Modeling the Circle Arrangement Problem with Tree Diagrams

Can Büyükuğurgör[1], Emre Yavuz[2]

Abstract

This project investigates how tree diagrams can be used to model and analyze a combinatorial circle arrangement problem. The problem concerns the placement of non-negative integers around a circle under fixed adjacency conditions, and it asks how many valid configurations can be formed for a given number of terms. The main aim of the study is not only to solve a particular counting problem, but also to evaluate whether tree diagrams can serve as an effective mathematical modeling tool for complex branching structures.

The study began with a literature review on counting problems, mathematical modeling, tree diagrams, and Catalan numbers. After the theoretical background had been established, the circle arrangement problem was examined for small values of n, and all valid arrangements were listed explicitly. These early cases were then transformed into a tree-diagram model, making the structure of the problem more visible and easier to analyze. As the number of terms increased, the tree-diagram method allowed the counting process to remain systematic and organized.

One of the most important findings of the project is that the sequence obtained for the main version of the problem matches the Catalan sequence for the computed values. This observation gives the project stronger mathematical significance by connecting the circle arrangement problem to a well-known combinatorial structure. In addition, a Python program was written to compute the number of valid arrangements for different values of n, making the method more practical and scalable.

The project also considers a generalized version of the problem in which the absolute difference between neighboring numbers is at most 2. This extension shows that the same modeling logic can be adapted to richer and more complex constraints. Overall, the study demonstrates that tree diagrams are valuable not only as visual aids, but also as tools for mathematical reasoning, pattern discovery, and computational verification.

Keywords: Tree Diagram, Counting Problem, Mathematical Modeling, Catalan Numbers, Combinatorics

Purpose

The purpose of this project is to determine how many valid arrangements can be formed when non-negative integers are written around a circle under a fixed difference condition between neighboring terms.

In the main version of the problem, the arrangement must begin with the number 1, and the absolute difference between every pair of neighboring numbers must be exactly 1. Starting from this definition, the project aims to:

- model the problem systematically,
- examine how the number of valid arrangements changes as n increases,

- evaluate the usefulness of tree diagrams as a method of analysis,
- identify numerical patterns in the results,
- and test whether the same approach can be extended to a more general version of the problem.

The project therefore has two complementary goals. The first is mathematical: solving and interpreting a specific counting problem. The second is methodological: showing that tree diagrams can be used as a practical and explanatory tool for combinatorial modeling.

Introduction

Counting problems play an important role in mathematics because they often reveal hidden patterns, recursive structures, and relationships between different areas of combinatorics. However, many counting problems become difficult to analyze when the number of possible cases grows rapidly. In such situations, a method that organizes the cases clearly can make a significant difference.

Tree diagrams are especially useful for this purpose. They make branching processes visible, allow the solver to track valid and invalid cases, and help transform a complicated counting task into a sequence of well-structured decisions. For this reason, tree diagrams are widely useful in probability, combinatorics, algorithm design, and mathematical modeling.

The circle arrangement problem studied in this report is a good example of a counting problem that benefits from a structured representation. At first glance, it appears to be a straightforward listing problem. However, once the number of positions on the circle increases, manual enumeration becomes increasingly inefficient. A more systematic approach is then required.

This project investigates whether the circle arrangement problem can be understood more clearly through a tree-diagram model. It also examines whether the resulting sequence of answers is related to a known combinatorial sequence. This led to one of the central outcomes of the study: the observed connection between the main version of the problem and the Catalan numbers.

At the same time, the project is not limited to this relationship alone. The broader contribution of the report lies in the modeling process itself: the problem is defined, explored through small cases, visualized through diagrams, generalized to a more flexible version, and supported computationally through code.

Literature Review

To provide a reliable foundation for the project, a literature review was conducted on four main topics: counting problems, mathematical modeling, tree diagrams, and Catalan numbers.

The review showed that counting problems are central in combinatorics because they often require the systematic organization of cases. A direct counting strategy may work for very small examples, but when the structure becomes more complicated, one typically needs recurrence relations, diagrams, or known combinatorial patterns. This was highly relevant to the present project because the circle arrangement problem quickly becomes difficult to solve by direct listing alone.

The review also emphasized the value of mathematical modeling. A mathematical model does not merely produce an answer; it also gives a structured representation of how the problem works. In this project, tree diagrams were selected as the main modeling tool because they represent the branching process step by step and make the transition structure between allowable values explicit.

