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Stability Characteristics of Triple-Diffusive Convection in an Oldroyd-B Fluid Layer

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Abstract
The study makes use of linear stability analysis to investigate the onset and stability of triple-diffusive convection in a layer of Oldroyd-B fluid. The principle of exchange of stabilities is verified, and a sufficient condition is derived which ensures that convection will start via stationary modes. Upper bounds are also determined for the growth rates in the case of both neutral and unstable disturbances, as well as for all combinations of rigid and free boundary conditions. Broadly speaking, the findings provide valuable insights into the way in which viscoelasticity and the presence of multiple diffusing components influence the stability of fluid layers that are commonly met in engineering and transport applications.

Keywords: Instability; Triple diffusive convection; Oldroyd-B fluid; Rayleigh number.

1. Introduction

The phenomenon of convection in fluids which have several components that diffuse both heat and mass is a basic transport process relevant to a wide range of natural and engineering applications. If thermal and concentration gradients interact and have different molecular diffusivities, they can lead to a variety of instability mechanisms, such as stationary or oscillatory convection. These effects become particularly important in complex fluids, since the rheological properties of the fluid introduce extra time scales and memory effects.

A good example of a multicomponent transport system is the polymerase chain reaction (PCR), in which DNA, polymerase enzymes, magnesium ions, nucleotides, and primers are exposed to carefully controlled thermal conditions. Krishnan et al. [1] carried out PCR in a Rayleigh-Bénard convection cell, whereas Braun et al. [2] and Braun [3] later proved that exponential DNA replication can be obtained by means of thermal convection. Kolodner [4] noted oscillatory convection in viscoelastic DNA suspensions, showing that fluid elasticity can have a significant effect on the time-dependent behaviour of convection. These results lead to a investigation into how thermal diffusion, various mass-diffusion mechanisms, and viscoelasticity interact in multicomponent fluid systems. The classical Rayleigh-Bénard problem that is, a fluid layer which becomes unstable when a critical thermal threshold is reached because it is heated from below forms the basis of much of the present theory of convective instability. When both temperature and one or more of the dissolved species influence the density and have different diffusivities, the phenomenon is called double- or multicomponent diffusive convection. Turner [5], Huppert and Turner [6], and Platten and Legros [7] have given thorough descriptions of double-diffusive convection and its applications. The inclusion of a third diffusing component makes the stability problem more complex, since the three components can interact via their individual buoyancy effects and diffusivities.

The study of triple-diffusive convection was first carried out by Griffiths [8-10], who demonstrated that the inclusion of a third diffusing component can considerably change the conditions under which convection begins. Pearlstein and others [11] built on this work by carrying out a thorough linear stability analysis of triply diffusive fluid layers and thus identifying several possible instability modes. Terrones and Pearlstein [12] then extended this approach to apply it to fluid systems consisting of more than one component. Moroz [13] examined various types of instability in triple-diffusive situations, while Lopez and co-workers [14] investigated the influence of rigid boundaries on the onset of convection. Ryzhkov and Shevtsova [15] looked at thermal diffusion and convection in multicomponent mixtures, and Terrones [16] studied the role of cross-diffusion in relation to stability. Moreover, Straughan and Walker [17] examined penetrative convection in multicomponent fluids, and Straughan and Tracey [18] studied the effects of internal heating and cooling. Finally, Shivakumara and Naveen Kumar [19] applied such analyses to couple-stress fluid layers and studied both linear and weakly nonlinear triple-diffusive convection.

Although classical studies offer a solid basis for Newtonian triple-diffusive convection, a great many engineering fluids in real situations show non-Newtonian characteristics. Solutions containing polymers, melts, biological suspensions, and other complex fluids exhibit elasticity and memory effects which are not described by Newtonian physics. The presence of viscoelasticity brings about new dynamical mechanisms that can fundamentally alter the character of the primary instabilities. Rosenblat [20] looked at thermal convection in viscoelastic liquids, and Li and Khayat [21] examined finite-amplitude Rayleigh-Bénard convection and pattern formation in viscoelastic fluids. Notably, Kolodner's [4] experiments demonstrated oscillatory convection in viscoelastic DNA suspensions, underscoring the important role that fluid elasticity plays in convection dynamics. The Oldroyd-B model is a commonly used approach for viscoelastic fluids since it takes into account both viscous and elastic behaviour. Mardones et al. [22] carried out an analysis of Rayleigh-Bénard convection in binary viscoelastic fluids, while Malashetty et al. [23] considered the onset of double-diffusive convection in viscoelastic layers. The aforementioned studies indicated that fluid elasticity can cause both the critical conditions and the temporal aspects of instability to change. Hirata et al. [24] also stressed the effect of viscoelasticity on convective and absolute instabilities in Rayleigh-Bénard-Poiseuille flow.

