely supported sequences and Cauchy–Schwarz. Thus both component matrix coefficients tend to zero. Strong convergence fails because $\|S^mx\|_\mathbb{D}=\|x\|_\mathbb{D}$ for every m. □

Definition 7.11. On $\ell^2(\mathscr{B}\mathbb{C})$, let $a_n=a_{n,1}\mathbf{e}_1+a_{n,2}\mathbf{e}_2$ be an unbounded coefficient sequence and define the diagonal operator

$D_ax=(a_nx_n)$, $D(D_a)=\{x:\Sigma_n|a_nx_n|_\mathbb{D}^2$ (7.4)

converges componentwise$\}$.

Proposition 7.12. D_a is densely defined and closed. It is bounded exactly when both coefficient sequences are bounded.

Proof. Finitely supported sequences lie in the domain, so it is dense. If $x^m\to x$ and $D_ax^m\to y$ in ℓ^2, coordinate continuity gives $a_nx_n^m\to a_nx_n$ and also $\to y_n$; hence $y_n=a_nx_n$ and $y\in\ell^2$, so $x\in D(D_a)$ and $D_ax=y$. Boundedness is equivalent to boundedness of each complex diagonal component, whose norm is the coefficient supremum. □

Theorem 7.13. Suppose the component multi-operator spaces $\mathcal{L}mb(X\_j,Y\_j)$ are complete in their multi-bounds. Then the bicomplex multi-operator space is complete in $\|\cdot\|mb,\mathbb{D}$.

Proof. A hyperbolic multi-bound Cauchy sequence is exactly a pair of component multi-bound Cauchy sequences. Their limits are component multi-bounded operators, which recombine. This assumption is automatic for many standard component categories but must be checked for the selected multi-norms. □

The examples distinguish three topologies. Operator-norm convergence controls the entire unit ball and preserves compactness under limits. Strong convergence controls each fixed vector but not the worst vector, as truncations show. Weak operator convergence controls only scalar observations and permits norm-preserving shifts to converge weakly to zero. Multi-bound convergence can be stronger still because it controls arbitrary finite families at every level.

Closed unbounded diagonals illustrate why the graph definition is needed beyond bounded operators. Their resolvents, domains, and adjoints are paired complex objects, but the domain must be decomposed explicitly. This is the natural starting point for bicomplex semigroup generators and unbounded observables.

7.14 Resolvents and Stability under Perturbation

Theorem 7.14. Let $T\in\mathcal{L}(X)$ and $\lambda\in\rho\mathscr{B}\mathbb{C}(T)$. If $E\in\mathcal{L}(X)$ satisfies $\|R(\lambda,T)E\|\mathbb{D}\prec\mathbb{D}1$, then $\lambda\in\rho\mathscr{B}\mathbb{C}(T+E)$ and

$$R(\lambda,T+E)=[I-R(\lambda,T)E]^{-1}R(\lambda,T). \qquad (7.5)$$

Proof. Factor $\lambda I-(T+E)=(\lambda I-T)[I-R(\lambda,T)E]$. The bracket has a componentwise convergent Neumann inverse because both norm coefficients are below one. Multiplying inverse factors yields the formula. □

Corollary 7.15. The resolvent depends continuously on T in operator norm inside a componentwise Neumann neighborhood. Specifically,

$$R(\lambda,T+E)-R(\lambda,T)=R(\lambda,T+E)ER(\lambda,T). \qquad (7.6)$$

Proof. Use the second resolvent identity, valid componentwise, or subtract the inverse factorizations. Taking norms gives a perturbation estimate with separate channel constants. □

Proposition 7.16. If $T\_m\to T$ in operator norm and every T_m is invertible with uniformly bounded inverse norms, then T is invertible provided the component lower bounds remain strictly positive.

Proof. Choose m so $\|T-T\_m\|\mathbb{D}\|T\_m^{-1}\|\mathbb{D}\prec\mathbb{D}1$. Then $T=T\_m[I+T\_m^{-1}(T-T\_m)]$ and the bracket is invertible by Neumann series. The strict two-component condition prevents loss of invertibility through one channel. □

Particularly crucial in the context of discretisation are perturbation estimates. A numerical approximation may be precise in one idempotent component and inaccurate in the other. The hyperbolic bound demonstrates that invertibility is maintained only when both relative errors are managed. The component form reveals which channel limits the permissible perturbation, but a real maximum norm can enforce a sufficient uniform threshold.

8. Interpretive Analysis and Applications

Paired complex transformations are represented by continuous bicomplex operators. Compact diagonals provide approximable operators, while multiplication and shift examples are fundamental in function and sequence spaces. Multi-boundedness uniformly regulates finite families and can denote the stability of discretised trajectories, frames, or batches.

Strong operator convergence is suitable for approximation schemes that are applied to each state, whereas norm convergence results in uniform error on the unit ball. Ergodic averages and variational problems exhibit weak convergence as a natural phenomenon. The direct application of complex convergence theorems is permitted by the component criteria; however, conclusions that involve a full bicomplex spectrum or positivity still necessitate explicit null-cone analysis.

Ordinary operator norms constrain Lipschitz behaviour channelwise for bicomplex neural or signal-processing layers. Richer multi-norms may detect geometry across families of signals, whereas minimum multi-norms do not add a firmer bound. The question of whether the additional geometry enhances an algorithm is empirical and model-dependent, rather than a direct result of continuity.

9. Open Problems and Limitations

The equivalence between boundedness and multi-boundedness was only established for operator-stable multi-norm pairs, such as the minimum structure. It is possible that a distinct theory may require compact behaviour uniformly across levels, as compactness was defined through the base topology. Nuclearity, weak compactness, complete continuity, and summing ideals are yet to be systematically defined.

Other open problems include close unbounded operators, semigroup generators, multi-bounded resolvents, Fredholm theory, strong and weak multi-operator topologies, approximation properties, and perturbation estimates. The components could be coupled and the direct reduction used throughout could be invalidated by nondecomposable multi-norms. Any such theory must continue to maintain a coherent positive cone and manage null-cone norm values without division.

10. In summary

Ordinary boundedness, global and sequential continuity, continuity at zero, and componentwise boundedness are equivalent for bicomplex-linear maps. Unless the selected multi-norms are operator stable, multi-boundedness is a more robust level-uniform property. A valid hyperbolic operator multi-norm is defined by component infima, and composition, sums, inverse results, compactness, closed graphs, open mappings, and uniform boundedness are derived from precise hypotheses. Complex convergence is precisely coupled with strong and weak operator convergence. The theory and its null-cone behaviour are illustrated through matrix, multiplication, shift, diagonal, and rank-one examples. The Hilbert and sequence-space analyses are prepared by these results, which also identify operator ideals and nondecomposable geometry as significant open directions.

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