Sources discussing tree diagrams highlighted that they reduce complexity, support visual reasoning, and help reveal repeated patterns in large decision structures. These features supported the choice of a tree-

diagram approach rather than relying only on raw case tables.
Catalan numbers became relevant during the literature review because they appear in many counting problems involving recursive branching and constrained paths. They are known to arise in valid parenthesizations, rooted binary trees, lattice paths that stay within a boundary, and several other classical combinatorial settings. Since the circle arrangement problem also creates a constrained branching process, the appearance of a Catalan-type pattern in the results became mathematically meaningful rather than accidental.
To strengthen the research, both online and printed sources were used. Introductory web resources helped clarify basic concepts, while academic sources provided a stronger theoretical context for interpreting the results. This combination of accessible and scholarly sources improved both the conceptual clarity and the academic reliability of the report.

Problem Definition
The main version of the problem studied in this project is defined as follows.
- Let $A_1, A_2, ..., A_n$ be non-negative integers arranged around a circle such that:
- $A_1 = 1$
- $|A_i - A_{i+1}| = 1$ for every $i = 1, 2, ..., n$, where $A_{n+1} = A_1$

The problem asks for the number of different valid arrangements that satisfy these conditions.
This means that every neighboring pair must differ by exactly 1, the arrangement must begin at 1, and because the terms are placed on a circle, the final term must also be consistent with the starting term under the same rule.
Several structural observations follow from this definition:
- each step changes the current value by either +1 or -1,
- no value is allowed to become negative,
- and the arrangement must be able to return to a value compatible with the starting value 1.

These conditions create a rich branching structure. For small values of n, the valid arrangements can be listed directly. For larger values, however, the number of possibilities grows quickly, which motivates the use of a more structured model.

Method
1. Initial Small-Case Exploration
The study began by solving the problem for small values of n. This stage was essential because it made the local structure of the problem visible. By writing down the shortest valid arrangements, it became possible to see how each value can be followed only by certain allowable next values.
This early exploration also revealed an important parity property. Except for the trivial starting situation, the arrangement must contain an even number of terms in order to return to the starting value 1 while changing by +1 or -1 at each step. This observation significantly narrows the class of cases that need to be studied.
2. Construction of the Tree Diagram
After the small examples were understood, the problem was represented by a tree diagram. In this model, each node corresponds to a possible value at a given stage, and each branch corresponds to a valid transition to the next stage.

Because the difference condition is exactly 1, each value can move only to the neighboring values that differ from it by 1, as long as the next value remains non-negative. This produces a branching structure that can be explored level by level.

For even values of n, the largest value that can appear in the main tree diagram is:

$n/2 + 1$

This upper bound determines how far the diagram must extend.

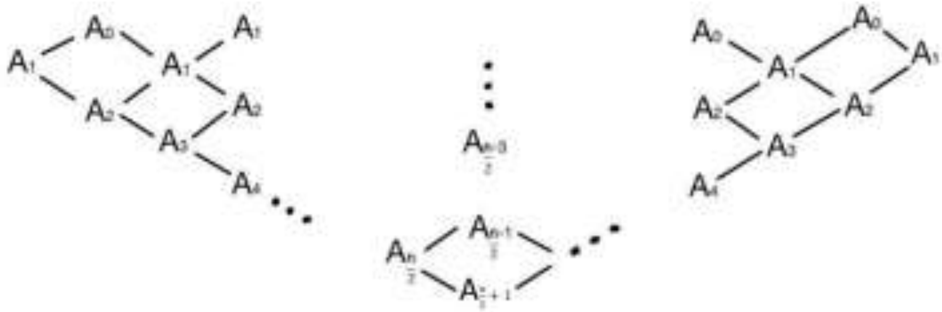

Figure 1. Tree structure used to build the main solution diagram.

3. Counting by Backward Summation

Once the tree diagram is built, the total number of valid arrangements can be obtained by working backward from the final stage. The last stage is constrained by the requirement that the arrangement must close consistently around the circle. After assigning the terminal values, the number of valid continuations at each earlier node is found by summing the counts of the nodes to which it is connected.