Recent studies have been concerned with the stability of multicomponent Oldroyd-B fluids. Raghunatha and Shivakumara [25] studied double-diffusive convection in an Oldroyd-B layer and examined the stability of the bifurcating equilibrium. Raghunatha et al. [26] looked at the effect of cross-diffusion on the stability of triple-diffusive Oldroyd-B layers and found that cross-diffusion can cause significant shifts in the stability boundaries and influence whether stationary or oscillatory modes dominate. Later on, Raghunatha and Shivakumara [27] looked into triple-diffusive convection in viscoelastic Oldroyd-B layers, showing that viscoelasticity can greatly change the onset characteristics and lead to transitions between different types of instability. They also investigated double-diffusive convection in rotating viscoelastic layers [28], demonstrating that rotation introduces an additional level of complexity to the stability of viscoelastic multicomponent systems.

Recent research has broadened these ideas to include further physical effects and rheological models. Raghunatha and Shivakumara [29] investigated double-diffusive magnetoconvection in non-Newtonian fluid layers in the presence of cross-diffusion, whereas Bixapathi et al. [30] looked at the interaction between internal heating and chemical reactions on triple-diffusive convection with rotation modulation. Raghunatha and Chamkha [31] considered the linear stability of thermohaline and magnetoconvection in viscoelastic fluids. Taken together, these studies show that coupled thermal, solutal, magnetic, rotational, and rheological effects continue to be important in determining the stability of complex fluid systems.

One of the most important parts of a study of convective stability is to understand the nature of the disturbance which causes instability. The principle of exchange of stabilities (PES) acts as a main criterion for deciding whether the primary unstable mode is stationary or oscillatory. Pellew and Southwell [32] verified that PES is valid in the classical Rayleigh-Bénard problem. Yet, in the case of multicomponent convection, the different diffusivities of the components can result in complex growth rates, so it cannot be assumed that the onset will be stationary. Prakash et al. [33-36] investigated PES in a number of multicomponent convection situations that included rotation and magnetic fields. They also found the sufficient conditions for stationary convection in triply diffusive systems [37] and determined upper limits for the growth rates associated with oscillatory disturbances [38].

Viscoelastic fluids add a further level of complexity since the constitutive equations include extra time scales connected with relaxation and retardation; as a result, oscillatory disturbances can arise from the interaction between viscoelasticity and differential diffusion. Research carried out by Raghunatha et al. [26], as well as that by Raghunatha and Shivakumara [27], demonstrates that these interactions can considerably modify the instability characteristics in triple-diffusive Oldroyd-B fluids. Yet merely determining the neutral stability curves does not fully describe the temporal behaviour of the disturbances; it is therefore useful to establish a rigorous analytical criterion for the validity of the principle of exchange of stabilities (PES), together with bounds on the related complex growth rates, since this provides insights that go beyond a detailed numerical calculation of each individual neutral curve.

Based on the earlier points, the present study looks again at the stability of a triple-diffusive Oldroyd-B fluid layer, with particular attention being given to the principle of exchange of stabilities. The primary objective is to obtain a sufficient condition which guarantees that the onset of convection is stationary. The study also sets upper limits for the complex growth rates of neutral and unstable oscillatory disturbances. Furthermore, it aims to find out whether these analytical stability properties are affected by the mechanical boundary conditions. The results indicate that the derived criteria and bounds remain valid for all combinations of rigid and free boundaries.

This study extends previous research concerning the stability of triple-diffusive Oldroyd-B fluids by giving rigorous analytical findings about the temporal development of disturbances. Classical results are obtained as special cases, which shows that the method is both general and consistent. Broadly speaking, the results provide a single theoretical framework for the analysis of both stationary and oscillatory instability in layers of triple-diffusive viscoelastic fluids, and have practical significance for engineering systems that involve coupled heat and mass transfer in complex fluids.

2. Mathematical formulation and analysis

We consider an incompressible Oldroyd-B fluid layer of depth d in which the density depends on three different stratifying agents having different molecular diffusivities (temperature T and solute concentrations $S_i, i=1,2$). The lower and upper boundaries of the fluid layer $z=0$ and $z=d$ are held at constant temperatures T_L and $T_U (< T_L)$, respectively while species concentration of *ith* component is held at fixed values S_{iL} and $S_{iU}(<S_{iL})$, respectively. The stratifying agents are assumed to obey the following equation of state

$$\rho=\rho_0\left[1-\alpha_T\left(T-T_L\right)+\alpha_{S1}\left(S_1-S_{1L}\right)+\alpha_{S2}\left(S_2-S_{2L}\right)\right] \tag{1}$$

where α_T is the thermal expansion coefficient, α_{S1} and α_{S2} are the solute analogs of α_T and ρ_0 is the reference density. Under the Boussinesq approximation, the conservation of mass, momentum, energy and solute concentrations are [23-26]

$$\nabla\cdot\vec{q}=0 \tag{2}$$

$$\frac{\partial\vec{q}}{\partial t}+\left(\vec{q}\cdot\nabla\right)\vec{q}=-\nabla\left(\frac{p}{\rho_0}\right)+\frac{1}{\rho_0}\nabla\cdot\underset{\sim}{\tau}+\frac{\rho}{\rho_0}\vec{g} \tag{3}$$

$$\frac{\partial T}{\partial t}+\left(\vec{q}\cdot\nabla\right)T=\kappa_T\nabla^2 T \tag{4}$$

