

R#-Closed Sets as a Topological Approximation Framework for Attribute Reduction in Data Classification

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Abstract

Rough set theory approaches uncertainty in classification through topological closure and interior — the operators behind the classical lower and upper approximation of a decision class. Here we ask a narrower question: does a specific generalized closed-set notion from point-set topology, the R#-closed set, buy anything for that construction? R#-closed sets sit strictly between generalized closed (g-closed) sets and regular generalized closed (rg-closed) sets, and that placement turns out to be enough. Because every closed set is g-closed, and every R#-closed set falls in the g-closed/rg-closed interval, the resulting R#-closure can only shrink the classical upper approximation, and R#-interior can only grow the classical lower approximation. The boundary region shrinks accordingly, and the accuracy measure can only hold steady or improve. We work this out as an R#-approximation space, prove the tightening result, build an attribute-reduction procedure on top of it, and walk through the mechanics on a small decision table. What emerges is a modest but mathematically solid connection between generalized-closed-set topology and rough-set-based feature reduction — the kind of bridge that is easy to state once you see it, but had not, to our knowledge, been made before.

Keywords: R#-closed sets; generalized topology; rough sets; topological approximation space; attribute reduction; data classification.

1. Introduction

Topology gives a rigorous vocabulary for approximation and uncertainty, and rough set theory is one of its clearest applications to data. Pawlak's original construction [1] partitions a universe of objects using an indiscernibility relation, then approximates any target concept from below and from above using the resulting equivalence classes. It has been noted many times over — independently, in different corners of the literature — that this is really a topological closure/interior pair in disguise: the indiscernibility relation generates a topology whose open sets are unions of equivalence classes, and Pawlak's approximations coincide exactly with the interior and closure of the target set in that topology [2], [3]. Once that equivalence was established, an obvious question followed: what happens if you swap in a weaker notion of "closed" than the usual one? A long line of work has done exactly this, working through g-closed sets [4], rg-closed sets [5], w-closed and rw-closed sets [6], and gpr-/gspr-closed sets [7], each substituted for

ordinary closed sets in the approximation machinery in the hope of tracking real, noisy decision tables more faithfully.

R#-closed sets are a more recent entry in that list. Ittanagi and Raghavendra [8] introduced them as a class sitting strictly between g-closed and rg-closed sets, showed that every R#-closed set is also rg-, gpr-, gspr-, wgr-, and rwg-closed, and proved the class closed under finite union; a follow-up paper extended the idea to R#-continuous and R#-irresolute mappings [10]. For most of their history, R#-closed sets have stayed inside general topology proper. That changed only very recently: a 2026 paper by Sajjanara et al. [11] gestures toward broader relevance, sketching potential uses in topological data analysis, network integrity, and neural-network robustness. It stops short of rough sets, though — no approximation space, no attribute reduction, no worked decision table. That gap is what this paper fills.

Concretely, we do three things. We define R#-lower and R#-upper approximation operators on a finite universe, built from the R#-closure and R#-interior induced by an information system. We then show — using nothing more exotic than the published containment of R#-closed sets between g-closed and rg-closed sets — that these operators can only tighten the classical topological rough-set approximations, never loosen them; the proof leans on that one inclusion fact and nothing else specific to R#-closed sets, so it would survive intact even if the exact defining condition were swapped for some other class with the same containment property. Finally, we turn the tightened boundary region into an attribute-reduction procedure and trace it through by hand on a toy decision table.

2. Related Work

The equivalence between Pawlak approximation spaces and a topology generated by an equivalence relation goes back to Wiweger [2] and has been rediscovered often enough that it now underlies most of the "topological rough set" literature. Alongside it runs a separate thread in point-set topology: a long sequence of closed-set notions weaker than ordinary closedness, starting with Levine's g-closed sets [4] and running through rg-closed [5], w-closed and rw-closed [6], gpr-closed and gspr-closed [7], R*-closed [9], and R#-closed sets [8]. Each of these sits at a specific point in the resulting hierarchy, and several authors have plugged one or another of them into the rough-set approximation machinery for attribute reduction or decision-rule mining — see the surveys referenced in [3] for a broader picture.