This changes the nature of the problem. Instead of repeatedly listing every full arrangement, the diagram records how many valid continuations are possible from each partial state. In this way, the solution becomes more systematic and less repetitive.

Figure 2. Backward accumulation of path counts in the tree diagram.

4. Pattern Detection and Computational Verification

When the counting process was repeated for increasing even values of n, the following sequence of answers emerged:

2, 5, 14, 42, 132, 429, ...

These values correspond to:

n = 2, 4, 6, 8, 10, 12, ...

After this pattern had been identified, a Python program was written to compute the answer for arbitrary admissible values of n. This computational step served two purposes. First, it confirmed the consistency of the values found through the tree-diagram model. Second, it allowed larger cases to be handled more efficiently.

5. Generalized Version of the Problem

To test the flexibility of the method, the project also considered a modified version of the problem in which the absolute difference between neighboring numbers is allowed to be at most 2 instead of exactly 1. In that case, the allowable step changes become:

-2, -1, 0, 1, 2

again subject to non-negativity and the circular return condition.

This extension required a modified tree diagram with more branches from each node. Even so, the same core strategy remained valid: determine the allowable value range, build the branching structure, apply the end condition, and count valid continuations by backward summation.

Project Work Schedule

Task Description	August	September	October	November	December	January
Literature Review		X	X	X	X	X
Field Study						
Data Collection and Analysis				X	X	X
Project Report Writing				X	X	X

The project is theoretical and computational in nature, so the field-study row remained empty in the original project schedule.

Findings

1. Basic Constraints

In the main version of the problem, the following conditions are imposed:

- the values A_1, A_2, ..., A_n are non-negative integers,
- $A_1 = 1$,
- the absolute difference between neighboring values is 1,
- and the arrangement is circular, so the last value must also satisfy the same difference rule with the first value.

Under these conditions, all possible cases were analyzed through tree diagrams and direct enumeration for the smallest values of n.

2. Example 1: Two Terms

Let A_1 and A_2 be non-negative integers placed around a circle such that:

- $A_1 = 1$
- $|A_1 - A_2| = 1$

Then the valid arrangements are:

A_1	A_2
1	0
1	2

Therefore, there are 2 valid arrangements.

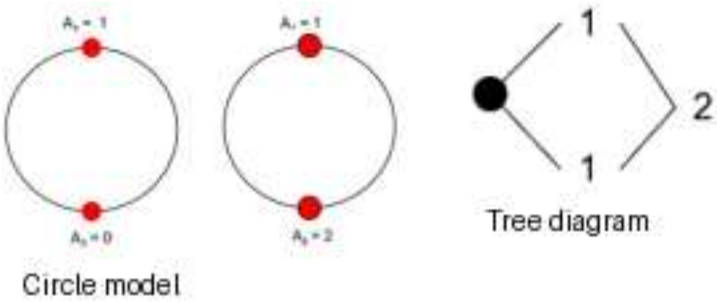

Figure 3. Example 1 represented as a circle model and a tree diagram.

This is the simplest nontrivial case and shows that the initial value 1 can move either downward to 0 or upward to 2, while still preserving the difference condition.

3. Example 2: Four Terms

Let A_1, A_2, A_3, A_4 be non-negative integers placed around a circle such that:

- $A_1 = 1$
- $|A_1 - A_2| = |A_2 - A_3| = |A_3 - A_4| = |A_4 - A_1| = 1$

Then the valid arrangements are:

A_1	A_2	A_3	A_4
1	0	1	0
1	2	1	0
1	0	1	2
1	2	1	2
1	2	3	2

Therefore, there are 5 valid arrangements.