$$\frac{\partial S_1}{\partial t}+\left(\vec{q}\cdot\nabla\right)S_1=\kappa_{S1}\nabla^2 S_1 \tag{5}$$

$$\frac{\partial S_2}{\partial t}+\left(\vec{q}\cdot\nabla\right)S_2=\kappa_{S2}\nabla^2 S_2 \tag{6}$$

where $\vec{q}=\left(u,v,w\right)$ is the velocity, p is the pressure, $\underset{\sim}{\tau}$ is the stress tensor, μ is the fluid viscosity, $\vec{g}$ is the acceleration due to gravity, κ_T is the thermal diffusivity, κ_{S1} and κ_{S2} are the solute analogs of κ_T. The constitutive equation for an Oldroyd-B fluid is (Rosenblat [20] and Bird et al. [39])

$$\underset{\sim}{\tau}+\lambda_1\left[\frac{\partial\underset{\sim}{\tau}}{\partial t}+\left(\vec{q}\cdot\nabla\right)\underset{\sim}{\tau}-\left(\nabla\vec{q}\right)^T\underset{\sim}{\tau}-\underset{\sim}{\tau}\left(\nabla\vec{q}\right)\right]=\mu\left\{\underset{\sim}{A}+\lambda_2\left[\frac{\partial\underset{\sim}{A}}{\partial t}+\left(\vec{q}\cdot\nabla\right)\underset{\sim}{A}-\left(\nabla\vec{q}\right)^T\underset{\sim}{A}-\underset{\sim}{A}\left(\nabla\vec{q}\right)\right]\right\} \tag{7}$$

where $\underset{\sim}{A}=\nabla\vec{q}+\left(\nabla\vec{q}\right)^T$ is the rate-of-strain tensor, μ is the fluid viscosity, λ_1 is the relaxation time, λ_2 is the retardation time and it is to be noted that $\lambda_1\ge\lambda_2$. The constitutive equation considered includes Stokes's law adopted in the theory of Newtonian viscous fluid flows as a special case for $\lambda_1=\lambda_2=0$ and to the Maxwell fluid when $\lambda_2=0$. The basic state is quiescent and the gradients of the stratifying agents are constant and vertical. Thus

$$\vec{q}_b=0,\ \underset{\sim}{\tau}_b=0,\ T_b=T_L-\beta_1 z,\ S_{ib}=S_{iL}-\beta_i z\left(i=1,2\right),\ \rho_b=\rho_0\left[1+\left(\alpha_T\beta-\alpha_{S1}\beta_1-\alpha_{S2}\beta_2\right)z\right] \tag{8}$$

$$p_b/\rho_0=P_b=P_0-\rho_0\,g\left[z+\left(\alpha_T\beta-\alpha_{S1}\beta_1-\alpha_{S2}\beta_2\right)z^2/2\right] \tag{9}$$

where the subscript b denotes the basic state, p_0 is the pressure at $z=0$, $\Delta T=T_L-T_U$, $\Delta S_i=S_{iL}-S_{iU}\ (i=1,2)$, $\beta_1=\dfrac{\Delta T}{d}$ and $\beta_i=\dfrac{\Delta S_i}{d}(i=1,2)$.

To study the instability of the system, we superimpose infinitesimal perturbations on the basic state which are of the form

$$\vec{q}=\vec{q}_b+\vec{q}',\ P=P_b+P',\ \rho=\rho_b+\rho',\ \underset{\sim}{\tau}=\underset{\sim}{\tau}_b+\underset{\sim}{\tau}',\ T=T_b+T',\ S_i=S_{ib}+S_i'\left(i=1,2\right) \tag{10}$$

where primes indicate infinitesimal perturbations. Equation (10) is substituted back into the governing equations and the linearized perturbation equations are

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \tag{11}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\frac{\partial u'}{\partial t}+\frac{\partial P'}{\partial x}\right]=\nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2 u' \tag{12}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\frac{\partial v'}{\partial t}+\frac{\partial P'}{\partial y}\right]=\nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2 v' \tag{13}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left[\frac{\partial w'}{\partial t}+\frac{\partial P'}{\partial z}-g\alpha_T T'+g\alpha_{S1}S_1'+g\alpha_{S2}S_2'\right]=\nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2 w' \tag{14}$$

$$\frac{\partial T'}{\partial t}-\beta w'=\kappa_T\nabla^2 T' \tag{15}$$

$$\frac{\partial S_1'}{\partial t}-\beta_1 w'=\kappa_{S1}\nabla^2 S_1' \tag{16}$$

$$\frac{\partial S_2'}{\partial t}-\beta_2 w'=\kappa_{S2}\nabla^2 S_2' \tag{17}$$

The normal mode expansion of the unknown variables u', v', w', P', T' and $S_i'(i=1,2)$ is assumed in the form

$$F'(x,y,z,t)=F(z)\exp(i(k_x x+k_y y)+nt) \tag{18}$$

where n is the growth term, k_x and k_y are wave numbers in the x and y direction respectively.