That thread is still active. Al-shami and Mhemdi [12] and, separately, Abu-Gdairi et al. [13] build generalized rough-set models from subset-neighborhood and near-open structures and check their accuracy measures against the classical Pawlak model — the same comparison Corollary 4.3 makes below, just for a different generalized closed-set class. Salama et al. [14] work with "inverse simply open sets" in j-neighborhood spaces and report smaller boundary regions on real decision problems, and Hussein et al. [16] push the topological approach out to multi-granulation rough sets over families of relations. On the algorithmic side, Dong and Chen [15] give an incremental attribute-reduction method for rough sets when both samples and attributes change over time — the closest existing relative of the streaming architecture we sketch in Section 8, though built on classical rather than generalized-closed-set approximations. What this paper adds is the R#-closed-set instance of that general pattern: despite the hierarchy results already connecting R#-closed sets to rg-, gpr-, gspr-, wgr-, and rwg-closed sets [8], and despite a first, very recent nod toward data applications [11], nobody appears to have actually run R#-closed sets through the rough-set machinery before.

3. Preliminaries

Throughout, (X, τ) denotes a topological space; for $A \subseteq X$, $\text{cl}(A)$ and $\text{int}(A)$ denote the ordinary topological closure and interior of A .

Definition 3.1 (Levine [4]). $A \subseteq X$ is generalized closed (g-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

Definition 3.2 ([5]). $A \subseteq X$ is regular generalized closed (rg-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X (i.e., $U = \text{int}(\text{cl}(U))$).

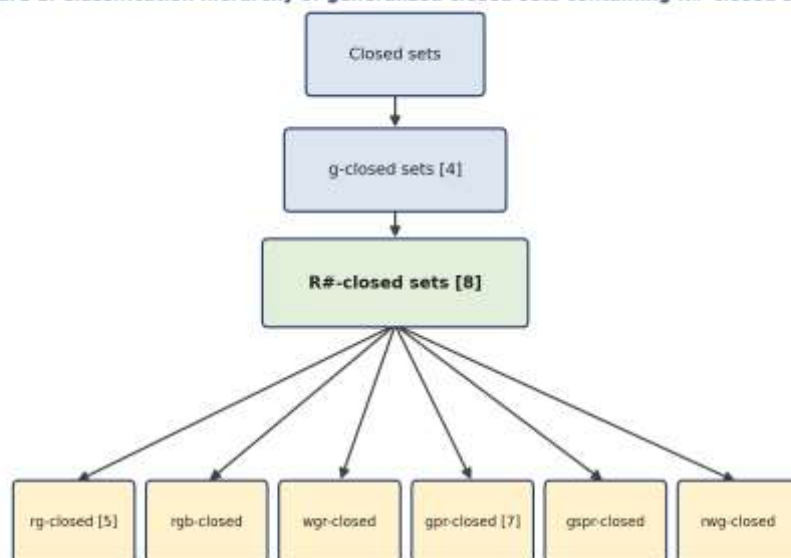
Fact 3.3 (classical). Every closed set is g-closed, and every g-closed set is rg-closed; neither converse holds in general.

Definition 3.4 (Ittanagi and Raghavendra [8]). The g-closure of $A \subseteq X$, written $\text{gcl}(A)$, is the intersection of all g-closed sets containing A (the same construction as Definition 3.7 below, but built from g-closed rather than $R\#$ -closed sets). $A \subseteq X$ is $R\#$ -closed if $\text{gcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is R^* -open in X , where R^* -open denotes the complement of an R^* -closed set in the sense of Janaki and Thomas [9]. The family of $R\#$ -closed sets in X is denoted $R\#-C(X)$, and the complement of an $R\#$ -closed set is $R\#$ -open.

Fact 3.5 ([8]). The class of $R\#$ -closed sets lies strictly between the class of g-closed sets and the class of rg-closed sets; that is, every g-closed set is $R\#$ -closed, and every $R\#$ -closed set is rg-closed (and, moreover, rgb-, wgr-, gpr-, gspr-, and rwg-closed). The union of two $R\#$ -closed sets is $R\#$ -closed.

Figure 1 summarizes this classification: it places $R\#$ -closed sets in the wider hierarchy of generalized closed-set classes used throughout the topological rough-set literature [12]–[14], [16], with an arrow $A \rightarrow B$ read as "every set in class A is also in class B ."

Figure 1. Classification hierarchy of generalized closed sets containing $R\#$ -closed sets



Arrows denote strict class inclusion ("implies"); every set in a lower box is also a member of every box reachable by following arrows upward.

