

A Monodromy Obstruction for One-Sided Difference Dirac Operators

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Abstract

Let $\nabla \mathbf{F}$ be the Clifford product of the del operator on $\mathbb{R}^m$ with a vector $\mathbf{F}$ of difference operators of step h . We decide for which s a fractional power $(\nabla \mathbf{F})^s$ exists as a Fourier multiplier, and the answer inverts the expectation suggested by lattice fermion doubling. For the central difference, whose symbol vanishes at the 2^m doubling points, the Clifford norm $\mathcal{N}$ arising from the identity $(\nabla \mathbf{F})(\mathbf{F} \nabla) = \Delta \circ \sum_j F_j^2$ is real and nonnegative; no branch question can arise, and $(\nabla \mathbf{F})^s$ exists for every $s \in \mathbb{C}$ on the whole cell minus the doubling points. For the one-sided differences, whose scalar symbol has no doubling zeros at all, $\mathcal{N}$ vanishes on a real variety of dimension $m - 2$ inside the cell and has winding number -1 on loops linking it; consequently no continuous single-valued $(\nabla \mathbf{F})^s$ exists on any domain linking that variety unless $s \in \mathbb{Z}$, and $(\nabla \mathbf{F})^s$ admits no absolutely convergent stencil. The obstruction is topological rather than pointwise: at each individual frequency every power exists. On the positive side we prove the invertibility threshold $\max_j h |\xi_j| < \pi/2$, a bandwidth rule that halves the Nyquist limit and cannot be improved, together with two-sided bounds on $\mathcal{N}$, an estimate for the inverse whose blow-up as the threshold is approached is of the correct order, and sectoriality on an explicit box.

Keywords: Difference Dirac Operator, Fermion Doubling, Monodromy, Fractional Power, Zero Divisor, Clifford Algebra, Functional Calculus

1. Introduction

A first-order differential operator on $\mathbb{R}^m$ is discretized by replacing each ∂_j with a difference quotient, and if the operator is the Dirac operator of Clifford analysis one must decide which difference quotient to use. The choice is usually made on grounds of accuracy or of symmetry, and the classical warning attached to it comes from lattice gauge theory: the symmetric choice suffers fermion doubling, its symbol acquiring spurious zeros in the Brillouin zone, while one-sided choices do not. This paper asks a different question of the same two schemes. Once the discrete operator is in hand, when does it possess a fractional power, or more generally a functional calculus? The answer turns out to run opposite to the doubling verdict, and the reason is a piece of topology rather than of algebra: the relevant obstruction is the winding of a Clifford-valued norm around a set where it vanishes, and it is the scheme without doubling that carries it.

When $\mathbf{F} = \nabla$ the geometric product collapses: $\nabla \wedge \nabla = 0$ and $\nabla^{2n} = (-\Delta)^n$, so that every even power of ∇ is a power of the Laplacian and the whole calculus is scalar. This paper shows that the collapse is special to $\mathbf{F} = \nabla$, and that replacing $\mathbf{F}$ by a vector of difference operators produces a bivector part which is algebraically harmless but analytically obstructive.

Let $\mathcal{C}\ell_{0,m}$ be the real Clifford algebra on generators $e_1, \dots, e_m$ subject to

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad 1 \leq j, k \leq m, \quad (1)$$

so $e_j^2 = -1$ and distinct generators anticommute. With $\nabla = \sum_j e_j \partial_j$ and $\mathbf{F} = \sum_j e_j F_j$ the geometric product splits into its scalar and bivector parts,

$$\nabla \mathbf{F} = -S + B, \quad S = \sum_j \partial_j \circ F_j, \quad B = \sum_{j < k} \Omega_{jk} e_j e_k, \quad \Omega_{jk} = \partial_j F_k - \partial_k F_j, \quad (2)$$

that is, $-S = \nabla \cdot \mathbf{F}$ and $B = \nabla \wedge \mathbf{F}$. Throughout we assume the F_j are Fourier multipliers, so that S and all Ω_{jk} commute pairwise (Hypothesis 3.2).

1.1 Results

(A) Doubling and monodromy point in opposite directions. For the central difference the symbol degenerates exactly at the 2^m points $t_j \in \{0, \pi\}$ of the Brillouin zone (Proposition 4.3): this is the fermion doubling pattern of Nielsen and Ninomiya [1, 2]. Nevertheless $\mathcal{N} \geq 0$ is real there, no winding is possible, and $(\nabla \mathbf{F})^s$ is a continuous single-valued symbol on all of $\mathbb{T}^m \setminus \mathcal{D}$ for every $s \in \mathbb{C}$ (Theorem 9.1). The one-sided differences show no doubling in their scalar symbol; yet their null set is a real variety of dimension $m - 2$ inside the cell (Theorem 4.1), $\mathcal{N}$ has winding number -1 on small loops linking it (Theorem 5.3), and no continuous single-valued $(\nabla \mathbf{F})^s$ exists on any domain containing such a loop unless $s \in \mathbb{Z}$ (Theorem 5.2 and Corollary 5.4). The scheme that doubles is the one that has a fractional calculus. The mechanism is the bivector part: at every nonzero null point $S \neq 0$ (Remark 4.4), and the cancellation in $\mathcal{N} = S^2 + Q$ is produced by $Q = -S^2$.

(B) The obstruction in operator form. Being a zero divisor does not by itself prevent roots; in $\mathbb{C} \oplus \mathbb{C}$ the element $(0, z)$ has k -th root $(0, z^{1/k})$, and $s \mapsto (0, z^s)$ even obeys the semigroup law (Remark 3.8). The genuine obstruction is global, and Theorem 6.1 states it without reference to symbols: no finite measure has $(\nabla \mathbf{F})^s$ as its Fourier–Stieltjes transform, so in particular $(\nabla \mathbf{F})^s$ is not convolution with an L^1 kernel and is not an infinite-order difference operator with absolutely summable coefficients. Remark 6.3 shows that this is sharp in the sense that bounded Borel realizations do exist: what the winding forbids is regularity, not size.

(C) A bandwidth rule. On band-limited data the one-sided operator is invertible as soon as $hK < \pi/2$, with an explicit bound on the inverse (Theorem 8.1), and the threshold $\pi/2$ cannot be improved (Theorem 7.1). Against the usual Nyquist condition $hK < \pi$ for a scalar one-sided scheme, the Clifford product costs a factor of two in admissible bandwidth. By Theorem 9.1 the restriction disappears for the central difference.

(D) Quantitative complement. On the box $\mathcal{B}_\theta = \{\max_j h|\xi_j| \leq \theta\}$ we prove the two-sided symbol bound (21), the estimate $\|(\nabla \mathbf{F})^{-1}(\xi)\| \leq \pi^2 m / (4 \cos \theta h |\xi|^2)$ (Theorem 7.5), whose $(\cos \theta)^{-1}$ growth is of the correct order as $\theta \uparrow \pi/2$ (Proposition 7.12), and sectoriality for θ up to the root $\theta_0 = 1.4044 \dots$ of $\theta^2 = 2 \sin \theta$ (Theorem 7.9), whence $(\nabla \mathbf{F})^s$ exists on $\mathcal{B}_\theta$ for every $s \in \mathbb{C}$ with the principal branch. Proposition 7.11 shows that the statement fails for $\theta \geq \pi/2$; that it holds on the whole range $\theta < \pi/2$ is supported numerically (Appendix A) but not proved.

Classical ingredients. The algebra used below is elementary and we claim no novelty for it: $\nabla \wedge \mathbf{F}$ is a wedge of two Clifford vectors, hence simple, so the unital algebra generated by $\nabla \mathbf{F}$ is two-dimensional over the scalars in every dimension and all integer powers are dimension-free, and the norm identity (6) is the Lagrange–Cauchy–Binet identity in operator form. Section 3 records these facts as lemmas; no part of the general theory of bivectors [3, 4, 5] is needed anywhere in this paper.

2. Related Work

2.1 Difference Dirac Operators

Dirac operators built from finite differences were introduced by Faustino and Kähler [6] and Faustino, Kähler and Sommen [7], where the structural difficulty is already visible: an operator assembled from purely forward differences does not factorize the discrete Laplacian, and a discrete curl appears. The Ghent school removed the difficulty rather than analysing it, through a doubled basis [8] and a discrete function theory on skew Weyl relations in which the splitting $e_j = e_j^+ + e_j^-$ restores the factorization exactly [9, 10]; earlier lattice Clifford calculus is due to Vaz, Sushch and Gürlebeck–Sprössig [11, 12, 13], and a symbol calculus for discrete pseudodifferential operators in the hypercomplex setting was developed by Cerejeiras, Kähler and Lucas [14]; the latter is the closest methodological precursor to Sections 4–6. What is not in this literature, so far as we have been able to determine, is an analysis of the bivector part in its own right: the null set of the resulting Clifford symbol, and the consequences of its vanishing for a functional calculus. The present paper takes the non-factorizing operator as its object rather than repairing it.

2.2 Fractional Powers and Sectoriality

Fractional powers of the scalar discrete Laplacian are standard [15, 16], and in the Clifford setting Faustino [17] constructs space- and time-fractional semidiscrete Dirac operators of Lévy–Leblond type under the sectorial restriction $|\vartheta| \leq \alpha\pi/2$. More generally T^s for Clifford- or quaternion-valued T is defined through the S -functional calculus and its H^∞ refinement [18, 19, 20, 21], sectoriality being the standing hypothesis there as it has been in the scalar case since McIntosh [22]; complex powers of the Dirac operator through the factorization $D = (-\Delta)^{1/2}H$ appear in the work of Brackx, De Bie and De Schepper [23, 24], and a Riesz-potential version is due to Bernstein [25]. In all of these the symbol is either the nonnegative scalar symbol of the discrete Laplacian, for which no branch question arises, or is assumed to remain in a sector, which is precisely a device for staying away from the region where it degenerates. Our symbol is Clifford-valued, vanishes on a positive-dimensional set inside the cell, and has nonzero winding around it: the obstruction of Section 5 is invisible in that theory.