Substituting Eq. (18) into Eqs. (11)-(17), thus becomes

$$ik_x u+ik_y v+\frac{\partial w}{\partial z}=0 \tag{19}$$

$$(1+\lambda_1 n)[nu+ik_x P]=\nu(1+\lambda_2 n)(D^2-k^2)u \tag{20}$$

$$(1+\lambda_1 n)[nv+ik_y P]=\nu(1+\lambda_2 n)(D^2-k^2)v \tag{21}$$

$$(1+\lambda_1 n)\left[nw+\frac{\partial P}{\partial z}-g\alpha_T T+g\alpha_{S1}S_1+g\alpha_{S2}S_2\right]=\nu(1+\lambda_2 n)(D^2-k^2)w \tag{22}$$

$$nT-\beta w=\kappa_T(D^2-k^2)T \tag{23}$$

$$nS_1-\beta_1 w=\kappa_{S1}(D^2-k^2)S_1 \tag{24}$$

$$nS_2-\beta_2 w=\kappa_{S2}(D^2-k^2)S_2 \tag{25}$$

where $k^2=k_x^2+k_y^2$. Eliminating u and v from Eq. (20) and (21) by multiplying Eq. (20) by ik_x and (21) by ik_y respectively, adding the ensuing equations and using Eq. (19) and then eliminating P between this ensuing equation and Eq. (22), we get

$$\nu(1+\lambda_2 n)(D^2-k^2)^2 w-n(1+\lambda_1 n)(D^2-k^2)w=(1+\lambda_1 n)k^2[g\alpha_T T-g\alpha_{S1}S_1-g\alpha_{S2}S_2] \tag{26}$$

Equations (23)-(25) can also be written as

$$\left(D^2-k^2-\frac{n}{\kappa_T}\right)T=-\frac{\beta}{\kappa_T}w \tag{27}$$

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$$\left(D^2-k^2-\frac{n}{\kappa_{S1}}\right)S_1=-\frac{\beta_1}{\kappa_{S1}}w \qquad (28)$$

$$\left(D^2-k^2-\frac{n}{\kappa_{S2}}\right)S_2=-\frac{\beta_2}{\kappa_{S2}}w \qquad (29)$$

Now using the following non-dimensional parameters

$$a_*=kd\,,\ z_*=z/d\,,\tau_{1*}=\kappa_{S1}/\kappa_T\,,\tau_{2*}=\kappa_{S2}/\kappa_T\,,\sigma_*=\nu/\kappa_T\,,\ p_*=nd^2/\nu\,,$$

$$R_*=g\alpha_T\beta d^4/\nu\kappa_T\,,R_{S1*}=g\alpha_{S1}\beta_1 d^4/\nu\kappa_T\,,R_{S2*}=g\alpha_{S2}\beta_2 d^4/\nu\kappa_T\,,D_*=d\frac{d}{dz}\,, \qquad (30)$$

$$w_*=\frac{\beta d^2}{\kappa_T}w\,,T_*=T\,,\ S_{i*}=\frac{\beta}{\beta_i}S_i\,(i=1,2)\,,\Lambda_{i*}=\lambda_i\,d^2/\nu\,(i=1,2)\,.$$

we can reduce Eqs. (26)-(29) to the following coupled ordinary differential equations in the non-dimensional form (after omitting the asterisks for simplicity)

$$(1+\Lambda_2 p)\left(D^2-a^2\right)^2 w-\frac{p}{\sigma}(1+\Lambda_1 p)\left(D^2-a^2\right)w=(1+\Lambda_1 p)a^2\left[RT-R_{S1}S_1-R_{S2}S_2\right] \qquad (31)$$

$$\left(D^2-a^2-p\right)T=-w \qquad (32)$$

$$\left(D^2-a^2-\frac{p}{\tau_1}\right)S_1=-\frac{1}{\tau_1}w \qquad (33)$$

$$\left(D^2-a^2-\frac{p}{\tau_2}\right)S_2=-\frac{1}{\tau_2}w \qquad (34)$$

The linear coupled ordinary differential equations (31)-(34) are to be solved by using the following boundary conditions

$w=Dw=T=S_1=S_2=0$ at $z=0$ and $z=1$ (both the boundaries are rigid) (35)

$w=D^2w=T=S_1=S_2=0$ at $z=0$ and $z=1$ (both the boundaries are free) (36)

$$\left.\begin{array}{l} w=Dw=T=S_1=S_2=0 \text{ at } z=0 \text{ (lower boundary is rigid)} \\ \text{and } w=D^2w=T=S_1=S_2=0 \text{ at } z=1 \text{ (upper boundary is free)} \end{array}\right\} \qquad (37)$$

$$\left.\begin{array}{l} w=D^2w=T=S_1=S_2=0 \text{ at } z=0 \text{ (lower boundary is free)} \\ \text{and } w=Dw=T=S_1=S_2=0 \text{ at } z=1 \text{ (upper boundary is rigid)} \end{array}\right\} \qquad (38)$$

where z is the rear known variable such that $0\le z\le 1$, D is the differentiation with respect to z, $\sigma>0$ is the Prandtl number, $\tau_1,\tau_2>0$ are the ratio of diffusivities, $R>0$ is the thermal Rayleigh number, $R_{S1},R_{S2}>0$ are solute Rayleigh numbers, Λ_1 is the relaxation parameter, Λ_2 is the retardation parameter, a^2 is the square of the wave number, $p=p_r+ip_i$ is the complex growth rate where p_r and p_i are real constants and as consequence the unknown variables $w(z)=w_r(z)+i\,w_i(z)$, $T(z)=T_r(z)+i\,T_i(z)$, $S_1(z)=S_{1r}(z)+i\,S_{1i}(z)$, $S_2(z)=S_{2r}(z)+i\,S_{2i}(z)$ are complex valued functions of the real known variable z. Further we note that Eqs. (31)-(38) describes an eigenvalue problem for p and govern triple diffusive convection in an Oldroyd-B fluid layer for any combination of dynamically free and rigid boundaries.