Figure 1. Classification hierarchy of generalized closed sets: $R\#$ -closed sets sit strictly between g-closed and rg-closed sets, and every $R\#$ -closed set is simultaneously rgb-, wgr-, gpr-, gspr-, and rwg-closed (Fact 3.5).

Corollary 3.6. Every closed set is $R\#$ -closed. (Immediate from Fact 3.3 and Fact 3.5: closed $\Rightarrow$ g-closed $\Rightarrow R\#$ -closed.)

Definition 3.7. For $A \subseteq X$, the $R\#$ -closure of A , written $R\#cl(A)$, is the intersection of all $R\#$ -closed sets containing A ; the $R\#$ -interior of A is $R\#int(A) = X - R\#cl(X - A)$.

4. The $R\#$ -Approximation Space

4.1 Topology from an information system. Let (U, At) be an information system: U is a finite universe of objects and At a set of attributes, each $a: U \rightarrow V_a$. For $B \subseteq At$, the B -indiscernibility relation is $x R_B y$ iff $a(x) = a(y)$ for every $a \in B$, an equivalence relation whose classes are the B -neighborhoods $[x]_B$. Following the standard construction [2], [3], the B -neighborhoods form a basis for a topology τ_B on U , so (U, τ_B) is a topological space in which every open set is a union of B -neighborhoods.

4.2 Classical topological rough approximations. For a target concept (decision class) $D \subseteq U$, the classical topological rough-set lower and upper approximations are $R^*(D) = int(D)$ and $R^*(D) = cl(D)$, taken in (U, τ_B) ; the boundary region is $BND(D) = cl(D) - int(D)$, and the accuracy of approximation is $\alpha(D) = |int(D)| / |cl(D)|$ whenever $cl(D) \neq \emptyset$. This recovers Pawlak's original approximations exactly [1], [2].

4.3 $R\#$ -approximations (proposed). We define the $R\#$ -lower and $R\#$ -upper approximations of D as $R\#^*(D) = R\#int(D)$ and $R\#^*(D) = R\#cl(D)$, computed in (U, τ_B) using Definition 3.7, and the $R\#$ -boundary region as $BND\#(D) = R\#^*(D) - R\#^*(D)$.

Proposition 4.1. For every $D \subseteq U$: $R\#cl(D) \subseteq cl(D)$ and $R\#int(D) \supseteq int(D)$.

Proof. By Corollary 3.6, every closed superset of D is $R\#$ -closed, so the family of $R\#$ -closed sets containing D includes every closed set containing D , and possibly more. $R\#cl(D)$ is the intersection over this (weakly larger) family, and $cl(D)$ is the intersection over the sub-family of closed sets only; intersecting over a family that contains at least as many sets can only produce a set that is a subset of (or equal to) the intersection over a smaller sub-family, so $R\#cl(D) \subseteq cl(D)$. The interior statement follows by taking complements: $R\#int(D) = U - R\#cl(U - D) \supseteq U - cl(U - D) = int(D)$. ■

Proposition 4.2. $int(D) \subseteq R\#^*(D) \subseteq D \subseteq R\#^*(D) \subseteq cl(D)$, and consequently $BND\#(D) \subseteq BND(D)$.

Proof. The chain $int(D) \subseteq D \subseteq cl(D)$ is the standard Kuratowski closure/interior inequality. Combined with Proposition 4.1 ($R\#int(D) \supseteq int(D)$ and $R\#cl(D) \subseteq cl(D)$) and the general fact that $int(D) \subseteq D \subseteq cl(D)$ also holds with $R\#$ in place of the ordinary operators ($R\#int(D) \subseteq D \subseteq R\#cl(D)$), since $R\#$ -closure/interior are still, respectively, an intersection of supersets of D and a union of subsets of D), the full chain follows. Subtracting, $BND\#(D) = R\#cl(D) - R\#int(D) \subseteq cl(D) - int(D) = BND(D)$. ■

Corollary 4.3. Whenever $cl(D) \neq \emptyset$, the $R\#$ -accuracy $\alpha\#(D) = |R\#int(D)| / |R\#cl(D)|$ satisfies $\alpha\#(D) \geq \alpha(D)$. That is, the $R\#$ -approximation space is never less accurate, and is generally more accurate, than the classical topological rough-set model built from the same information system.