2.3 Doubling, Bivectors, Zero Divisors

The vanishing of a first-order lattice symbol at points of the Brillouin zone other than the origin is the fermion doubling phenomenon, and the no-go theorem of Nielsen and Ninomiya [1, 2], with the proofs of Friedan [26] and Rabin [27] and the Ginsparg–Wilson response [28], is its standard form; Proposition 4.3 exhibits exactly that pattern for the central difference, and Section 10 explains why our results are of a different kind. On the algebraic side, the decomposition of a bivector into commuting simple bivectors [4, 3, 5] and the zero-divisor variety of a Clifford algebra [29, 30, 31, 32] are classical; none of it is used here, since $\nabla \wedge \mathbf{F}$ is a wedge of two vectors (Lemma 3.3). What does not appear to be present in that literature is the use of the zero-divisor variety as an analytic obstruction, as in Theorem 5.2.

3. The Symbol Algebra

We use the Fourier transform $\hat{u}(\xi) = \int_{\mathbb{R}^m} u(x) e^{-i\langle x, \xi \rangle} dx$ acting componentwise on $\mathcal{C}\ell_{0,m}$ -valued functions, so that ∂_j corresponds to multiplication by $i\xi_j$. The symbol of ∇ is the Clifford vector $\underline{\xi} := i \sum_j e_j \xi_j$, and by (1),

$$\underline{\xi}^2 = (\sum_j e_j (i\xi_j))^2 = -\sum_j (i\xi_j)^2 = |\xi|^2. \quad (3)$$

More generally, for any Clifford vector $x = \sum_j e_j x_j$ with scalar entries, $x^2 = -\sum_j x_j^2$.

Definition 3.1. For $h > 0$ let

$$\Delta_{h,j}u(x) = u(x + he_j) - u(x), \quad \nabla_{h,j}u(x) = u(x) - u(x - he_j),$$

$$\delta_{h,j}u(x) = 1/2 \left(u(x + he_j) - u(x - he_j) \right)$$

be the forward, backward and central differences in the j -th direction. Their symbols are

$$f_j^+(\xi) = e^{ih\xi_j} - 1, \quad f_j^-(\xi) = 1 - e^{-ih\xi_j}, \quad f_j^0(\xi) = i\sin(h\xi_j).$$

We write $t_j = h\xi_j$ and regard $t = (t_1, \dots, t_m)$ as a point of the torus $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$, i.e. of the fundamental cell $|\xi_j| \leq \pi/h$.

Hypothesis 3.2. The operators S and Ω_{jk} , $1 \leq j < k \leq m$, commute pairwise.

Hypothesis 3.2 holds automatically whenever the F_j are Fourier multipliers, since then ∂_j and F_k all commute and S, Ω_{jk} are polynomials in commuting operators; in particular it holds for all three difference operators of Definition 3.1. It is not a technical convenience: without it S fails to be central and already the binomial expansion of $(-S + B)^n$ acquires commutator remainders, so that none of the closed formulas below survives.

Lemma 3.3. Let $a = \sum_j e_j a_j$ and $b = \sum_j e_j b_j$ be Clifford vectors whose scalar entries commute pairwise, and set

$$B = a \wedge b = \sum_{j < k} \omega_{jk} e_j e_k, \quad \omega_{jk} = a_j b_k - a_k b_j.$$

Then $B \wedge B = 0$ and

$$B^2 = -Q, \quad Q := \sum_{j < k} \omega_{jk}^2, \tag{4}$$

a scalar, in every dimension m .

Proof. For bivectors A, B the geometric product splits as $AB = \langle AB \rangle_0 + \langle AB \rangle_2 + \langle AB \rangle_4$ with $\langle AB \rangle_2 = 1/2(AB - BA)$, so $\langle B^2 \rangle_2 = 0$ and only grades zero and four survive. In $\langle B^2 \rangle_0 = \sum_{j < k} \sum_{l < p} \omega_{jk} \omega_{lp} \langle e_j e_k e_l e_p \rangle_0$ a term is scalar exactly when $\{j, k\} = \{l, p\}$, and then $e_j e_k e_j e_k = -e_j e_j e_k e_k = -1$ by (1); hence $\langle B^2 \rangle_0 = -Q$, the entries commuting by hypothesis. For grade four, associativity of the exterior product and $a \wedge a = 0$ give $\langle B^2 \rangle_4 = B \wedge B = (a \wedge b) \wedge (a \wedge b) = -a \wedge a \wedge b \wedge b = 0$, the sign coming from transposing the two middle grade-one factors. Therefore $B^2 = -Q$. $\square$

In particular $\mathcal{A} := \text{alg}\langle \nabla \mathbf{F} \rangle = \text{span}\{1, B\}$ has dimension two over the scalars for every m , whereas for a general bivector the corresponding algebra has dimension $2^{\lfloor m/2 \rfloor}$ and its analysis needs the invariant decomposition of Roelfs and De Keninck [3]. This is why every formula below is dimension-free.

Lemma 3.4 (Integer powers). Assume Hypothesis 3.2 and write $D = \nabla \mathbf{F} = -S + B$. Then for every $n \in \mathbb{N}$,

$$D^n = \alpha_n(S, Q) + \mathfrak{b}_n(S, Q) B, \tag{5}$$

where

$$\begin{aligned} a_n(S, Q) &= \sum_{j \geq 0} (-1)^{n+j} \binom{n}{2j} Q^j S^{n-2j}, \\ b_n(S, Q) &= \sum_{j \geq 0} (-1)^{n+j+1} \binom{n}{2j+1} Q^j S^{n-2j-1}. \end{aligned}$$

Proof. By Hypothesis 3.2 the scalar S commutes with B , so the binomial theorem gives the expansion $D^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} S^{n-k} B^k$. By (4) and induction, $B^{2j} = (-Q)^j$ and $B^{2j+1} = (-Q)^j B$; substituting $k = 2j$ in the even terms and $k = 2j + 1$ in the odd ones and collecting the two families yields (5). $\square$

Remark 3.5. The coefficients a_n, b_n are polynomials in S and Q , so (5) is valid over any commutative ring, in particular for the complex-valued symbols of Definition 3.1; writing $z_{\pm} = -S \pm i\sqrt{Q}$ one has $a_n = 1/2 (z_+^n + z_-^n)$ and $b_n \sqrt{Q} = 1/2i (z_+^n - z_-^n)$, so the formula does not depend on the branch. The familiar real form $\rho^n \cos n\vartheta + \rho^n \sin n\vartheta \hat{B}$ is not available here, because S and Q are complex and Re, Im are not algebraic operations.

The next identity is the Lagrange–Cauchy–Binet identity

$$\left(\sum_j a_j^2 \right) \left(\sum_j b_j^2 \right) - \left(\sum_j a_j b_j \right)^2 = \sum_{j < k} (a_j b_k - a_k b_j)^2$$

with $a_j = \partial_j$ and $b_j = F_j$, and as such is classical. We give the Clifford derivation because what is needed below is not the scalar identity but the factorization (6) that produces it, together with the identification of $\mathcal{N}$ as a multiplicative norm on $\mathcal{A}$ (Lemma 5.1).

Proposition 3.6. Assume Hypothesis 3.2. Then in every dimension m ,

$$(\nabla \mathbf{F})(\mathbf{F} \nabla) = \nabla^2 \mathbf{F}^2 = \Delta \circ \sum_{j=1}^m F_j^2, \quad (6)$$

equivalently

$$\mathcal{N} := S^2 + Q = \left(\sum_j \partial_j^2 \right) \circ \left(\sum_j F_j^2 \right). \quad (7)$$

On the symbol side, with f_j the symbol of F_j ,

$$\mathcal{N}(\xi) = -|\xi|^2 \sum_{j=1}^m f_j(\xi)^2. \quad (8)$$

Proof. We first record that

$$\mathbf{F} \nabla = \sum_{j,k} e_j e_k F_j \partial_k = \sum_j e_j^2 F_j \partial_j + \sum_{j < k} e_j e_k (F_j \partial_k - F_k \partial_j) = -S - B, \quad (9)$$

using $e_j^2 = -1$, $e_k e_j = -e_j e_k$, and Hypothesis 3.2 to write $F_j \partial_k = \partial_k F_j$, so that $F_j \partial_k - F_k \partial_j = -\Omega_{jk}$.

Now $\mathbf{F}$ is a Clifford vector with commuting scalar entries, so by the remark after (3), $\mathbf{F}^2 = -\sum_j F_j^2$ is a scalar operator; by Hypothesis 3.2 it commutes with ∇ . Associativity therefore gives

$$(\nabla \mathbf{F})(\mathbf{F} \nabla) = \nabla(\mathbf{F} \mathbf{F}) \nabla = \nabla \mathbf{F}^2 \nabla = \mathbf{F}^2 \nabla^2.$$

Since $\nabla^2 = -\Delta$ and $\mathbf{F}^2 = -\sum_j F_j^2$, this is $\Delta \circ \sum_j F_j^2$, which is (6). On the other hand, by (2) and (9),

$$(\nabla \mathbf{F})(\mathbf{F} \nabla) = (-S + B)(-S - B) = S^2 + SB - BS - B^2 = S^2 - B^2 = S^2 + Q$$

by (4) and the centrality of S . Comparing the two evaluations gives (7). Passing to symbols and using (3) together with $\mathbf{F}(\xi)^2 = -\sum_j f_j(\xi)^2$ yields (8). $\square$

Proposition 3.7. Fix ξ and write $D = D(\xi)$, $S = S(\xi)$, $Q = Q(\xi)$, $\mathcal{N} = \mathcal{N}(\xi)$. Then:

(i) D satisfies $D^2 + 2SD + \mathcal{N} = 0$, whose roots are

$$z_{\pm} = -S \pm i\sqrt{Q}. \quad (10)$$

(ii) D is invertible in $\mathcal{C}\ell_{0,m} \otimes \mathbb{C}$ if and only if $\mathcal{N} \neq 0$, and then

$$D^{-1} = \frac{-S-B}{\mathcal{N}}. \quad (11)$$

(iii) If $Q \neq 0$ then $\mathcal{A} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, the two characters $\chi_{\pm}$ being determined by $\chi_{\pm}(B) = \pm i\sqrt{Q}$, and $\chi_{\pm}(D) = z_{\pm}$. For every $s \in \mathbb{C}$ and any choice of branches at the point ξ ,

$$D^s = \frac{z_+^s + z_-^s}{2} + \frac{z_+^s - z_-^s}{2i\sqrt{Q}} B, \quad (12)$$

independently of the sign of $\sqrt{Q}$.