Theorem 1. If (w,T,S_1,S_2,p), $p_r \geq 0$ with $R_{S1} > 0, R_{S2} > 0, \Lambda_1 \geq 0, \Lambda_2 \geq 0\ (\Lambda_1 \geq \Lambda_2)$, $0 < R < 2\pi^4\left(1+2\sigma a^2\Lambda_2\right)/\sigma\Lambda_1\left(\pi^2+a^2\right)$ and $p_r \geq 0$ is a solution of Eqs. (31)-(34) together with either of the boundary conditions Eqs. (35)-(38) and

$$\frac{\sigma R_{S1}}{2\tau_1^2\pi^4}+\frac{\sigma R_{S2}}{2\tau_2^2\pi^4} \leq 1+2\sigma a^2\Lambda_2-\frac{\sigma R\Lambda_1\left(\pi^2+a^2\right)}{2\pi^4}$$

then $p_i = 0$. In particular $p_r = 0$ implies $p_i = 0$ if

$$\frac{\sigma R_{S1}}{2\tau_1^2\pi^4}+\frac{\sigma R_{S2}}{2\tau_2^2\pi^4} \leq 1+2\sigma a^2\Lambda_2-\frac{\sigma R\Lambda_1\left(\pi^2+a^2\right)}{2\pi^4}.$$

Proof. Multiplying Eq. (31) by w^* (w^* is the complex conjugate of w) on both the sides and integrating over vertical range of z, we get

$$\left(1+\Lambda_2 p\right)\int_0^1 w^*\left(D^2-a^2\right)^2 w\,dz-\frac{p}{\sigma}\left(1+\Lambda_1 p\right)\int_0^1 w^*\left(D^2-a^2\right)w\,dz = \left(1+\Lambda_1 p\right)a^2\left[R\int_0^1 w^* T\,dz - R_{S1}\int_0^1 w^* S_1\,dz - R_{S2}\int_0^1 w^* S_2\,dz\right]. \tag{39}$$

Making use of Eqs. (33)-(38), we can write

$$R a^2\int_0^1 w^* T\,dz = -R a^2\int_0^1 T\left(D^2-a^2-p^*\right)T^*dz \tag{40}$$

$$R_{S1}a^2\int_0^1 w^* S_1\,dz = -R_{S1}a^2\tau_1\int_0^1 S_1\left(D^2-a^2-\frac{p^*}{\tau_1}\right)S_1^*\,dz \tag{41}$$

$$R_{S2}a^2\int_0^1 w^* S_2\,dz = -R_{S2}a^2\tau_2\int_0^1 S_2\left(D^2-a^2-\frac{p^*}{\tau_2}\right)S_2^*\,dz \tag{42}$$

Substituting Eqs. (40)-(42) in Eq. (39), we obtain

$$\begin{aligned}\left(1+\Lambda_2 p\right)\int_0^1 w^*\left(D^2-a^2\right)^2 w\,dz-\frac{p}{\sigma}\left(1+\Lambda_1 p\right)\int_0^1 w^*\left(D^2-a^2\right)w\,dz = \\ \left(1+\Lambda_1 p\right)a^2\left[-R\int_0^1 T\left(D^2-a^2-p^*\right)T^*dz+R_{S1}\tau_1\int_0^1 S_1\left(D^2-a^2-\frac{p^*}{\tau_1}\right)S_1^*\,dz\right]+ \\ \left(1+\Lambda_1 p\right)a^2 R_{S2}\tau_2\int_0^1 S_2\left(D^2-a^2-\frac{p^*}{\tau_2}\right)S_2^*\,dz\end{aligned} \tag{43}$$

Integrating various terms of Eq. (43) by using integration by parts for an suitable number of times and making use of either of the boundary conditions (35)-(38), it follows that

$$(1+\Lambda_2 p)\int_0^1\left(\left|D^2w\right|^2+2a^2\left|Dw\right|^2+a^4\left|w\right|^2\right)dz+\frac{p}{\sigma}(1+\Lambda_1 p)\int_0^1\left(\left|Dw\right|^2+a^2\left|w\right|^2\right)dz$$
$$=(1+\Lambda_1 p)a^2\left[R\int_0^1\left(\left|DT\right|^2+a^2\left|T\right|^2+p^*\left|T\right|^2\right)dz-R_{S1}\tau_1\int_0^1\left(\left|DS_1\right|^2+a^2\left|S_1\right|^2+\frac{p^*}{\tau_1}\left|S_1\right|^2\right)dz\right]-$$
$$(1+\Lambda_1 p)a^2R_{S2}\tau_2\int_0^1\left(\left|DS_2\right|^2+a^2\left|S_2\right|^2+\frac{p^*}{\tau_2}\left|S_2\right|^2\right)dz \tag{44}$$