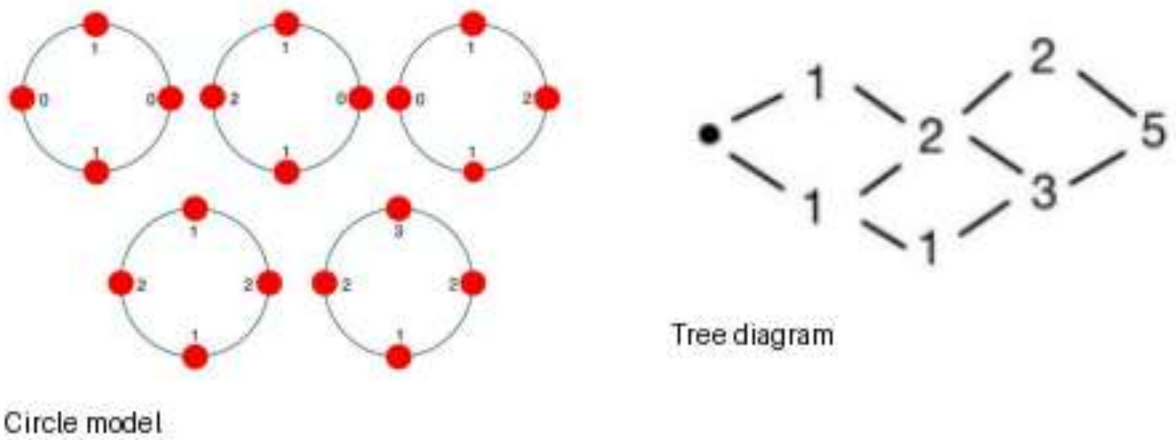

Figure 4. Example 2 represented as a circle model and a tree diagram.

This case already shows the value of the tree-diagram approach. Although the number of terms is still small, the number of valid cases has grown noticeably, and the branching structure is no longer trivial.

4. Example 3: Six Terms

Let A_1, A_2, A_3, A_4, A_5, A_6 be non-negative integers placed around a circle such that:

- $A_1 = 1$

- $|A_1 - A_2| = |A_2 - A_3| = |A_3 - A_4| = |A_4 - A_5| = |A_5 - A_6| = |A_6 - A_1| = 1$

Then the valid arrangements are:

A_1	A_2	A_3	A_4	A_5	A_6
1	0	1	0	1	0
1	0	1	0	1	2
1	0	1	2	1	0
1	2	1	0	1	0
1	0	1	2	1	2
1	2	1	0	1	2
1	2	1	2	1	0
1	2	1	2	1	2
1	2	1	2	3	2
1	0	1	2	3	2
1	2	3	2	1	2
1	2	3	2	3	2
1	2	3	2	1	0
1	2	3	4	3	2

Therefore, there are 14 valid arrangements.