(iv) If $Q = 0$ and $S \neq 0$ then $B^2 = 0$ and $D^s = (-S)^s(1 - sB/S)$.

Proof. (i) By (2) and (4), $(D + S)^2 = B^2 = -Q$, i.e. $D^2 + 2SD + S^2 + Q = 0$; the roots of $\lambda^2 + 2S\lambda + \mathcal{N}$ are $-S \pm \sqrt{S^2 - \mathcal{N}} = -S \pm \sqrt{-Q} = -S \pm i\sqrt{Q}$.

(ii) By (9) and the proof of Proposition 3.6, $(-S + B)(-S - B) = \mathcal{N}$, so if $\mathcal{N} \neq 0$ then (11) is a two-sided inverse, the algebra $\mathcal{A}$ being commutative. Conversely, suppose D is invertible and $\mathcal{N} = 0$. By (i), $D(D + 2S) = D^2 + 2SD = -\mathcal{N} = 0$, and multiplying on the left by D^{-1} gives $D + 2S = S + B = 0$. A bivector equal to a scalar vanishes, so $B = 0$ and $S = 0$, whence $D = 0$, which is not invertible. Hence $\mathcal{N} \neq 0$.

(iii) Since $B^2 = -Q$ with $Q \neq 0$, the assignment $B \mapsto X$ identifies $\mathcal{A} \otimes \mathbb{C}$ with $\mathbb{C}[X]/(X^2 + Q)$, and $X^2 + Q = (X - i\sqrt{Q})(X + i\sqrt{Q})$ has distinct roots, so by the Chinese remainder theorem $\mathcal{A} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ with the stated characters. Under this isomorphism D corresponds to (z_+, z_-) , and the holomorphic functional calculus in $\mathbb{C} \oplus \mathbb{C}$ acts componentwise, giving $D^s \cong (z_+^s, z_-^s)$. Reassembling in the basis $\{1, B\}$, an element $a + bB$ has $\chi_{\pm}(a + bB) = a \pm ib\sqrt{Q}$, so $a = 1/2(z_+^s + z_-^s)$ and $b = 1/2i\sqrt{Q}(z_+^s - z_-^s)$, which is (12). Replacing $\sqrt{Q}$ by $-\sqrt{Q}$ interchanges z_+ and z_- and simultaneously changes the sign of the denominator, leaving (12) unchanged.

(iv) If $Q = 0$ then $B^2 = 0$, so $D = -S(1 - B/S)$ with $(B/S)^2 = 0$, and the binomial series for $(1 - X)^s$ terminates after two terms. $\square$

Remark 3.8. Proposition 3.7 is a pointwise statement, and it is important not to over-read it. At a point where $\mathcal{N} = 0$ but $D \neq 0$, the element D is a nonzero zero divisor and has no inverse, so D^s does not exist for $\operatorname{Re} s < 0$. It does not follow that D has no roots. In $\mathbb{C} \oplus \mathbb{C}$ the element $(0, z)$ with $z \neq 0$ satisfies $(0, z^{1/k})^k = (0, z)$, and the family $s \mapsto (0, z^s)$ satisfies the semigroup law; what fails is only that its limit as $s \rightarrow 0$ is the idempotent $(0, 1)$ rather than the identity. Accordingly the pointwise degeneracy on the

null set $\mathcal{Z}_h$ of Section 4 is not by itself an obstruction to fractional powers. The genuine obstruction is global and is the subject of Section 5.

4. The Null Set for Difference Operators

Theorem 4.1 (One-sided differences). Let $F_j = \Delta_{h,j}$; the backward case follows by $t \mapsto -t$. Put $w(t) = \sin^2(t/2) e^{it}$, $t \in \mathbb{T}$.

Then

$$\mathcal{N}(\xi) = 4|\xi|^2 \sum_{j=1}^m w(t_j), \quad \text{so} \quad \mathcal{N}(\xi) = 0 \Leftrightarrow \xi = 0 \text{ or } \sum_{j=1}^m w(t_j) = 0. \quad (13)$$

The set $\mathcal{Z}_m = \{t \in \mathbb{T}^m : \sum_j w(t_j) = 0\}$ is the common zero set of the two real equations

$$\sum_{j=1}^m (1 - \cos t_j) \cos t_j = 0, \quad \sum_{j=1}^m (1 - \cos t_j) \sin t_j = 0, \quad (14)$$

and has dimension $m - 2$ at its regular points. Moreover

$$\mathcal{Z}_1 = \{0\}, \quad \mathcal{Z}_2 = \{0\} \cup \{\pm(\pi/2, -\pi/2)\},$$

and for $m \geq 3$ the set $\mathcal{Z}_m$ is positive-dimensional.

Proof. Write $f_j = e^{it_j} - 1$. From $e^{it} - 1 = 2i \sin(t/2) e^{it/2}$ we get $f_j^2 = -4 \sin^2(t_j/2) e^{it_j} = -4w(t_j)$, so $\sum_j f_j^2 = -4 \sum_j w(t_j)$ and (8) gives $\mathcal{N}(\xi) = 4|\xi|^2 \sum_j w(t_j)$, which is (13). Writing $w(t) = 1/2 (1 - \cos t) e^{it}$ and separating real and imaginary parts turns $\sum_j w(t_j) = 0$ into (14); these are two real equations on the m -torus, so at any point where their differential has rank two the zero set is a manifold of dimension $m - 2$, and Lemma 4.2 exhibits such a point.

For $m = 1$: the second equation gives $\cos t_1 = 1$ or $\sin t_1 = 0$, and the first gives $\cos t_1 \in \{0, 1\}$; the case $\cos t_1 = 0$ makes $(1 - \cos t_1) \sin t_1 = \pm 1 \neq 0$, contradicting the second. Hence $t_1 = 0$. For $m = 2$: the condition is $w(t_1) = -w(t_2)$; comparing moduli gives $\sin^2(t_1/2) = \sin^2(t_2/2)$, so $t_2 = \pm t_1$ in $\mathbb{T}$, and if both do not vanish then comparing arguments gives $t_1 \equiv t_2 + \pi \pmod{2\pi}$, which rules out $t_2 = t_1$ and leaves $t_2 = -t_1$ with $2t_1 \equiv \pi$, i.e. $t_1 = \pm\pi/2$; both points do satisfy $w(t_1) + w(t_2) = 0$ because $w(\pm\pi/2) = \pm i/2$. For $m \geq 3$ the point $p = (\pi/2, -\pi/2, 0, \dots, 0)$ lies in $\mathcal{Z}_m$ by the case $m = 2$ and $w(0) = 0$, and by Lemma 4.2 the differential has rank two there, so near p the set $\mathcal{Z}_m$ is a manifold of dimension $m - 2 \geq 1$. $\square$

In frequency variables we write $\mathcal{Z}_h = \{\xi \neq 0 : h\xi \in \mathcal{Z}_m\}$, the torus being identified with the fundamental cell $|\xi_j| \leq \pi/h$; by (13) this is exactly the set of nonzero ξ in the cell at which $\mathcal{N}(\xi) = 0$.

Lemma 4.2. Let $G(t) = \sum_j w(t_j)$ and $p = (\pi/2, -\pi/2, 0, \dots, 0)$. Then

$$w'(t) = e^{it} (1/2 \sin t + i/2 (1 - \cos t)), \quad w'(\pi/2) = i - 1/2, \quad w'(-\pi/2) = 1 + i/2,$$

and the Jacobian matrix of G , viewed as a map into $\mathbb{C} \cong \mathbb{R}^2$, restricted to the variables (t_1, t_2) at p is

$$J = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \det J = -1/2 < 0.$$

In particular $dG(p)$ has real rank two, and reverses orientation on the (t_1, t_2) -plane.

Proof. From $w(t) = 1/2 (1 - \cos t) e^{it}$,

$$w'(t) = 1/2 \sin t e^{it} + i/2 (1 - \cos t) e^{it} = e^{it} (1/2 \sin t + i/2 (1 - \cos t)).$$

At $t = \pi/2$ we have $e^{i\pi/2} = i$, $\sin t = 1$, $\cos t = 0$, so the bracket is $1/2 + i/2$ and $w'(\pi/2) = i \cdot 1 + i/2 = i - 1/2$. At $t = -\pi/2$ we have $e^{-i\pi/2} = -i$, $\sin t = -1$, $\cos t = 0$, so the bracket is $-1/2 + i/2$ and $w'(-\pi/2) = -i \cdot -1 + i/2 = 1 + i/2$. Since $\partial G / \partial t_j = w'(t_j)$, writing the two complex numbers $w'(\pi/2) = -1/2 + i/2$ and $w'(-\pi/2) = 1/2 + i/2$ as columns $(\text{Re}, \text{Im})^T$ produces J , whose determinant is $1/4((-1)(1) - (1)(1)) = -1/2$. $\square$