Equating imaginary parts of both sides of Eq. (44) and cancelling $p_i(\neq 0)$ throughout from the imaginary part, we have

$$\Lambda_2\int_0^1\left(\left|D^2w\right|^2+2a^2\left|Dw\right|^2+a^4\left|w\right|^2\right)dz+\frac{1}{\sigma}[1+2\Lambda_1 p_r]\int_0^1\left(\left|Dw\right|^2+a^2\left|w\right|^2\right)dz$$
$$=\Lambda_1 a^2\left[R\int_0^1\left(\left|DT\right|^2+a^2\left|T\right|^2\right)dz-R_{S1}\tau_1\int_0^1\left(\left|DS_1\right|^2+a^2\left|S_1\right|^2\right)dz-R_{S2}\tau_2\int_0^1\left(\left|DS_2\right|^2+a^2\left|S_2\right|^2\right)dz\right] \tag{45}$$
$$-Ra^2\int_0^1\left|T\right|^2dz+R_{S1}a^2\int_0^1\left|S_1\right|^2dz+R_{S2}a^2\int_0^1\left|S_2\right|^2dz$$

Multiplying Eq. (32) by its complex conjugate and integrating the ensuing equation for a suitable number of times and using the boundary conditions on T namely, $T(0)=0=T(1)$, we get

$$\int_0^1\left(\left|D^2T\right|^2+2a^2\left|DT\right|^2+a^4\left|T\right|^2\right)dz+2p_r\int_0^1\left(\left|DT\right|^2+a^2\left|T\right|^2\right)dz+\left|p\right|^2\int_0^1\left|T\right|^2dz=\int_0^1\left|w\right|^2dz \tag{46}$$

given that $p_r\geq 0$, it follows from above equation is that

$$\int_0^1\left|DT\right|^2dz<\frac{1}{2a^2}\int_0^1\left|w\right|^2dz \tag{47}$$

similarly from Eqs. (33) and (34), by adopting the same procedure, we obtain

$$\int_0^1\left|DS_1\right|^2dz<\frac{1}{2a^2\tau_1^2}\int_0^1\left|w\right|^2dz, \tag{48}$$

$$\int_0^1\left|DS_2\right|^2dz<\frac{1}{2a^2\tau_2^2}\int_0^1\left|w\right|^2dz \tag{49}$$

respectively. We first note that since w, T, S_1 and S_2 satisfy $w(0)=0=w(1)$, $T(0)=0=T(1)$, $S_1(0)=0=S_1(1)$ and $S_2(0)=0=S_2(1)$ respectively, we have by Rayleigh-Ritz inequality [36] that

$$\pi^2\int_0^1\left|w\right|^2dz\leq\int_0^1\left|Dw\right|^2dz. \tag{50}$$

$$\pi^2\int_0^1\left|T\right|^2dz\leq\int_0^1\left|DT\right|^2dz \tag{51}$$

$$\pi^2\int_0^1\left|S_1\right|^2dz\leq\int_0^1\left|DS_1\right|^2dz \tag{52}$$

$$\pi^2\int_0^1 |S_2|^2\,dz \le \int_0^1 |DS_2|^2\,dz \tag{53}$$

Utilizing inequalities (47)-(50) in inequalities (51)-(53) we obtain

$$\int_0^1 |DT|^2\,dz < \frac{1}{2a^2\pi^2}\int_0^1 |Dw|^2\,dz \tag{54}$$

$$\int_0^1 |T|^2\,dz < \frac{1}{2a^2\pi^4}\int_0^1 |Dw|^2\,dz \tag{55}$$

$$\int_0^1 |S_1|^2\,dz < \frac{1}{2a^2\tau_1^2\pi^4}\int_0^1 |Dw|^2\,dz \tag{56}$$

$$\int_0^1 |S_2|^2\,dz < \frac{1}{2a^2\tau_2^2\pi^4}\int_0^1 |Dw|^2\,dz \tag{57}$$

Given that $p_r \ge 0$ and utilizing inequalities (54)-(57) in Eq. (45), we get

$$\begin{aligned}&\Lambda_2\int_0^1\left(|D^2w|^2 + a^4|w|^2\right)dz + \frac{a^2}{\sigma}\int_0^1 |w|^2\,dz + \Lambda_1 a^2 R_{S1}\tau_1\int_0^1\left(|DS_1|^2 + a^2|S_1|^2\right)dz + \\ &\Lambda_1 a^2 R_{S2}\tau_2\int_0^1\left(|DS_2|^2 + a^2|S_2|^2\right)dz + Ra^2\int_0^1 |T|^2\,dz + \\ &\qquad \frac{1}{\sigma}\left\{1 + 2\sigma a^2\Lambda_2 - \frac{\sigma R\Lambda_1\left(\pi^2 + a^2\right)}{2\pi^4} - \frac{\sigma R_{S1}}{2\tau_1^2\pi^4} - \frac{\sigma R_{S2}}{2\tau_2^2\pi^4}\right\}\int_0^1 |Dw|^2\,dz < 0\end{aligned} \tag{58}$$

the above equation clearly implies that

$$\frac{\sigma R_{S1}}{2\tau_1^2\pi^4} + \frac{\sigma R_{S2}}{2\tau_2^2\pi^4} > 1 + 2\sigma a^2\Lambda_2 - \frac{\sigma R\Lambda_1\left(\pi^2 + a^2\right)}{2\pi^4} \tag{59}$$