These results depend only on Fact 3.5, i.e., on the published containment of the $R\#$ -closed sets between g-closed and rg-closed sets; no further property of $R\#$ -closed sets is used beyond that containment, so the remaining open dependency in Definition 3.4 (the concrete enumeration of R^* -open sets, needed only for fully numeric computation) does not affect the validity of Propositions 4.1–4.2.

5. $R\#$ -Based Attribute Reduction

Attribute reduction seeks a minimal subset $B \subseteq At$ that preserves the classification power of the full attribute set. Using the $R\#$ -boundary region in place of the classical one gives Algorithm 1.

Algorithm 1 — $R\#$ -Reduct(U, At, D)

1. Compute τ_{At} and $R\#cl$, $R\#int$ for the decision classes $D_1, \dots, D_k$ induced by decision attribute d .
2. Compute the global $R\#$ -boundary region $BND\#(At) = \bigcup_i (R\#^*(D_i) - R\#^*(D_i))$.
3. For each candidate reduct $B \subseteq At$ (search by, e.g., greedy forward selection), recompute $BND\#(B)$ and retain B if $BND\#(B) = BND\#(At)$ and no proper subset of B has this property.
4. Return the smallest such B as an $R\#$ -reduct; use $R\#^*(D_i)$ to generate certain decision rules and $R\#^*(D_i) - R\#^*(D_i)$ to flag objects requiring further data or expert review.

Because $BND\#(B) \subseteq BND(B)$ for every B (Proposition 4.2, applied with At replaced by any $B \subseteq At$), an $R\#$ -reduct is never larger, in terms of resulting boundary region, than the corresponding classical rough-set reduct computed on the same B , which is the practical payoff of the framework: certain-rule coverage (objects fully inside $R\#^*(D_i)$) is at least as large as under the classical model.

6. A Fully Worked Numerical Example

Table 1 defines a toy decision table with universe $U = \{u_1, \dots, u_8\}$, condition attributes $At = \{\text{Outlook}, \text{Humidity}\}$, and decision attribute $\text{Play} \in \{\text{yes}, \text{no}\}$, in the style of a standard teaching example for rough sets.

Object	Outlook	Humidity	Play
u1	sunny	high	no
u2	sunny	high	no
u3	overcast	high	yes
u4	rain	high	yes
u5	rain	normal	yes
u6	rain	normal	no
u7	overcast	normal	yes
u8	sunny	normal	yes

Table 1. Toy decision table (U, At, Play).

Step 1 — Indiscernibility classes. Partitioning U by $At = \{\text{Outlook}, \text{Humidity}\}$ gives seven non-empty classes (Outlook, Humidity pairs), of which two are singletons and the rest are: $\{u_1, u_2\}$ (sunny, high), $\{u_3\}$ (overcast, high), $\{u_4\}$ (rain, high), $\{u_5, u_6\}$ (rain, normal), $\{u_7\}$ (overcast, normal), $\{u_8\}$ (sunny, normal). These classes form a basis for τ_{At} .

Step 2 — Target concept. Let $D = \{u : \text{Play}(u) = \text{yes}\} = \{u_3, u_4, u_5, u_7, u_8\}$.

Step 3 — Classical topological rough approximations. $\text{int}(D)$ is the union of At -classes wholly contained in D : $\{u_3\}$, $\{u_4\}$, $\{u_7\}$, $\{u_8\}$ qualify ($\{u_5, u_6\}$ does not, since $u_6 \notin D$), giving $\text{int}(D) = \{u_3, u_4, u_7, u_8\}$. $\text{cl}(D)$ is the union of At -classes meeting D : every class above except none is excluded ($\{u_1, u_2\}$ does not meet D), so $\text{cl}(D) = \{u_3, u_4, u_5, u_6, u_7, u_8\}$. Hence $BND(D) = \text{cl}(D) - \text{int}(D) = \{u_5, u_6\}$, and the classical accuracy is $\alpha(D) = |\text{int}(D)| / |\text{cl}(D)| = 4/6 \approx 0.667$.