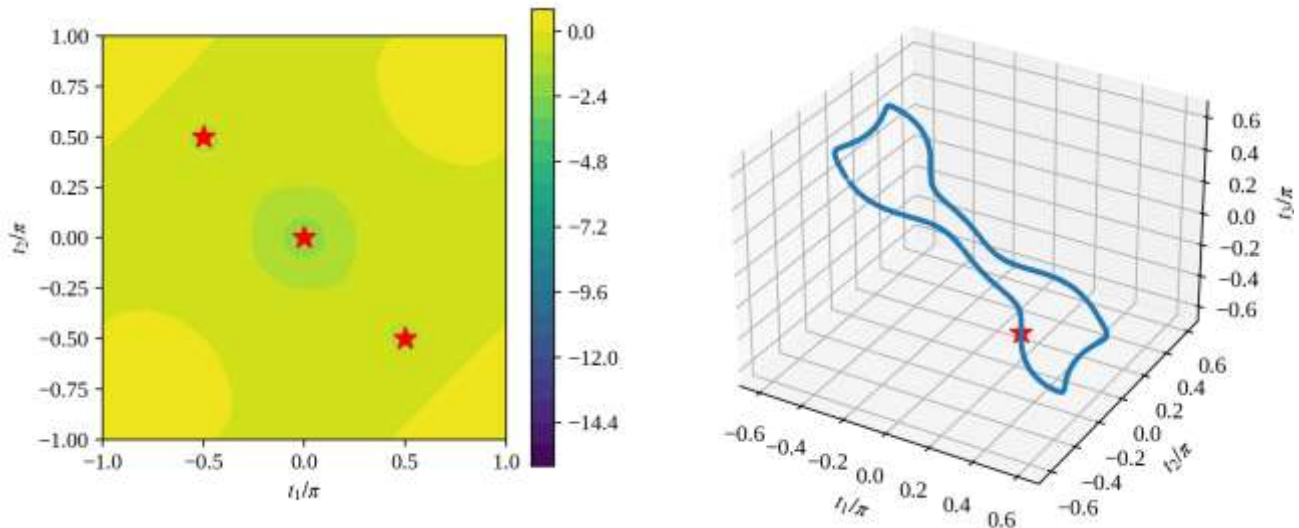


Figure 1: (a) $\log_{10}|\sum_j w(t_j)|$ on the fundamental cell for $m = 2$; the three zeros (stars) are the origin and $\pm(\pi/2, -\pi/2)$, as in Theorem 4.1. (b) the null set $Z_3 \subset \mathbb{T}^3$, shown by 2712 points obtained by minimum-norm Gauss–Newton projection of random points onto (14); it is a curve, of dimension $m - 2 = 1$; numerically the largest coordinate modulus on Z_3 is $t_* = \arccos(-1/3)$, attained at $(t_*, -\arccos 1/3, -\arccos 1/3)$ and its images under permutation and sign reversal (Appendix A); the star marks the point $(\pi/2, -\pi/2, 0)$ of Lemma 4.2.

Proposition 4.3 (Central differences). Let $F_j = \delta_{h,j}$. Then $\sum_j f_j^0(\xi)^2 = -\sum_j \sin^2 t_j$, so that $\mathcal{N}(\xi) = |\xi|^2 \sum_j \sin^2 t_j$ and $\mathcal{N}(\xi) = 0$ if and only if $\xi = 0$ or $t_j \in \{0, \pi\}$ for every j . The null set is therefore the finite set of 2^m points, of dimension zero, and meets the open cell $\max_j |t_j| < \pi$ only at the origin.

Proof. Here $f_j^0 = i \sin t_j$, so $(f_j^0)^2 = -\sin^2 t_j$, and (8) gives

$$\mathcal{N} = |\xi|^2 \sum_j \sin^2 t_j.$$

This is a sum of nonnegative terms; it vanishes if and only if $\sin t_j = 0$ for every j , that is, $t_j \in \{0, \pi\}$ in $\mathbb{T}$. $\square$

Remark 4.4. For $m \geq 3$ the one-sided schemes therefore possess a null set of dimension $m - 2 \geq 1$, whereas the symmetric scheme has a null set of dimension zero. The mechanism is the bivector part: at every point of Z_h the scalar part S does not vanish, and the cancellation in $\mathcal{N} = S^2 + Q$ is produced by $Q = -S^2$. Indeed, by (24) below, $\text{Re}(-S) = h^{-1} \sum_j t_j \sin t_j$ with every summand nonnegative on the cell,

so $\operatorname{Re}(-S) = 0$ only if every $t_j \in \{0, \pi\}$; at such a point $\sum_j w(t_j) = -\#\{j: t_j = \pi\}$, which is nonzero unless $t = 0$. The points of Example 4.5 are instances. Note also that for the central difference $\mathcal{N} \geq 0$ is real, so no branch question can arise; the phenomenon of Section 5 is specific to the one-sided schemes.

Example 4.5 (An explicit zero divisor). Let $m = 2$, $h = 1$, $\xi_0 = (\pi/2, -\pi/2)$, so that $f_1 = -1 + i$ and $f_2 = -1 - i$. Then $S = -\pi$ and $\Omega_{12} = -i\pi$, whence $Q = \Omega_{12}^2 = -\pi^2$ and $\mathcal{N} = S^2 + Q = 0$, confirming Theorem 4.1. Here $D = \pi + B$ and $\bar{D} = \pi - B$ with $B = -i\pi e_1 e_2$, and $D\bar{D} = \mathcal{N} = 0$ with $D \neq 0$ and $\bar{D} \neq 0$: the element $\bar{D}$ is an explicit nonzero annihilator of D . Since $\sqrt{Q} = \pm i\pi$ the two spectral values are $\{0, 2\pi\}$, so exactly one character annihilates D and D lies in a proper ideal without being zero. The same occurs at interior points in higher dimensions: for $m = 3$, $h = 1$ and $\xi = (-\pi/2, 0, \pi/2)$ one has $\mathcal{N} = 0$ while $|S| = \pi \neq 0$, so $Q = -S^2 \neq 0$ and $D \neq 0$. Figure 1 shows the null set for $m = 2$ and $m = 3$.

5. The Monodromy Obstruction

Let $\bar{\cdot}: \mathcal{A} \rightarrow \mathcal{A}$ be the algebra automorphism fixing the scalars and sending $B \mapsto -B$ (the restriction to $\mathcal{A}$ of the main involution composed with reversion), and define the norm map

$$\mathfrak{N}(X) = X\bar{X}, \quad X \in \mathcal{A}. \quad (15)$$

Lemma 5.1. For $X = a + bB \in \mathcal{A}$ one has $\mathfrak{N}(X) = a^2 + b^2Q$, a scalar. The map $\mathfrak{N}$ is multiplicative, $\mathfrak{N}(XY) = \mathfrak{N}(X)\mathfrak{N}(Y)$, and when $Q \neq 0$ it satisfies $\mathfrak{N}(X) = \chi_+(X)\chi_-(X)$. In particular $\mathfrak{N}(D) = \mathcal{N}$.

Proof. $\mathfrak{N}(a + bB) = (a + bB)(a - bB) = a^2 - b^2B^2 = a^2 + b^2Q$ by (4). Since $\mathcal{A}$ is commutative and $\bar{\cdot}$ is an algebra automorphism, $\mathfrak{N}(XY) = XY\bar{X}\bar{Y} = XY\bar{X}\bar{Y} = (X\bar{X})(Y\bar{Y})$. The involution sends $B \mapsto -B$, hence interchanges the two characters of Proposition 3.7(iii): $\chi_{\pm}(\bar{X}) = \chi_{\mp}(X)$. Therefore $\mathfrak{N}(X) = \chi_+(X\bar{X}) = \chi_+(X)\chi_+(\bar{X}) = \chi_+(X)\chi_-(X)$. Finally $\mathfrak{N}(D) = (-S + B)(-S - B) = S^2 + Q = \mathcal{N}$. $\square$

Theorem 5.2 (Monodromy). Let $U \subseteq \mathbb{R}^m$ be open and connected with $\mathcal{N} \neq 0$ on U , and let $s \in \mathbb{C}$. Suppose there exists a continuous map

$$U \rightarrow \mathcal{A} \otimes \mathbb{C}, \quad \xi \mapsto D(\xi)^{(s)},$$

which at every point equals one of the branches (12) of $D(\xi)^s$ (at a point where $Q(\xi) = 0$, one of the values $(-S)^s(1 - sB/S)$ of Proposition 3.7(iv)). Then for every loop γ in U ,

$$s \cdot \operatorname{wind}(\mathcal{N} \circ \gamma) \in \mathbb{Z}, \quad (16)$$

where wind denotes the winding number about $0 \in \mathbb{C}$.

Proof. Set $g := \mathfrak{N}(D^{(s)})$. By Lemma 5.1, g is a continuous scalar function on U , and at each point with $Q \neq 0$ we have $g = \chi_+(D^{(s)})\chi_-(D^{(s)}) = z_+^s z_-^s$ for the branches chosen there, while at a point with $Q = 0$ Lemma 5.1 gives $g = ((-S)^s)^2$, where $S^2 = \mathcal{N} \neq 0$. Since $z_+ z_- = \mathcal{N} \neq 0$ on U , both $z_{\pm} \neq 0$ and g is nowhere zero; in either case g is, at each point, some value of $\mathcal{N}^s$ (a sum of logarithms of z_+ and z_- , or twice a logarithm of $-S$, being a logarithm of $\mathcal{N}$), that is

$$g(\xi) = \exp(s \ell(\xi)) \quad \text{for some } \ell(\xi) \in \log \mathcal{N}(\xi). \quad (17)$$

Let $\gamma: [0,1] \rightarrow U$ be a loop, $\gamma(0) = \gamma(1)$, and put $n = \text{wind}(\mathcal{N} \circ \gamma)$. Because $\mathcal{N} \circ \gamma$ is continuous and nowhere zero, it admits a continuous argument $\vartheta: [0,1] \rightarrow \mathbb{R}$ with

$$\mathcal{N}(\gamma(\tau)) = |\mathcal{N}(\gamma(\tau))| e^{i\vartheta(\tau)}, \quad \vartheta(1) - \vartheta(0) = 2\pi n.$$

Define $L(\tau) = \ln|\mathcal{N}(\gamma(\tau))| + i\vartheta(\tau)$, a continuous logarithm of $\mathcal{N}$ along γ , and

$$R(\tau) = \frac{g(\gamma(\tau))}{\exp(sL(\tau))}.$$

Then R is continuous on $[0,1]$. By (17), at each τ the exponents $\ell(\gamma(\tau))$ and $L(\tau)$ are two logarithms of the same nonzero number, so they differ by $2\pi i k(\tau)$ with $k(\tau) \in \mathbb{Z}$; hence

$$R(\tau) \in \{e^{2\pi i s k} : k \in \mathbb{Z}\} =: \Sigma.$$

The set Σ is countable. A continuous image of the connected set $[0,1]$ is connected, and a connected subset of $\mathbb{C}$ containing more than one point is uncountable; therefore $R([0,1])$ is a single point and R is constant. (This step is needed because for irrational real s the set Σ is dense in the unit circle, so one cannot argue by discreteness.)