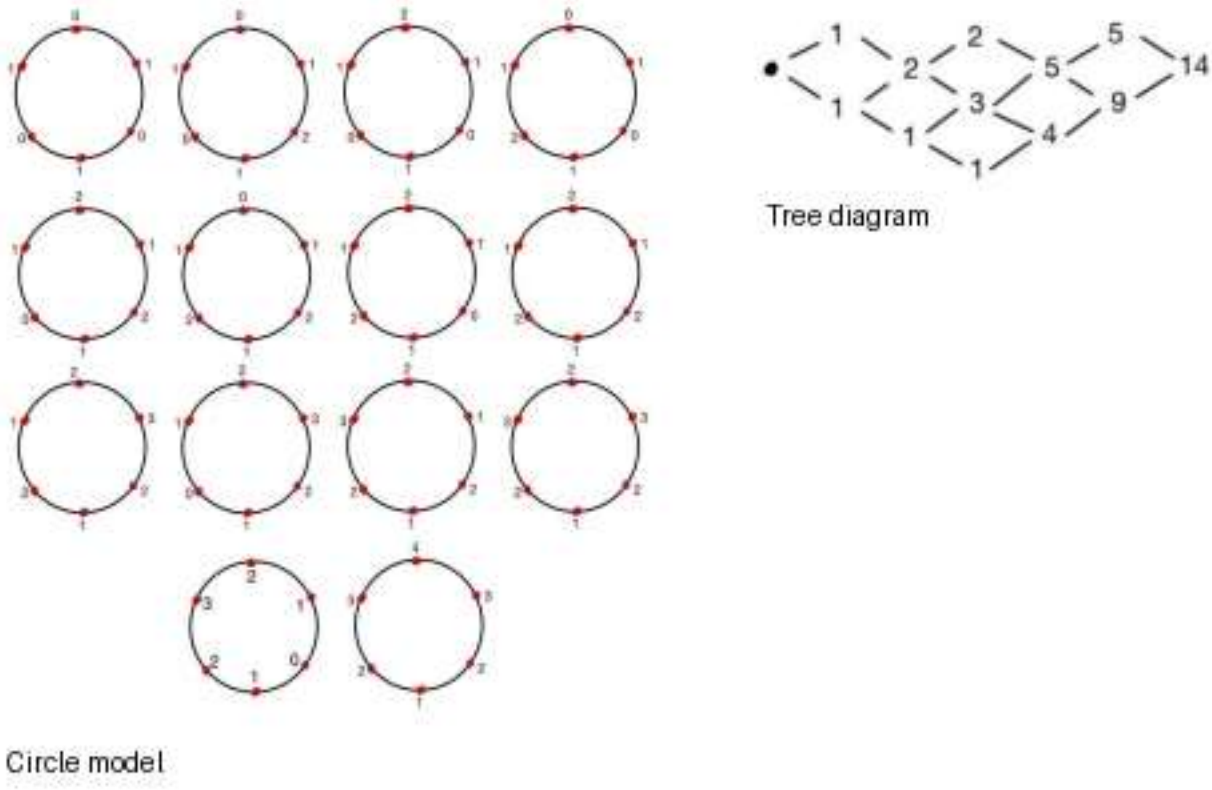

Figure 5. Example 3 represented as a circle model and a tree diagram.

At this stage, the limits of direct case listing become clear. Although it is still possible to write all arrangements explicitly, the process becomes time-consuming. The tree diagram offers a clearer and more manageable route to the answer.

5. Larger Cases

The same reasoning can be extended to larger even values of n. In the original project document, the tree-diagram solutions were continued for larger cases such as n = 8, n = 10, and n = 12.

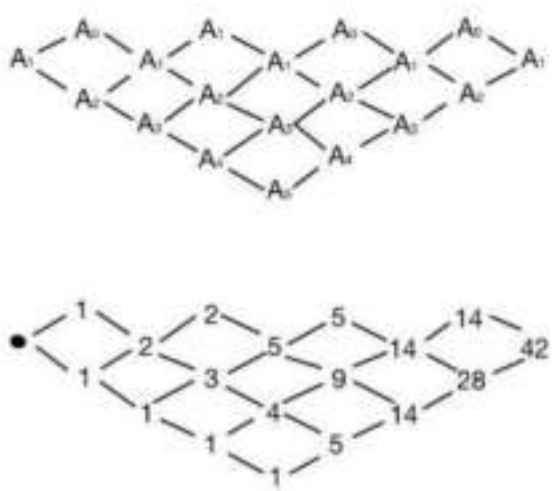

Figure 6. Tree-diagram solution for a larger even case.

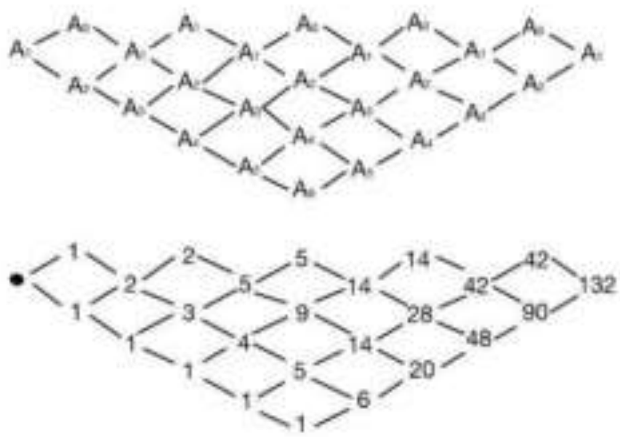

Figure 7. Tree-diagram solution extended further.

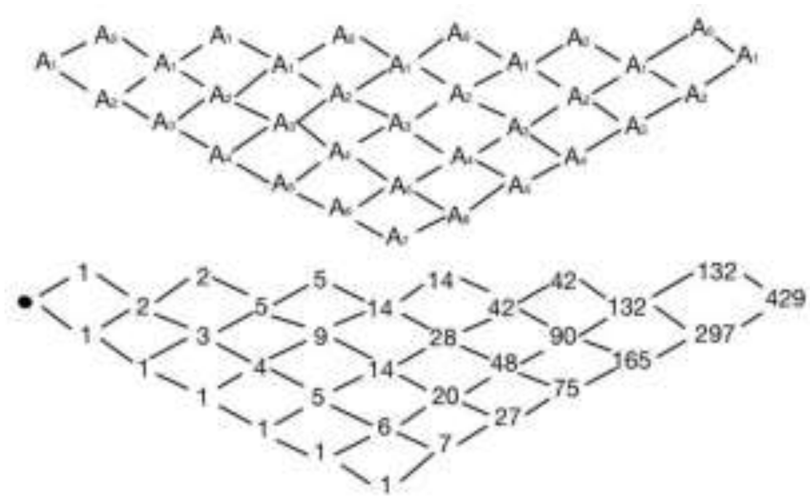

Figure 8. Tree-diagram solution for the twelve-term case.