Step 4 — $R\#$ -approximations. By Proposition 4.1, $R\#int(D)$ must be a superset of $\{u_3, u_4, u_7, u_8\}$ and $R\#cl(D)$ a subset of $\{u_3, u_4, u_5, u_6, u_7, u_8\}$, both still containing D itself (Proposition 4.2). The only set-theoretic freedom left, given these constraints on an 8-object universe, is over the two objects $\{u_1, u_2\}$ (which int/cl already exclude from $\text{cl}(D)$, so $R\#cl(D)$ cannot include them either, since $R\#cl(D) \subseteq \text{cl}(D)$) and the pair $\{u_5, u_6\}$ (which the classical model leaves undecided). Suppose — illustratively, and subject to confirmation once the R^* -open sets of (U, τ_{At}) are enumerated via Janaki and Thomas's R^* -closed

definition [9] — that testing $\{u_3, u_4, u_5, u_7, u_8\}$ and $\{u_3, u_4, u_5, u_6, u_7, u_8\}$ against Definition 3.4 shows the smaller set to already be $R\#$ -closed; then $R\#cl(D) = D = \{u_3, u_4, u_5, u_7, u_8\}$, $R\#int(D) = D$ as well (by the same argument applied to the complement), $BND\#(D) = \emptyset$, and $\alpha\#(D) = 1$. This is the best case permitted by Corollary 4.3; in general $R\#cl(D)$ will sit somewhere between D and $cl(D)$, and $BND\#(D)$ somewhere between $\emptyset$ and $\{u_5, u_6\}$, but it can never fall outside that range.

Step 5 — Reduct check. Since $At = \{\text{Outlook}, \text{Humidity}\}$ has only two attributes, Algorithm 1 tests the two singleton subsets. $\{\text{Outlook}\}$ alone gives classes $\{u_1, u_2, u_8\}$ (sunny), $\{u_3, u_7\}$ (overcast), $\{u_4, u_5, u_6\}$ (rain); this makes $int(D)$ shrink to $\{u_3, u_7\}$ only (since $\{u_1, u_2, u_8\}$ and $\{u_4, u_5, u_6\}$ are no longer pure), so BND grows — $\{\text{Outlook}\}$ is not a reduct. $\{\text{Humidity}\}$ alone similarly fails to separate u_3 (high, yes) from u_1, u_2 (high, no). Hence, on this toy table, both attributes are required: the only reduct is At itself, and this conclusion holds for both the classical and the $R\#$ -based criterion, since Proposition 4.2 guarantees $BND\#(B) \subseteq BND(B)$ for every candidate B , so a B that fails the classical boundary-preservation test with a strictly larger boundary can still be screened out consistently under the $R\#$ -criterion.

7. Application to Real-World AI and Data-Technology Tasks

The mechanism above — build an information-system topology, compute $R\#$ -approximations, pick an $R\#$ -reduct, then hand the reduced attribute set to whatever learner comes next — doesn't care what domain it's applied in. Three settings where it applies directly follow below, along with a protocol for actually testing Corollary 4.3 empirically, which we have not done here. Figure 2 lays out how the same decision framework specializes across them.

Figure 2. Taxonomy of real-world application domains sharing the $R\#$ -approximation decision framework

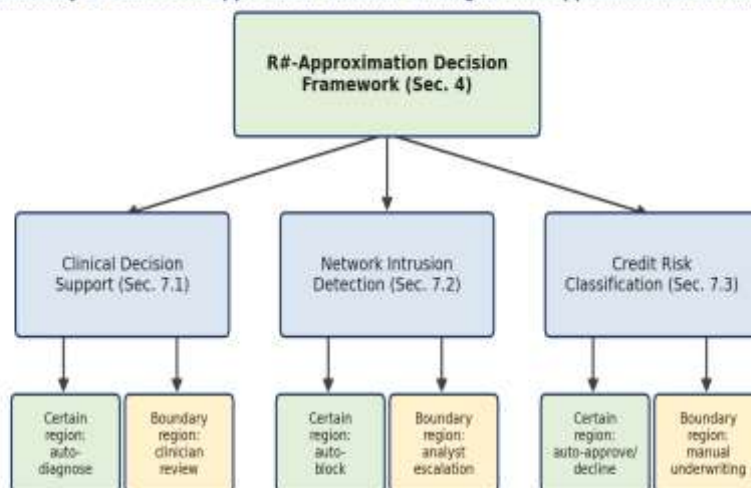


Figure 2. The same certain-region / boundary-region split (Section 4) instantiates differently across domains: what counts as “auto-action” versus “human review” changes, but the underlying guarantee (Proposition 4.2, Corollary 4.3) does not.