Now $\gamma(1) = \gamma(0)$ gives $g(\gamma(1)) = g(\gamma(0))$, while $L(1) = L(0) + 2\pi i n$. Hence

$$R(1) = \frac{g(\gamma(0))}{\exp(sL(0) + 2\pi i s n)} = R(0) e^{-2\pi i s n}.$$

Since R is constant and nowhere zero, $e^{-2\pi i s n} = 1$, i.e. $s n \in \mathbb{Z}$, which is (16). □

Theorem 5.3 (Winding number). Let $F_j = \Delta_{h,j}$, $m \geq 2$, and let

$$\xi_0 = h^{-1}(\pi/2, -\pi/2, 0, \dots, 0) \in \mathcal{Z}_h.$$

Let γ_ε denote the positively oriented circle of radius ε about ξ_0 inside the two-plane through ξ_0 spanned by the ξ_1 - and ξ_2 -directions. Then for all sufficiently small $\varepsilon > 0$,

$$\text{wind}(\mathcal{N} \circ \gamma_\varepsilon) = -1.$$

Proof. By (13), $\mathcal{N}(\xi) = 4|\xi|^2 G(t)$ with $G(t) = \sum_j w(t_j)$ and $t = h\xi$, and $t_0 := h\xi_0 = (\pi/2, -\pi/2, 0, \dots, 0)$ satisfies $G(t_0) = 0$. Differentiating the product at ξ_0 , the term in which $|\xi|^2$ is differentiated carries the factor $G(t_0) = 0$ and therefore drops out, so as real linear maps $\mathbb{R}^2 \rightarrow \mathbb{C} \cong \mathbb{R}^2$ on the (ξ_1, ξ_2) -plane,

$$d\mathcal{N}(\xi_0)|_{(\xi_1, \xi_2)} = 4|\xi_0|^2 h dG(t_0)|_{(t_1, t_2)} = 4h|\xi_0|^2 J,$$

with J as in Lemma 4.2, the factor h arising from $t = h\xi$. The determinant of this 2×2 real matrix is $(4h|\xi_0|^2)^2 \det J < 0$ by Lemma 4.2, so it is invertible and orientation-reversing.

By the inverse function theorem, $\mathcal{N}$ restricted to that two-plane is therefore a diffeomorphism from some neighbourhood V of ξ_0 onto a neighbourhood W of $0 \in \mathbb{C}$, and it reverses orientation. Choose $\varepsilon_0 > 0$ with $\overline{B(\xi_0, \varepsilon_0)} \subset V$. For $0 < \varepsilon \leq \varepsilon_0$ the curve $\mathcal{N} \circ \gamma_\varepsilon$ is a simple closed curve in $W \setminus \{0\}$ bounding the image of the disc, which contains 0 ; since the map is an orientation-reversing diffeomorphism, the induced degree at 0 is -1 , i.e. $\text{wind}(\mathcal{N} \circ \gamma_\varepsilon) = -1$. □

Corollary 5.4. Let U be a domain contained in the fundamental cell on which $\mathcal{N} \neq 0$, and suppose U contains a circle γ_ε as in Theorem 5.3. Then a continuous single-valued family $(\nabla \mathbf{F})^s$ on U exists only if

$s \in \mathbb{Z}$. In particular, for non-integer s there is no fractional power of $\nabla \mathbf{F}$ as a Fourier multiplier on the fundamental cell with the null set of $\mathcal{N}$ removed, and $(\nabla \mathbf{F})^s$ belongs to no Hörmander class $S_{1,0}^\mu$ there, membership in such a class presupposing a single-valued smooth symbol.

Proof. Combine Theorems 5.2 and 5.3: (16) with $n = -1$ gives $-s \in \mathbb{Z}$. □

Remark 5.5. The obstruction is of a different nature from the pointwise degeneracy of Section 4. The pointwise powers (12) exist at every $\xi \notin \mathcal{Z}_h$; what fails is their assembly into a single-valued symbol. Note also that Theorem 5.2 is a statement about U , not about $\mathcal{Z}_h$: the obstruction is felt on every domain that links the null set, however far from it.

Remark 5.6. Take $s = 1/2$ and the circle γ_ε of Theorem 5.3 with $m = 2$, $h = 1$, $\varepsilon = 1/4$; Corollary 5.4 predicts the monodromy factor $e^{-\pi i} = -1$. Continuing the two spectral values by continuity, z_+ returns with argument unchanged while z_- returns with argument decreased by 2π , so $z_-^{1/2} \mapsto -z_-^{1/2}$ after one circuit: the branch of $(\nabla \mathbf{F})^{1/2}$ continued once around γ_ε is the negative of the branch one started from. Numerically the ratio of final to initial value is $-1.000000 - 0.000000i$; Figure 2 displays both curves.

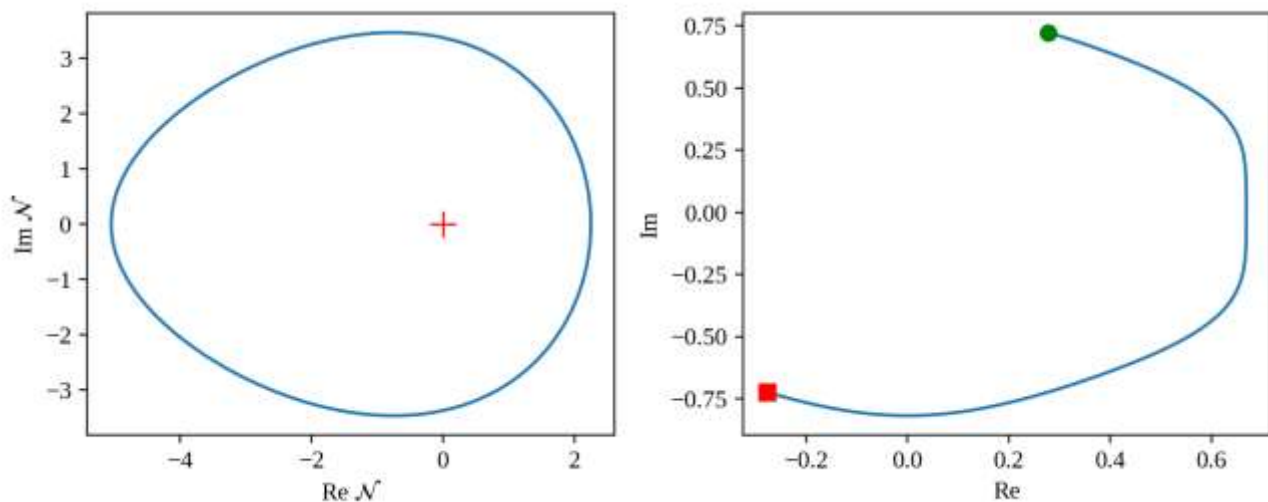


Figure 2: (a) the image of $\mathcal{N} \circ \gamma_\varepsilon$ in $\mathbb{C}$ for $m = 2$, $h = 1$, $\varepsilon = 1/4$, encircling the origin once in the negative sense, as in Theorem 5.3. (b) the corresponding continuation of $z_-^{1/2}$; the circuit starts at the disc and ends at the square, at the antipodal point, as in Remark 5.6.

6. The Obstruction in Operator Form

Corollary 5.4 is a statement about symbols. We now record what it says about operators, and, just as importantly, what it does not say.

Theorem 6.1. Let U be a domain contained in the fundamental cell on which $\mathcal{N} \neq 0$ and which contains a circle γ_ε as in Theorem 5.3, and let $s \in \mathbb{C} \setminus \mathbb{Z}$. Then there is no finite $\mathcal{C}\ell_{0,m} \otimes \mathbb{C}$ -valued Borel measure μ on $\mathbb{R}^m$ whose Fourier–Stieltjes transform $\hat{\mu}(\xi) = \int_{\mathbb{R}^m} e^{-i\langle x, \xi \rangle} d\mu(x)$ is, at every $\xi \in U$, one of the values

(12) of $D(\xi)^s$. In particular, on no such U is $(\nabla \mathbf{F})^s$ realized by convolution with a kernel $k \in L^1(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m} \otimes \mathbb{C}$, nor by an infinite-order difference operator

$$Tu(x) = \sum_{n \in \mathbb{Z}^m} a_n u(x - hn), \quad \sum_{n \in \mathbb{Z}^m} \|a_n\| < \infty. \quad (18)$$

Proof. Since μ is finite, dominated convergence makes $\hat{\mu}$ continuous on $\mathbb{R}^m$, and by hypothesis it takes at every point of U one of the values (12), which lie in $\mathcal{A} \otimes \mathbb{C}$. Theorem 5.2 applied to $\xi \mapsto \hat{\mu}(\xi)$ and the loop γ_ε therefore gives $s \cdot \text{wind}(\mathcal{N} \circ \gamma_\varepsilon) \in \mathbb{Z}$, which by Theorem 5.3 reads $-s \in \mathbb{Z}$, contrary to $s \notin \mathbb{Z}$. The two special cases are the finite measures $d\mu = k dx$ and $\mu = \sum_n a_n \delta_{hn}$; for the latter, $\hat{\mu}(\xi) = \sum_n a_n e^{-ih\langle n, \xi \rangle}$ is the symbol of (18). $\square$

Corollary 6.2. Let U and s be as in Theorem 6.1. Then no bounded Fourier multiplier on $L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m} \otimes \mathbb{C}$ whose symbol is continuous on U realizes $(\nabla \mathbf{F})^s$ there.