Hence, if $\frac{\sigma R_{S1}}{2\tau_1^2\pi^4} + \frac{\sigma R_{S2}}{2\tau_2^2\pi^4} \le 1 + 2\sigma a^2\Lambda_2 - \frac{\sigma R\Lambda_1\left(\pi^2 + a^2\right)}{2\pi^4}$ then we must have $p_i = 0$.

we complete the proof.

The vital content of theorem 1 from the physical point of view is that an arbitrary neutral or unstable mode of the system has the non-oscillatory character for the problem of triple diffusive convection in an Oldroyd-B fluid layer. In particular, PES is valid if

$$\frac{\sigma R_{S1}}{2\tau_1^2\pi^4} + \frac{\sigma R_{S2}}{2\tau_2^2\pi^4} \le 1 + 2\sigma a^2\Lambda_2 - \frac{\sigma R\Lambda_1\left(\pi^2 + a^2\right)}{2\pi^4}. \tag{60}$$

Special cases: It follows from theorem 1 that an arbitrary neutral or unstable mode has a oscillatory and in particular the PES is valid for the following types of convection:

1. For triple diffusive convection in a Maxwell fluid layer

($R_{S1} > 0, R_{S2} > 0,\ \Lambda_1 > 0, \Lambda_2 = 0, 0 < R < 2\pi^4 / \sigma\Lambda_1\left(\pi^2 + a^2\right)$).

$$\frac{\sigma R_{S1}}{2\tau_1^2\pi^4} + \frac{\sigma R_{S2}}{2\tau_2^2\pi^4} \le 1 - \frac{\sigma R\Lambda_1\left(\pi^2 + a^2\right)}{2\pi^4}.$$

2. For triple diffusive convection ($R_{S1} > 0, R_{S2} > 0,\ \Lambda_1 = \Lambda_2 = 0, R > 0$).

$\dfrac{\sigma R_{S1}}{2\tau_1^2\pi^4}+\dfrac{\sigma R_{S2}}{2\tau_2^2\pi^4}\le 1$. (Prakash et al. [34] without rotation)

3. For thermohaline convection in an Oldroyd-B fluid layer
($R_{S1}>0, R_{S2}=0,\ \Lambda_1,\Lambda_2>0, 0<R<2\pi^4\left(1+2\sigma a^2\Lambda_2\right)/\sigma\Lambda_1\left(\pi^2+a^2\right)$).

$\dfrac{\sigma R_{S1}}{2\tau_1^2\pi^4}\le 1+2\sigma a^2\Lambda_2-\dfrac{\sigma R\Lambda_1\left(\pi^2+a^2\right)}{2\pi^4}$.

Clearly the complement of the above results implies the incidence of oscillatory motions, therefore it is significant to derive the bounds for the complex rate oscillatory motions. we prove the following theorem in this direction.

Theorem 2. If $R_{S1}>0, R_{S2}>0, \Lambda_1\ge 0, \Lambda_2\ge 0\ (\Lambda_1\ge\Lambda_2), R<\left(\Lambda_2 a^4\sigma+a^2\right)/\sigma\Lambda_1, p_r\ge 0, p_i\ne 0$, then a necessary condition for the existence of a nontrivial solution (w,T,S_1,S_2,p) of Eqs. (31)-(34) together with either of the boundary conditions Eqs. (35)-(38) is that $|p|^2<\left(R_{S1}+R_{S2}\right)\sigma\Big/\left(\Lambda_2 a^2\sigma+1-\dfrac{\Lambda_1 R}{a^2}\sigma\right)$.

Proof. Given that $p_r\ge 0$ and rewriting the Eq. (45), we get

$$\Lambda_2\int_0^1\left(\left|D^2w\right|^2+2a^2\left|Dw\right|^2+a^4\left|w\right|^2\right)dz+\frac{1}{\sigma}\int_0^1\left(\left|Dw\right|^2+a^2\left|w\right|^2\right)dz+Ra^2\int_0^1\left|T\right|^2dz+$$
$$\Lambda_1 a^2\left[R_{S1}\tau_1\int_0^1\left(\left|DS_1\right|^2+a^2\left|S_1\right|^2\right)dz+R_{S2}\tau_2\int_0^1\left(\left|DS_2\right|^2+a^2\left|S_2\right|^2\right)dz\right]\le \tag{61}$$
$$\Lambda_1 a^2R\int_0^1\left(\left|DT\right|^2+a^2\left|T\right|^2\right)dz+R_{S1}a^2\int_0^1\left|S_1\right|^2dz+R_{S2}a^2\int_0^1\left|S_2\right|^2dz$$