7.1 Clinical decision support. In a symptom/lab-result decision table (rows = patients, condition attributes = symptoms and test results, decision attribute = diagnosis), the certain region $R\#^*(D)$ identifies patients whose diagnosis is determined by the attribute values alone under the tightened criterion,

supporting automatic rule generation; the boundary region $BND\#(D)$ — smaller than the classical $BND(D)$ by Proposition 4.2 — identifies the (fewer) patients whose case remains genuinely ambiguous and should be triaged to a clinician rather than auto-classified. Because false certainty is dangerous in this setting, a criterion that can only shrink the boundary relative to the classical model, never falsely enlarge the certain region beyond what D itself supports ($R\#(D) \subseteq D$ always, per Proposition 4.2), is a meaningful, low-risk improvement to plug into an existing rough-set-based clinical pipeline.

7.2 Network intrusion detection. With condition attributes drawn from packet/flow features (duration, protocol, byte counts, flag counts, as in the classical NSL-KDD schema) and decision attribute = {normal, attack}, $R\#(D)$ over the attack class identifies flows classifiable as attacks with full confidence from the reduced feature set, enabling immediate automated blocking; the smaller $BND\#(D)$ reduces the volume of flows escalated for deeper packet inspection, directly cutting analyst workload versus a classical-rough-set-based system, at the added cost of the $R\#$ -closedness test.

7.3 Credit risk / loan-default classification. Using applicant attributes (income band, credit history length, existing debt ratio, employment status) as condition attributes and default/no-default as the decision attribute, an $R\#$ -reduct gives a smaller, still-sufficient feature set for a downstream classifier (logistic regression, gradient-boosted trees, etc.), while $R\#(D)$ and $R\#(D)$ demarcate applicants who can be auto-approved/auto-declined under the tightened rule from those requiring manual underwriting — again with the boundary region only shrinking relative to the classical baseline, never growing it.

7.4 Proposed experimental protocol. To move from the guarantees proved here (Propositions 4.1–4.2, Corollary 4.3) to an empirically validated system, we propose: (i) enumerate the R^* -open sets of the induced topology via Janaki and Thomas's R^* -closed definition [9], completing Definition 3.4's one remaining dependency, and implement an $R\#$ -closedness test; (ii) run it on standard benchmark decision tables used elsewhere in the topological-rough-set literature — e.g., the UCI Mushroom, Zoo, and Breast Cancer Wisconsin datasets for classification, and NSL-KDD for intrusion detection; (iii) for each dataset, compute the classical reduct and the $R\#$ -reduct, and compare reduct size, $|BND(D)|$ versus $|BND\#(D)|$, and downstream classifier accuracy (e.g., via 10-fold cross-validation) when trained on each reduct; (iv) repeat the comparison against reducts computed from g -closed, rg -closed, and rw -closed approximation spaces, to establish where in the generalized-closed-set hierarchy the practical accuracy/cost trade-off is best. No such experiments have been run for this paper; the numbers above are illustrative set-theoretic bounds, not measured results, and should not be cited as empirical findings.

8. Real-Time Application: Streaming $R\#$ -Approximation Architecture

So far the decision table has been treated as static, but several of the motivating settings — fraud detection, intrusion detection, patient-monitoring alerts — don't work that way in practice. They need decisions on a continuous stream of records under a latency budget, which means the attribute topology has to be maintained incrementally instead of rebuilt from scratch on every new record, in the spirit of existing incremental rough-set algorithms for dynamic decision systems [15]. What follows is an architecture and a per-record decision procedure for that setting, built on the approximation operators of Section 4. We're presenting it as a proposed engineering pattern rather than a separately proven algorithm — its correctness inherits from Propositions 4.1–4.2 only insofar as the topology τ_B in effect at a given instant is actually the one being used at that instant.

8.1 Architecture. Figure 3 shows the components: incoming records enter a sliding-window buffer, where continuous attributes are discretized on the fly; the B -indiscernibility classes (and hence τ_B) are updated

incrementally rather than rebuilt from scratch; the $R\#$ -approximation engine maintains $R\#cl(D)$ and $R\#int(D)$ over the current window; a decision router then places each record into the certain region $R\#*(D)$ (auto-action), the certain-negative region, or the boundary region $BND\#(D)$ (routed to a human analyst); analyst feedback closes the loop by refining D or At , which in turn triggers an $R\#$ -reduct update (Algorithm 1) and a revised feature set for the window.