6. Observed Numerical Pattern

The counts obtained from the tree-diagram model for the main problem are summarized below:

Number of Terms n	Number of Valid Arrangements
2	2
4	5
6	14
8	42
10	132
12	429

These results show that the number of valid arrangements grows rapidly. More importantly, they reveal a recognizable pattern rather than a random sequence.

7. Relationship with Catalan Numbers

One of the strongest results of the project is the observed relationship between the main circle arrangement problem and the Catalan sequence.

The values

2, 5, 14, 42, 132, 429, ...

match the Catalan numbers

$C_2, C_3, C_4, C_5, C_6, C_7, \ldots$

for the corresponding computed cases.

Equivalently, if a(n) denotes the number of valid arrangements in the main problem, the computed data support the observed rule

$a(2k) = C_{k+1}$ for $k = 1, 2, 3, \ldots$

for the cases tested in this project.

This connection is supported not only by the numerical pattern, but also by the structure of the model itself. In the tree diagram, every step changes the current value by +1 or -1, the values must remain non-negative, and the path must return to a state compatible with the initial value. This creates a structure similar to classical Catalan-type objects, especially constrained lattice paths.

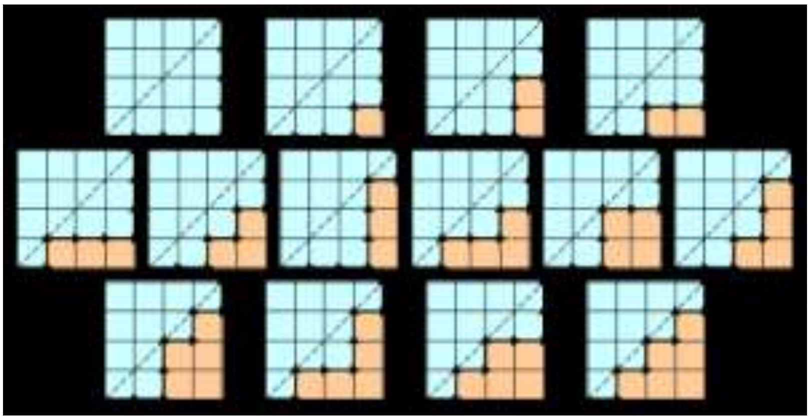

Figure 9. Lattice-path model associated with Catalan numbers.

For this reason, the Catalan connection is not merely a numerical coincidence within the project. It shows that the tree-diagram model reveals a deeper combinatorial structure inside the original circle arrangement problem.

8. Generalized Version: Difference at Most 2

To show that the method is not limited to a single special case, the project also examined a modified version in which the absolute difference between neighboring numbers is allowed to be at most 2.

In this version, the allowable transitions are:

- x -> x – 2
- x -> x – 1
- x -> x
- x -> x + 1
- x -> x + 2

provided that the resulting value remains non-negative and the circular condition is still respected.

This generalization makes the branching structure richer and more complicated. The original tree diagram must therefore be modified to include more branches from each node. Even so, the same basic logic still applies: determine the relevant range of values, build the branching structure, and count valid continuations through backward summation.