Proof. Such a symbol would be a continuous single-valued branch on U , so Theorem 5.2 gives $-s \in \mathbb{Z}$. $\square$

Corollary 6.2 is the form in which the obstruction is usually wanted, but it is weaker than Theorem 6.1, not stronger: once continuity of the symbol is assumed the conclusion is immediate from Theorem 5.2 and boundedness plays no role at all, whereas Theorem 6.1 places its hypothesis on the kernel and rules out an entire class of operators without any assumption on the symbol. The class (18) is exactly the class of lattice operators with an absolutely convergent stencil, and it contains no realization of $(\nabla \mathbf{F})^s$ near the null set. Remark 6.3 shows that the continuity hypothesis in Corollary 6.2 cannot simply be dropped.

Remark 6.3. The regularity hypothesis cannot be traded for a bound, and it is worth being explicit about this, since the obstruction is sometimes expected to manifest itself as unboundedness. Fix $R, \delta > 0$ and let $E \subseteq \{\xi: |\xi| \leq R, |Q(\xi)| \geq \delta, |\mathcal{N}(\xi)| \geq \delta\}$ be Borel. Taking the argument in $(-\pi, \pi]$ gives Borel branches of the square root and of the logarithm on $\mathbb{C} \setminus \{0\}$, hence Borel functions $\xi \mapsto z_\pm(\xi)$ and $\xi \mapsto z_\pm(\xi)^s$, and (12) then defines a Borel realization of D^s on E . On E the continuous functions S and Q are bounded, say $|S| + |Q|^{1/2} \leq C$, so $|z_\pm| \leq C$; and since $z_+ z_- = \mathcal{N}$ with $|\mathcal{N}| \geq \delta$, also $|z_\pm| \geq \delta/C$. Hence $|z_\pm^s|$ is bounded above and below for every $s \in \mathbb{C}$, and $|Q| \geq \delta$ keeps the coefficient of B in (12) bounded as well: the resulting Fourier multiplier is bounded on L^2 . What Theorem 5.2 forbids is therefore not a bounded realization but a regular one, and with regularity go all the properties that presuppose it – a continuous or smooth symbol, a Hörmander estimate, an absolutely convergent stencil, and a kernel that is a finite measure.

7. Quantitative Estimates and Sectoriality

Corollary 5.4 rules out global fractional powers. We now show that on a box strictly inside the threshold everything is well behaved, with explicit constants. Throughout $t = h\xi$ and $\mathcal{B}_\theta = \{\xi: \max_j h|\xi_j| \leq \theta\}$.

Theorem 7.1 (Sharp threshold). Let $F_j = \Delta_{h,j}$. If $\max_j |t_j| < \pi/2$ and $\xi \neq 0$ then $\mathcal{N}(\xi) \neq 0$. The bound is sharp: $(\pi/2, -\pi/2, 0, \dots, 0) \in \mathcal{Z}_m$ for every $m \geq 2$. Moreover

$$\min\{|t|: t \in \mathcal{Z}_m \setminus \{0\}\} = \frac{\pi}{\sqrt{2}},$$

attained exactly at the points obtained from $(\pi/2, -\pi/2, 0, \dots, 0)$ by permutation and sign reversal.

Proof. By (13) it suffices to consider $G(t) = \sum_j w(t_j)$. Its real part is

$$\operatorname{Re} G(t) = \sum_j \sin^2(t_j/2) \cos t_j. \quad (19)$$

If $|t_j| < \pi/2$ for every j then $\cos t_j > 0$ and $\sin^2(t_j/2) \geq 0$, so every summand is nonnegative, and the j -th summand is strictly positive whenever $t_j \neq 0$. Hence $\operatorname{Re} G(t) > 0$ unless $t = 0$, and in particular $G(t) \neq 0$. Sharpness follows from Theorem 4.1.

For the minimal radius we use both equations of (14), not merely the real one: Example 7.2 below shows that the real equation alone does not suffice. Write the null condition $\sum_j w(t_j) = 0$ of (13) as

$$\sum_j a_j e^{it_j} = 0, \quad a_j := \sin^2(t_j/2) \geq 0, \quad (20)$$

and let $\Sigma = \{j: t_j \neq 0\}$, so that $a_j > 0$ exactly for $j \in \Sigma$, and $\Sigma \neq \emptyset$ when $t \neq 0$. We claim the directions $\{e^{it_j}: j \in \Sigma\}$ lie in no open half-plane: if $\langle e^{it_j}, u \rangle > 0$ for every $j \in \Sigma$ and some unit vector u , then pairing (20) with u gives $\sum_{j \in \Sigma} a_j \langle e^{it_j}, u \rangle = 0$ with every summand strictly positive, which is absurd.

Let $\alpha = \min_{j \in \Sigma} t_j$ and $\beta = \max_{j \in \Sigma} t_j$, the representatives being taken in $(-\pi, \pi]$. If $\beta - \alpha < \pi$, then $u = e^{i(\alpha+\beta)/2}$ satisfies $|t_j - \alpha + \beta/2| \leq \beta - \alpha/2 < \pi/2$ and hence $\langle e^{it_j}, u \rangle = \cos(t_j - \alpha + \beta/2) > 0$ for every $j \in \Sigma$, contradicting the claim. Therefore $\beta - \alpha \geq \pi$. Since $\alpha, \beta \in (-\pi, \pi]$ and $0 \notin \Sigma$, this forces $\alpha \leq \beta - \pi \leq 0$ and $\beta \geq \alpha + \pi \geq 0$, hence $\alpha < 0 < \beta$, so α and β are attained at two distinct indices and $|t|^2 \geq \alpha^2 + \beta^2 \geq 1/2(\beta - \alpha)^2 \geq \pi^2/2$,

the middle inequality being $2(\alpha^2 + \beta^2) - (\beta - \alpha)^2 = (\alpha + \beta)^2 \geq 0$. Thus $|t| \geq \pi/\sqrt{2}$. Equality forces $(\alpha + \beta)^2 = 0$ and $\beta - \alpha = \pi$, i.e. $\alpha = -\pi/2$ and $\beta = \pi/2$, and $t_j = 0$ for every other j ; conversely $w(\pi/2) + w(-\pi/2) = i/2 - i/2 = 0$, so those points do lie in Z_m . $\square$

Example 7.2. Both equations are needed. Fix small $\varepsilon > 0$, put $t_1 = \pi/2 + \varepsilon$ and choose $t_2, \dots, t_m$ small with $\sum_{j \geq 2} \sin^2(t_j/2) \cos t_j = -\sin^2(t_1/2) \cos t_1$, which is possible since the right-hand side is $O(\varepsilon)$ and positive. The real equation of (14) then holds while $|t|^2 = \pi^2/4 + O(\varepsilon)$, far below $\pi^2/2$. What such a configuration violates is the imaginary equation: the first term $\sin^2(t_1/2) \sin t_1$ is bounded below by $1/2 - O(\varepsilon)$, whereas the remaining terms are $O(|t_j|^3)$ and cannot cancel it. Any proof of the minimal radius must therefore use both equations, as the argument above does.

Theorem 7.3 (Two-sided symbol bound). Let $F_j = \Delta_{h,j}$ and $0 < \theta < \pi/2$. Then for every $\xi \in \mathcal{B}_\theta$,

$$\frac{4\cos\theta}{\pi^2} h^2 |\xi|^4 \leq |\mathcal{N}(\xi)| \leq h^2 |\xi|^4. \quad (21)$$

Proof. By (13), $|\mathcal{N}(\xi)| = 4|\xi|^2 |G(t)|$. For the lower bound, $|G| \geq \operatorname{Re} G$ and by (19) and $\cos t_j \geq \cos \theta > 0$,

$$\operatorname{Re} G(t) \geq \cos \theta \sum_j \sin^2(t_j/2).$$

Jordan's inequality $\sin u \geq 2u/\pi$ on $[0, \pi/2]$, applied with $u = |t_j|/2 \leq \pi/2$, gives $\sin(|t_j|/2) \geq |t_j|/\pi$, so $\sum_j \sin^2(t_j/2) \geq |t|^2/\pi^2$. Hence $|G(t)| \geq \cos \theta |t|^2/\pi^2$ and, using $|t|^2 = h^2 |\xi|^2$,

$$|\mathcal{N}| \geq 4|\xi|^2 \cdot \frac{\cos\theta}{\pi^2} h^2 |\xi|^2 = \frac{4\cos\theta}{\pi^2} h^2 |\xi|^4.$$

For the upper bound, $|w(t)| = \sin^2(t/2) \leq t^2/4$, so $|G(t)| \leq 1/4 |t|^2 = 1/4 h^2 |\xi|^2$ and $|\mathcal{N}| \leq h^2 |\xi|^4$. $\square$

Lemma 7.4. For $F_j = \Delta_{h,j}$ and $\max_j |t_j| \leq \pi$,

$$|S(\xi)| \leq h|\xi|^2, \quad \sum_{j < k} |\Omega_{jk}(\xi)| \leq (m-1)h|\xi|^2.$$

Proof. We have $|f_j| = |e^{it_j} - 1| = 2|\sin(t_j/2)| \leq |t_j| = h|\xi_j|$, hence $|S| = |\sum_j i \xi_j f_j| \leq \sum_j |\xi_j| h|\xi_j| = h|\xi|^2$ and $|\Omega_{jk}| \leq |\xi_j||f_k| + |\xi_k||f_j| \leq 2h|\xi_j||\xi_k|$. By Cauchy-Schwarz, $\sum_{j < k} |\xi_j||\xi_k| = 1/2 [(\sum_j |\xi_j|)^2 - |\xi|^2] \leq 1/2 [m|\xi|^2 - |\xi|^2] = m - 1/2 |\xi|^2$, so $\sum_{j < k} |\Omega_{jk}| \leq (m-1)h|\xi|^2$. $\square$