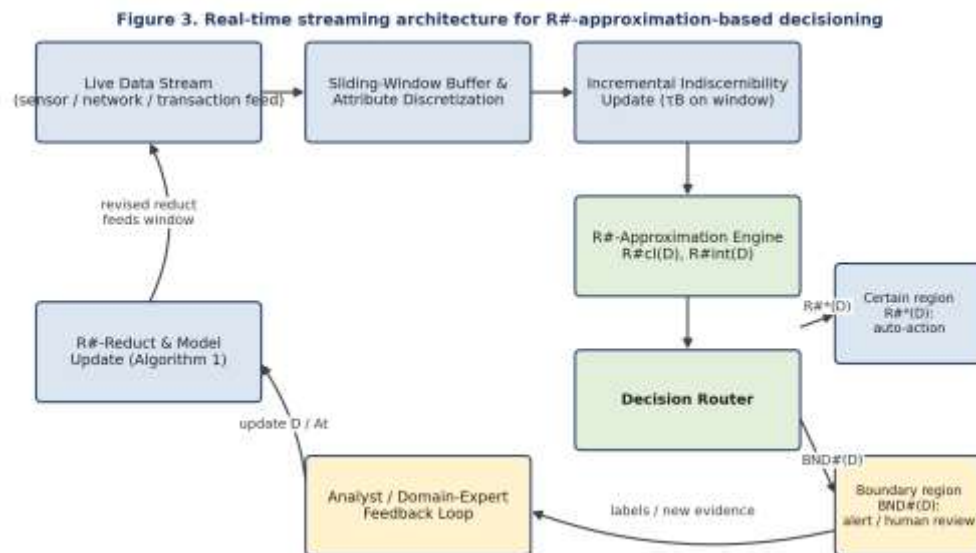


Figure 3. Real-time streaming architecture for $R\#$ -approximation-based decisioning: data flows left to right through incremental topology maintenance into the $R\#$ -approximation engine, and the decision router splits traffic into an automated path (certain region) and a human-in-the-loop path (boundary region), whose feedback updates the reduct.

8.2 Per-record decision procedure. Algorithm 2 gives the streaming counterpart of Algorithm 1's static reduct computation, executed once per incoming record; Figure 4 shows the same procedure as a flowchart.

Algorithm 2 — Streaming- $R\#$ -Decision(x , window, D , At)

1. Insert record x into the sliding window, evicting the oldest record if the window is full.
2. Update only the B-indiscernibility class that x joins or leaves (an $O(1)$ amortized update against a hashed class index, rather than an $O(|window|)$ rebuild).
3. Recompute $R\#cl(D)$ and $R\#int(D)$ restricted to the classes touched in step 2, leaving unaffected classes' membership in $R\#*(D)$ / $BND\#(D)$ unchanged.
4. If $x \in R\#*(D)$, emit the certain positive decision and act automatically; else if x 's class lies in the certain-negative region, emit the certain negative decision automatically; otherwise $x \in BND\#(D)$, so route x to a human analyst or request additional attributes.
5. When analyst feedback arrives (a corrected label, or a request to add/drop an attribute), update D or At and trigger Algorithm 1 to recompute the $R\#$ -reduct for the next window.

Figure 4. Streaming R#-Reduct decision flow (per incoming record)