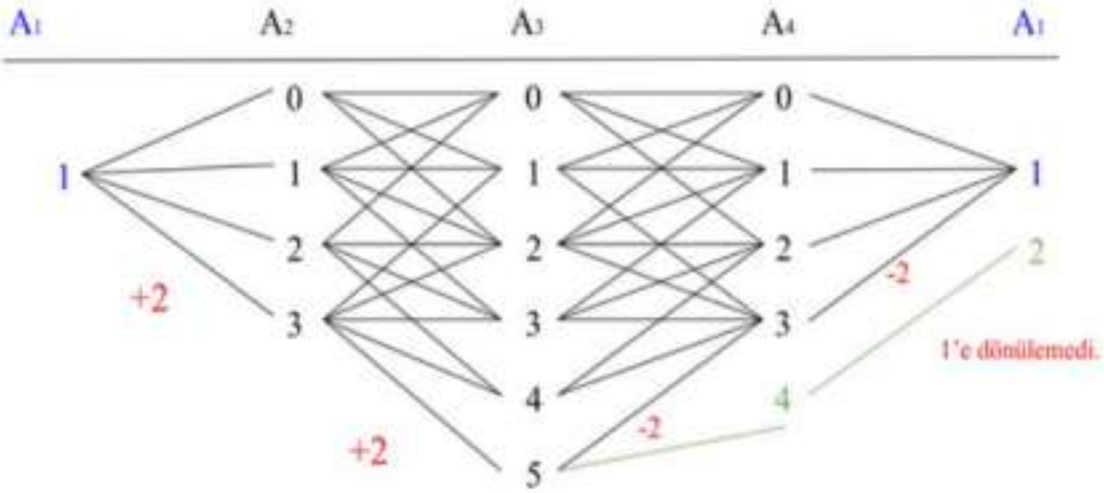

Figure 10. Modified tree diagram for the generalized problem.

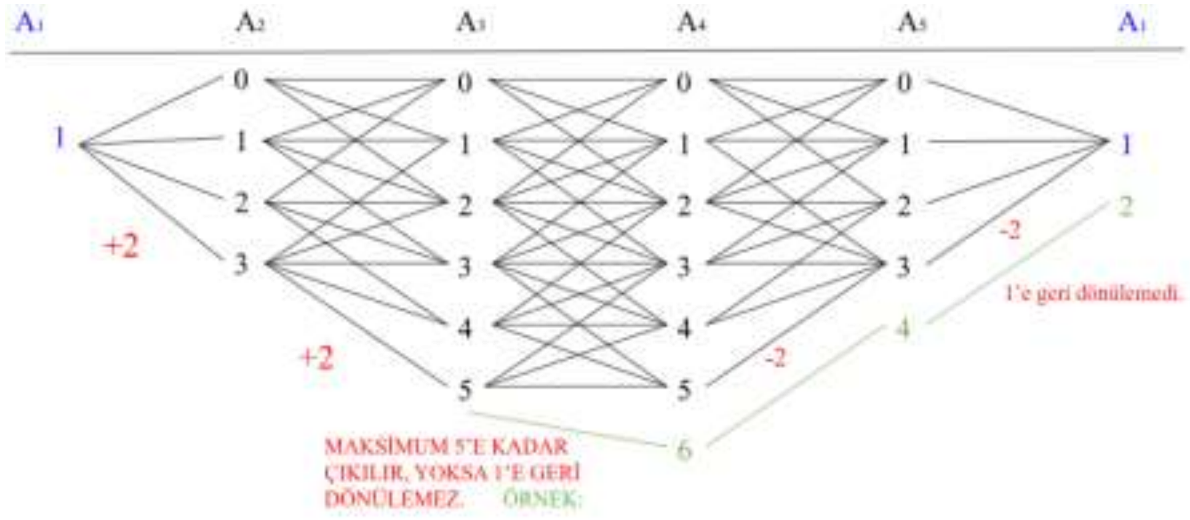

Figure 11. Extended boundary structure in the generalized problem.