Theorem 7.5 (Operator bounds). Let $F_j = \Delta_{h,j}$ and $0 < \theta < \pi/2$. For $\xi \in \mathcal{B}_\theta \setminus \{0\}$,

$$\|(\nabla F)^{-1}(\xi)\| \leq \frac{\pi^2 m}{4\cos\theta} \cdot \frac{1}{h|\xi|^2}. \quad (22)$$

Consequently, if $u \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ has $\text{supp } \hat{u} \subset \mathcal{B}_\theta \cap \{|\xi| \geq \rho\}$ then

$$\|(\nabla F)^{-1}u\|_{L^2} \leq \frac{\pi^2 m}{4\cos\theta h\rho^2} \|u\|_{L^2}. \quad (23)$$

Proof. Give $\mathcal{C}\ell_{0,m}$ the norm induced by the orthonormal blade basis, for which each blade acts by left multiplication as an isometry. By (11), the triangle inequality and Lemma 7.4,

$$\|D^{-1}\| \leq \frac{|S| + \sum_{j < k} |\Omega_{jk}|}{|\mathcal{N}|} \leq \frac{h|\xi|^2 + (m-1)h|\xi|^2}{|\mathcal{N}|} = \frac{mh|\xi|^2}{|\mathcal{N}|},$$

and Theorem 7.3 bounds $|\mathcal{N}|$ below by $4\cos\theta/\pi^2 h^2 |\xi|^4$, which gives (22). Estimate (23) follows by Plancherel's theorem. $\square$

We now show that the two spectral roots stay in a half-plane, which is what licenses the principal branch.

Lemma 7.6. For $F_j = \Delta_{h,j}$,

$$-S = \frac{1}{h} \sum_j t_j \sin t_j + \frac{i}{h} \sum_j t_j (1 - \cos t_j), \quad \text{hence} \quad \text{Re}(-S) = \frac{1}{h} \sum_j t_j \sin t_j. \quad (24)$$

If $\max_j |t_j| \leq \theta \leq \pi$ then

$$\text{Re}(-S) \geq \frac{\sin\theta}{\theta} \cdot \frac{|t|^2}{h} \geq 0, \quad (25)$$

and if $\theta < \pi$ then $\text{Re}(-S) > 0$ for every $t \neq 0$.

Proof. With $\partial_j \mapsto i\xi_j = it_j/h$ and $f_j = e^{it_j} - 1$,

$$-S = -\frac{i}{h} \sum_j t_j (\cos t_j - 1 + i \sin t_j) = \frac{1}{h} \sum_j t_j \sin t_j + \frac{i}{h} \sum_j t_j (1 - \cos t_j),$$

since $-i(\cos t_j - 1) = i(1 - \cos t_j)$ and $-i \cdot i \sin t_j = \sin t_j$; this is (24). Since $u \mapsto \sin u/u$ is even and decreasing on $[0, \pi]$, for $|t_j| \leq \theta \leq \pi$ we have $t_j \sin t_j = t_j^2 \cdot \frac{\sin t_j}{t_j} \geq \frac{\sin \theta}{\theta} t_j^2 \geq 0$; summing gives (25). If $\theta < \pi$ then $\sin \theta / \theta > 0$, so the middle term is positive for $t \neq 0$. $\square$

Lemma 7.7. Define $\psi(t) = \frac{e^{it}-1}{2it}$ for $t \neq 0$ and $\psi(0) = 1/2$. Then ψ is entire, $|\psi'(t)| \leq 1/4$ for every $t \in \mathbb{R}$ with strict inequality for $t \neq 0$, and

$$\Omega_{jk} = -\frac{2}{h} t_j t_k (\psi(t_k) - \psi(t_j)), \quad \text{whence} \quad |\Omega_{jk}| \leq \frac{1}{2h} |t_j| |t_k| |t_j - t_k|. \quad (26)$$

Proof. Since $f_j = e^{it_j} - 1 = 2it_j\psi(t_j)$ and $\partial_j \mapsto it_j/h$,

$$\Omega_{jk} = \frac{i}{h} (t_j f_k - t_k f_j) = -\frac{2}{h} t_j t_k (\psi(t_k) - \psi(t_j)).$$

Differentiating $2it\psi(t) = e^{it} - 1$, multiplying by t and eliminating ψ gives $2t^2\psi'(t) = -i(ite^{it} - e^{it} + 1)$. Writing $ite^{it} - (e^{it} - 1) = \int_0^t i e^{is} (e^{i(t-s)} - 1) ds$ and using $|e^{i\tau} - 1| \leq |\tau|$ for real τ gives $|ite^{it} - e^{it} + 1| \leq \int_0^{|t|} |t-s| ds = t^2/2$, whence $|\psi'(t)| \leq 1/4$; for $t \neq 0$ the inequality $|e^{i\tau} - 1| < |\tau|$ is strict for $\tau \neq 0$, so the integral bound and hence $|\psi'(t)| < 1/4$ are strict. The estimate in (26) is then the mean value inequality. $\square$

Lemma 7.8. For every $t \in \mathbb{R}^m$, writing $\mu = \max_j t_j^2$,

$$\sum_{j < k} t_j^2 t_k^2 (t_j - t_k)^2 = |t|^2 \sum_j t_j^4 - (\sum_j t_j^3)^2 \leq \mu |t|^4. \quad (27)$$

The constant is sharp, equality being attained at $t = (1, -1, 0, \dots, 0)$.

Proof. Put $v = (t_1^2, \dots, t_m^2)$ and $w = (t_1, \dots, t_m)$, so that $v_j w_k - v_k w_j = t_j t_k (t_j - t_k)$ and the left-hand side is $\sum_{j < k} (v_j w_k - v_k w_j)^2$, which by the Lagrange identity equals $|v|^2 |w|^2 - \langle v, w \rangle^2$; this is the middle expression. Dropping $(\sum_j t_j^3)^2 \geq 0$ and using $\sum_j t_j^4 \leq \mu \sum_j t_j^2 = \mu |t|^2$ gives the bound; at $t = (1, -1, 0, \dots, 0)$ both the left side and $\mu |t|^4$ equal 4. $\square$

Theorem 7.9 (Sectoriality). Let $\theta_0 = 1.4044148 \dots$ denote the unique root in $(0, \pi)$ of

$$\theta^2 = 2 \sin \theta, \quad (28)$$

and let $0 < \theta \leq \theta_0$. Then for every $\xi \in \mathcal{B}_\theta \setminus \{0\}$ both roots $z_\pm$ of (10) satisfy $\operatorname{Re} z_\pm > 0$.

Proof. The map $\theta \mapsto \theta^2/(2 \sin \theta)$ is continuous and strictly increasing on $(0, \pi)$ from 0 to $+\infty$, so (28) has a unique root there. By Lemma 7.7 and then Lemma 7.8 with $\mu \leq \theta^2$,

$$|Q| = \left| \sum_{j < k} \Omega_{jk}^2 \right| \leq \sum_{j < k} |\Omega_{jk}|^2 \leq \frac{1}{4h^2} \sum_{j < k} t_j^2 t_k^2 (t_j - t_k)^2 \leq \frac{\theta^2}{4h^2} |t|^4,$$

so $|Q|^{1/2} \leq \frac{\theta}{2h} |t|^2$, while $\operatorname{Re}(-S) \geq \frac{\sin \theta}{\theta} \cdot \frac{|t|^2}{h}$ by Lemma 7.6. Since $\theta \leq \theta_0$ and $\theta \mapsto \theta^2/(2 \sin \theta)$ is increasing, $\theta^2 \leq 2 \sin \theta$, i.e. $\theta/2 \leq \sin \theta / \theta$, so $|Q|^{1/2} \leq \operatorname{Re}(-S)$. This inequality is strict for $t \neq 0$. If $\theta < \theta_0$ then $\theta/2 < \sin \theta / \theta$ and $|t| > 0$. If $\theta = \theta_0$ and equality held, then equality would hold in (25),

which, $u \mapsto \sin u/u$ being strictly decreasing on $(0, \pi]$, forces $|t_j| \in \{0, \theta\}$ for every j . If the nonzero t_j all have the same sign they are all equal, so every Ω_{jk} vanishes by (26), $Q = 0$, and $|Q|^{1/2} = 0 < \operatorname{Re}(-S)$ by Lemma 7.6. Otherwise some pair has $t_j = \theta$, $t_k = -\theta$; for it $|\psi(t_k) - \psi(t_j)| = |\int_{t_j}^{t_k} \psi'| < 1/4 |t_j - t_k|$ by the strict form of Lemma 7.7, so the second inequality in the display above is strict, and again $|Q|^{1/2} < \operatorname{Re}(-S)$. As $z_{\pm} = -S \pm i\sqrt{Q}$ and $|\operatorname{Re}(\pm i\sqrt{Q})| \leq |Q|^{1/2}$, we get $\operatorname{Re} z_{\pm} \geq \operatorname{Re}(-S) - |Q|^{1/2} > 0$ for $t \neq 0$. $\square$

Corollary 7.10. Let $0 < \theta \leq \theta_0$. For every $s \in \mathbb{C}$ the principal branch defines, through (12), a continuous single-valued $\mathcal{C}\ell_{0,m} \otimes \mathbb{C}$ -valued symbol $(\nabla \mathbf{F})^s$ on $\mathcal{B}_{\theta} \setminus \{0\}$, and the resulting family satisfies $(\nabla \mathbf{F})^s (\nabla \mathbf{F})^{s'} = (\nabla \mathbf{F})^{s+s'}$ there. In particular $(\nabla \mathbf{F})^s$ is a bounded Fourier multiplier on $\{u: \operatorname{supp} \hat{u} \subset \mathcal{B}_{\theta}, |\xi| \geq \rho\}$ for every $\rho > 0$.