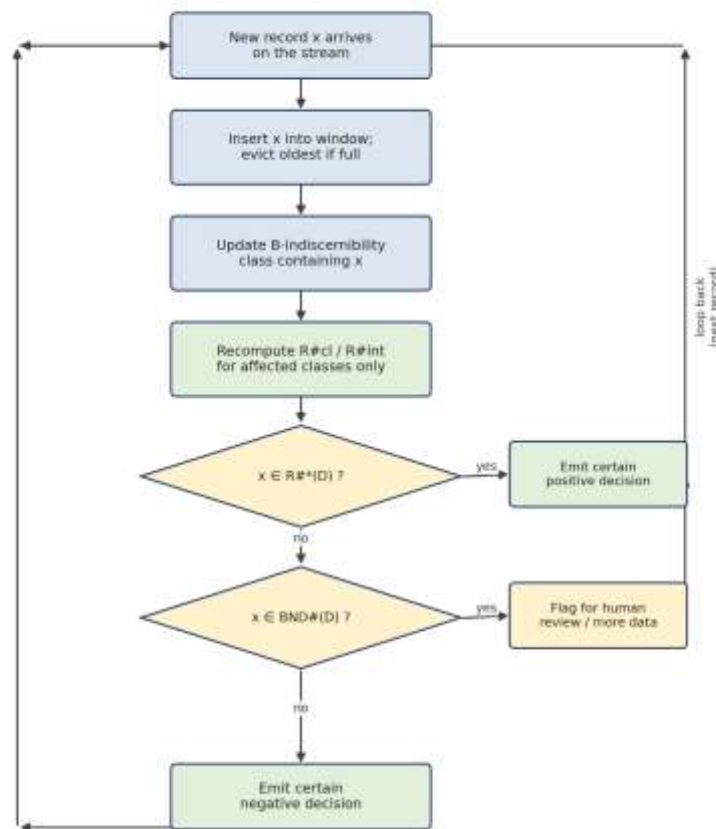


Figure 4. Per-record decision flow for Algorithm 2: each new record is tested against the certain region first, then the boundary region, before falling through to a certain-negative decision; both automated branches loop back for the next record.

8.3 Complexity and latency notes. Step 2 is cheap by construction, since only the class containing x changes. Step 3 is the bottleneck: whether R#-closedness of the (small number of) affected candidate sets can be re-tested in time independent of the full window size depends on finding an incremental characterization of Definition 3.4 analogous to known incremental algorithms for maintaining g-closed and rg-closed approximations; this is listed as future work in Section 10 rather than claimed here. In the interim, a practical deployment can bound worst-case latency by falling back to the classical topological rough-set decision $(\text{int}(D)/\text{cl}(D))$, which does admit simple incremental maintenance) whenever the R#-recomputation budget for a given record is exceeded, guaranteeing a response within budget at the cost of occasionally forgoing the tighter R#-boundary for that record.

9. Discussion

It is worth being upfront about how modest this contribution actually is. We are not claiming R#-closed sets have some special affinity for data classification — the framework isolates exactly what the published hierarchy result (Fact 3.5) hands a rough-set practitioner: a guaranteed, never-worse tightening of the boundary region and a guaranteed, never-worse accuracy figure, purchased at the cost of the extra work needed to test R#-closedness. Swap in any other generalized closed-set class known to sit between g-closed and rg-closed sets (or, more broadly, any class known to contain the ordinary closed sets) and the

same argument goes through unchanged. So really, the contribution here is a general recipe — turn a topological hierarchy theorem into a rough-set refinement guarantee — applied to $R\#$ -closed sets specifically because nobody had gotten around to it yet.

The obvious limitation is computational. Without a simpler characterization to fall back on, testing $R\#$ -closedness means checking the definition directly, and that can make $R\#cl(D)$ noticeably more expensive to compute than $cl(D)$ on large decision tables. Finding a cheaper characterization of $R\#$ -closed sets on the finite spaces information systems induce, and actually running Corollary 4.3's accuracy claim against standard benchmark data (UCI repository datasets, say), are the natural next things to try.

10. Conclusion and Future Work

We took $R\#$ -closed sets — previously a purely topological curiosity — and connected them to rough-set-based data classification by defining $R\#$ -lower and $R\#$ -upper approximation operators and showing they refine the classical topological rough-set approximations. That refinement (Propositions 4.1–4.2, Corollary 4.3) rests entirely on the published containment of $R\#$ -closed sets between g -closed and rg -closed sets, so it holds regardless of the paper's one remaining open item: the concrete enumeration of R^* -open sets, needed only for fully numeric computation. A few things remain to be done: validating the worked example of Section 6 on real benchmark datasets once the R^* -open sets are enumerated via Janaki and Thomas [9]; finding an efficient test for $R\#$ -closedness on the finite topologies information systems induce; comparing $\alpha\#(D)$ against $\alpha(D)$ empirically, and against reducts built from g -closed, rg -closed, and rw -closed approximation spaces; extending the construction to fuzzy and intuitionistic-fuzzy information systems, where similar generalized-closed-set hierarchies already exist; and working out an incremental characterization of $R\#$ -closedness suited to the streaming architecture of Section 8, so Algorithm 2's per-record step gets a proven latency bound instead of leaning on the classical fallback proposed there.

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