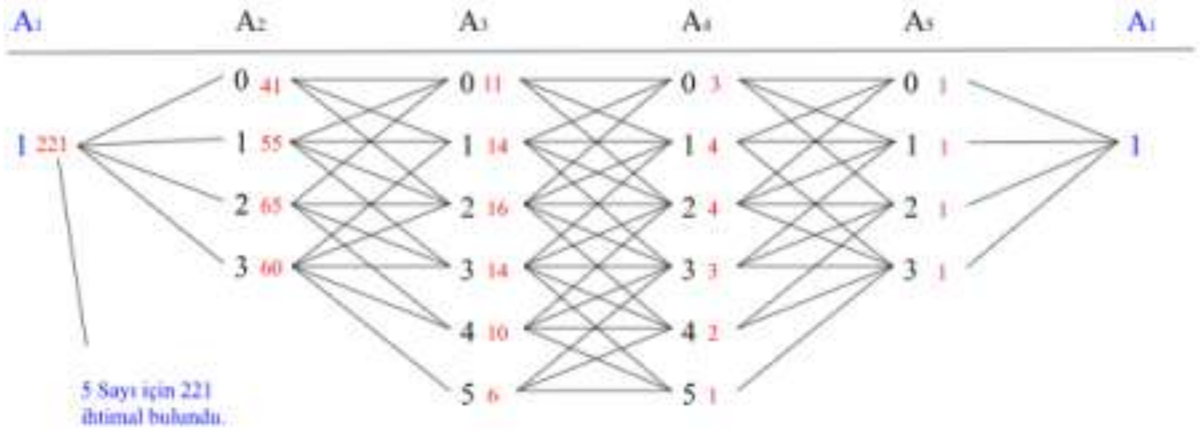

Figure 12. Counting structure used in the generalized problem.

This extension is important because it demonstrates the flexibility of the method. The project does not rely on a single narrow rule; instead, it shows that tree-diagram modeling can adapt to more general constraints while preserving the same analytical framework.

The original exploratory notes also compared a relaxed version of this extension with OEIS entry A005191. In the present report, that comparison is kept only as an exploratory remark rather than a formal identification, because the exact modeling constraints are not identical in every version considered.

9. Broader Interpretation

The project also suggests that tree diagrams may be useful beyond olympiad-style mathematics problems. Any system that evolves through constrained local choices may benefit from a similar representation.

For example, the original report discusses the possibility of optimizing the positions of lighting poles placed around a boundary while respecting local constraints on spacing or height differences. In such settings, tree diagrams can help count possible configurations and compare them systematically.

This broader interpretation strengthens the project by showing that it is not merely a narrow counting exercise. It is also a study of how structured visual models can support mathematical reasoning in more general contexts.

Discussion

The findings show that the circle arrangement problem becomes increasingly difficult when approached only through direct enumeration. Even though the first few cases can be solved by listing all possibilities, the complexity grows quickly as more positions are added.

The tree-diagram method addresses this difficulty effectively. It preserves the logic of the original problem while organizing the cases into a structure that supports both visualization and calculation. In this sense, the tree diagram is not only a way of presenting the answer; it is part of the reasoning process that leads to the answer.

The connection with Catalan numbers is especially important because it places the project within a broader combinatorial context. However, the project should not be reduced to this relationship alone.

The report's larger contribution is its demonstration of how a difficult counting problem can be modeled, interpreted, generalized, and supported computationally.

At the same time, the present report has natural limitations. It identifies and explains the Catalan relationship, but it does not attempt to give a fully formal proof of that correspondence. Likewise, the generalized difference <= 2 case has been modeled successfully, but its relationship to known integer sequences remains open for further study. These are not weaknesses of the project; rather, they show that the study has clear directions for future development.

Conclusion

This project has shown that tree diagrams can be used as an effective method for modeling and solving the circle arrangement problem. Beginning from small cases, the study developed a structured representation of the problem that made it possible to count valid arrangements in a clearer and more systematic way than direct listing alone.

The project also showed that the computed answers for the main version of the problem match the Catalan sequence for the tested even values of n. This gives the study additional mathematical depth and connects it to a broader family of combinatorial structures.

In addition, the Python implementation confirmed that the method can be supported computationally and applied efficiently to larger cases. The extension to the difference <= 2 version further demonstrated that the same modeling strategy remains useful when the branching rules become more complex.

Overall, the project achieved its main objective. It translated a difficult counting problem into a structured mathematical model, clarified the logic of the solution, linked the results to known combinatorial patterns, and showed how the same approach can be extended and applied more broadly.

Recommendations

Recommendation 1: The relationship between the circle arrangement problem and Catalan numbers could be developed further through a formal proof. This would strengthen the theoretical contribution of the project. Recommendation 2: The generalized version with $|A_i - A_{i+1}| <= 2$ could be investigated more deeply to determine whether its sequence matches another known combinatorial pattern.

Recommendation 3: The Python code developed for this project could be expanded into an interactive application that automatically generates the relevant tree diagrams and computes the number of valid arrangements for selected values of n.

Recommendation 4: The same modeling approach could be applied to other counting problems such as combinations, permutations, Fibonacci-type recurrences, and constrained path problems.
Recommendation 5: In future presentations of the project, the visual figures should continue to be integrated directly into the report, since they play an important role in making the method understandable and convincing.

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