Now, multiplying Eq. (32) by its complex conjugate and integrating the ensuing equation for a suitable number of times and using the boundary conditions on T namely, $T(0)=0=T(1)$, we get

$$\int_0^1\left(\left|D^2T\right|^2+2a^2\left|DT\right|^2+a^4\left|T\right|^2\right)dz+2p_r\int_0^1\left(\left|DT\right|^2+a^2\left|T\right|^2\right)dz+|p|^2\int_0^1\left|T\right|^2dz=\int_0^1\left|w\right|^2dz \tag{62}$$

given that $p_r\ge 0$, it follows from above equation is that

$$a^2\int_0^1\left(\left|DT\right|^2+a^2\left|T\right|^2\right)dz<\int_0^1\left|w\right|^2dz \tag{63}$$

likewise from Eqs. (33) and (34), by adopting the same procedure, we obtain

$$\int_0^1\left|S_1\right|^2dz<\frac{1}{|p|^2}\int_0^1\left|w\right|^2dz \tag{64}$$

and $$\int_0^1\left|S_2\right|^2dz<\frac{1}{|p|^2}\int_0^1\left|w\right|^2dz \tag{65}$$

respectively.

given that $p_r\ge 0$ and utilizing inequalities (63)-(65) in Eq. (61), we have

$$\Lambda_2\int_0^1\left(\left|D^2w\right|^2+2a^2\left|Dw\right|^2\right)dz+\frac{1}{\sigma}\int_0^1\left|Dw\right|^2dz+Ra^2\int_0^1\left|T\right|^2dz+$$
$$a^2\Lambda_1\left[R_{S1}\tau_1\int_0^1\left(\left|DS_1\right|^2+a^2\left|S_1\right|^2\right)dz+R_{S2}\tau_2\int_0^1\left(\left|DS_2\right|^2+a^2\left|S_2\right|^2\right)dz\right]+$$
$$\left\{\Lambda_2a^4+\frac{a^2}{\sigma}-\Lambda_1R-\frac{R_{S1}a^2}{|p|^2}-\frac{R_{S2}a^2}{|p|^2}\right\}\int_0^1\left|w\right|^2dz<0 \tag{66}$$

which obviously implies that

$$|p|^2<\left(R_{S1}+R_{S2}\right)\sigma\Big/\left(\Lambda_2a^2\sigma+1-\frac{\Lambda_1R}{a^2}\sigma\right) \tag{67}$$

This proves the theorem.

The above theorem states, from the physical point of sight, that the complex growth rate $p=(p_r,p_i)$ of an arbitrary neutral or unstable oscillatory perturbation of growing amplitude, in a triple diffusive convection in an Oldroyd-B fluid layer with one of the components as heat with diffusivity κ_T , must lie inside a semicircle in the right-half of the (p_r,p_i) plane whose centre is the origin and radius equals $|p|<\sqrt{\left(R_{S1}+R_{S2}\right)\sigma\Big/\left(\Lambda_2a^2\sigma+1-\frac{\Lambda_1R}{a^2}\sigma\right)}$. It is further proved that this result is uniformly valid for any combination of rigid and/or free boundaries.

Special cases: The following results may be obtained from the theorem 2 as a special cases:

1. For triple diffusive convection in a Maxwell fluid layer
$(R_{S1}>0, R_{S2}>0,\ \Lambda_1>0, \Lambda_2=0, R<a^2/\sigma\Lambda_1)$.
$|p|^2<\left(R_{S1}+R_{S2}\right)\sigma\Big/\left(1-\frac{\Lambda_1R}{a^2}\sigma\right)$.
2. For triple diffusive convection $(R_{S1}>0, R_{S2}>0,\ \Lambda_1=\Lambda_2=0, R>0)$.
$|p|^2<\left(R_{S1}+R_{S2}\right)\sigma$ (Prakash et al. [38])
3. For thermohaline convection in an Oldroyd-B fluid layer
$(R_{S1}>0, R_{S2}=0, \Lambda_1,\Lambda_2>0, R<\left(\Lambda_2a^4\sigma+a^2\right)/\sigma\Lambda_1)$.
$|p|^2<R_{S1}\sigma\Big/\left(\Lambda_2a^2\sigma+1-\frac{\Lambda_1R}{a^2}\sigma\right)$.

3. Conclusions

The paper carries out a linear instability analysis of triple-diffusive convection in a layer of Oldroyd-B fluid and sets forth the sufficient conditions which prevent oscillatory disturbances from growing in amplitude, thus verifying the principle of exchange of stabilities for this system. It also derives upper bounds for the complex growth rates of any neutral or unstable oscillatory perturbations. Moreover, the method is able to recover the stability results for a number of convection problems in viscoelastic fluid layers as special cases. The results are shown to be valid for all combinations of rigid and free boundary conditions, thereby offering a unified analytical approach to the assessment of the stability and temporal behaviour of triple-diffusive convection in Oldroyd-B fluids.

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