Proof. By Theorem 7.9 both $z_{\pm}(\xi)$ lie in the open right half-plane, on which the principal logarithm is holomorphic and single-valued, so $z_{\pm}^s$ is unambiguous and, on the open set where $Q \neq 0$, continuous in ξ together with (12). At a point where $Q = 0$ we have $S \neq 0$ by Lemma 7.6, and Proposition 3.7(iv) gives the value $(-S)^s(1 - sB/S)$, which is the limit of (12) as $Q \rightarrow 0$; hence the symbol extends continuously across $\{Q = 0\}$. The semigroup law holds in each character and therefore in $\mathcal{A}$, and boundedness follows from Theorem 7.3 and Lemma 7.4, which bound $|z_{\pm}|$ above and below on the indicated set. $\square$

Proposition 7.11 (The sharp threshold is $\pi/2$). Let $m \geq 2$, $\tau \in (0, \pi)$ and $t = (\tau, -\tau, 0, \dots, 0)$. Then

$$Q = -\frac{16\tau^2}{h^2} \sin^4(\tau/2), \quad z_{\mp} = \frac{2\tau}{h} [\sin \tau \mp (1 - \cos \tau)],$$

both real. Consequently $\operatorname{Re} z_{-} > 0$ for $\tau < \pi/2$, $z_{-} = 0$ at $\tau = \pi/2$, and $\operatorname{Re} z_{-} < 0$ for $\tau > \pi/2$. Hence the conclusion of Theorem 7.9 fails on $\mathcal{B}_{\theta}$ for every $\theta \geq \pi/2$, so its range cannot be extended beyond $\pi/2$. Whether it holds for every $\theta < \pi/2$ is not settled here; numerically (Appendix A) the minimum of $\min_{\pm} \operatorname{Re} z_{\pm}/(|t|^2/h)$ over $\mathcal{B}_{\theta}$, $\theta_0 \leq \theta < \pi/2$, is attained on the present family and is positive.

Proof. With $t_3 = \dots = t_m = 0$ only Ω_{12} is nonzero. Since $\psi(\pm\tau) = (e^{\pm i\tau} - 1)/(\pm 2i\tau)$,

$$\psi(-\tau) - \psi(\tau) = -\frac{e^{i\tau} + e^{-i\tau} - 2}{2i\tau} = \frac{1 - \cos \tau}{i\tau} = -\frac{2i}{\tau} \sin^2(\tau/2),$$

so (26) gives $\Omega_{12} = \frac{2\tau^2}{h} (\psi(-\tau) - \psi(\tau)) = -\frac{4i\tau}{h} \sin^2(\tau/2)$ and $Q = \Omega_{12}^2 = -\frac{16\tau^2}{h^2} \sin^4(\tau/2)$. By (24), $-S = 2\tau \sin \tau/h$, the imaginary part cancelling because $t_1 \sin t_1$ and $t_2 \sin t_2$ are equal while $t_1(1 - \cos t_1)$ and $t_2(1 - \cos t_2)$ are opposite. As $Q < 0$, $i\sqrt{Q} = \mp \frac{4\tau}{h} \sin^2(\tau/2) = \mp \frac{2\tau}{h} (1 - \cos \tau)$, which gives the stated $z_{\mp}$. Finally $\sin \tau - (1 - \cos \tau) > 0 \Leftrightarrow \sin \tau + \cos \tau > 1 \Leftrightarrow 0 < \tau < \pi/2$ on $(0, \pi)$. $\square$

Proposition 7.12 (The factor $(\cos \theta)^{-1}$ is of the correct order). Let $m \geq 2$, $h > 0$, $\tau \in (0, \pi/2)$ and $\xi_{\tau} = h^{-1}(\tau, -\tau, 0, \dots, 0)$. Then, with the blade norm of Theorem 7.5,

$$h|\xi_{\tau}|^2 \parallel (\nabla \mathbf{F})^{-1}(\xi_{\tau}) \parallel = \frac{2\tau \sin(\tau/2)}{(1 + \sin \tau - \cos \tau)(\sin \tau + \cos \tau - 1)}, \quad (29)$$

independently of m and of h . The right-hand side tends to $+\infty$ as $\tau \uparrow \pi/2$, and

$$\lim_{\tau \uparrow \pi/2} \cos \tau \cdot h |\xi_\tau|^2 \|(\nabla \mathbf{F})^{-1}(\xi_\tau)\| = \frac{\pi}{2\sqrt{2}}.$$

Consequently the growth $(\cos \theta)^{-1}$ in (22) is of the correct order as $\theta \uparrow \pi/2$, and only the constant is open: the proof gives $\pi^2 m/4$, while this family forces at least $\pi/(2\sqrt{2})$, a ratio of $\pi m/\sqrt{2}$.

Proof. With $t_3 = \dots = t_m = 0$ only Ω_{12} is nonzero, and by Proposition 7.11

$$-S = \frac{2\tau \sin \tau}{h} =: a, \quad \Omega_{12} = -\frac{4i\tau}{h} \sin^2(\tau/2), \quad |\Omega_{12}| = \frac{2\tau(1 - \cos \tau)}{h} =: b,$$

so that $Q = \Omega_{12}^2 = -b^2$ and $\mathcal{N} = S^2 + Q = a^2 - b^2 = (a - b)(a + b)$. By (11) and the definition of the blade norm, $\|D^{-1}\| = (a^2 + b^2)^{1/2}/|\mathcal{N}|$. Now

$$a^2 + b^2 = \frac{4\tau^2}{h^2} [\sin^2 \tau + (1 - \cos \tau)^2] = \frac{8\tau^2}{h^2} (1 - \cos \tau) = \frac{16\tau^2}{h^2} \sin^2(\tau/2),$$

so $(a^2 + b^2)^{1/2} = 4\tau \sin(\tau/2)/h$, while $a \pm b = (2\tau/h)(\sin \tau \pm (1 - \cos \tau))$. Since $h|\xi_\tau|^2 = 2\tau^2/h$, multiplying out gives (29); both h and m cancel.

For the limit put $c = \cos \tau \downarrow 0$. Then $\sin \tau = \sqrt{1 - c^2} = 1 - 1/2 c^2 + O(c^4)$, so the second factor in the denominator is $\sin \tau + c - 1 = c - 1/2 c^2 + O(c^4)$ and $c/(\sin \tau + c - 1) \rightarrow 1$, while the first factor tends to 2, the numerator to $2 \cdot \pi/2 \cdot \sin(\pi/4) = \pi/\sqrt{2}$. Hence the product with $\cos \tau$ tends to $(\pi/\sqrt{2})/2 = \pi/(2\sqrt{2})$. $\square$

8. A Bandwidth Rule for One-Sided Schemes

Theorems 7.1, 7.3 and 7.5 combine into a restriction that a user of a one-sided difference Dirac operator must respect.

Theorem 8.1. Let $u \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ be band-limited, $\text{supp } \hat{u} \subset \{\rho \leq |\xi|, \max_j |\xi_j| \leq K\}$, and let $F_j = \Delta_{h,j}$. If the step h satisfies

$$h < \frac{\theta}{K} \quad \text{for some } \theta < \frac{\pi}{2},$$

then $\nabla \mathbf{F}$ is invertible on the frequency support of u and

$$\|(\nabla \mathbf{F})^{-1}u\|_{L^2} \leq \frac{\pi^2 m}{4 \cos \theta h \rho^2} \|u\|_{L^2}.$$

If moreover $\theta \leq \theta_0$, then $(\nabla \mathbf{F})^s u$ is defined for every $s \in \mathbb{C}$ by Corollary 7.10.

Proof. The hypothesis gives $\max_j h|\xi_j| \leq hK < \theta$ on $\text{supp } \hat{u}$, so $\text{supp } \hat{u} \subset \mathcal{B}_\theta$ and Theorems 7.1 and 7.5 apply. $\square$

The rule $hK < \pi/2$ should be compared with the usual Nyquist condition $hK < \pi$, which is what a scalar one-sided scheme requires: the Clifford product costs an additional factor of two in the admissible bandwidth, and Theorem 7.1 shows this factor cannot be improved. By Theorem 9.1 the restriction disappears if the central difference is used instead.

8.1 Delimiting the Sectorial Hypotheses

Fractional powers of hypercomplex and lattice operators are ordinarily constructed under sectoriality [22, 18, 20, 21, 17], and it is natural to ask how much that hypothesis excludes. Theorem 7.9 and Proposition

7.11 answer this for the present family: sectoriality holds on $\max_j h|\xi_j| \leq \theta_0$, fails as soon as $\max_j h|\xi_j| \geq \pi/2$ is allowed, and holds numerically in between; and Corollary 5.4 shows that beyond the null set the failure is not repairable by any choice of branch, since it is topological. In this sense the sectorial hypothesis in that literature is not merely convenient but necessary: dropping it does not weaken the conclusion, it destroys the object.

9. The Symmetric Scheme Carries No Obstruction

The contrast of Remark 4.4 extends from the null set to the functional calculus, and in the opposite direction to what the doubling phenomenon might suggest.

Theorem 9.1. Let $F_j = \delta_{h,j}$ be the central differences and let $\mathcal{D} = \{t \in \mathbb{T}^m: t_j \in \{0, \pi\} \text{ for all } j\}$ be the set of 2^m doubling points. Then, with $t = h\xi$:

- (i) $\Omega_{jk} = -1/h (t_j \sin t_k - t_k \sin t_j)$ is real, hence $Q = \sum_{j < k} \Omega_{jk}^2 \geq 0$ is real;
- (ii) $-S = 1/h \sum_j t_j \sin t_j$ is real and nonnegative, and is strictly positive off $\mathcal{D}$;
- (iii) $z_{\pm} = -S \pm i\sqrt{Q}$ satisfy $\operatorname{Re} z_{\pm} = -S > 0$ off $\mathcal{D}$;
- (iv) $\mathcal{N} = S^2 + Q \geq 0$ is real, so $\operatorname{wind}(\mathcal{N} \circ \gamma) = 0$ for every loop γ in $\mathbb{T}^m \setminus \mathcal{D}$.

Consequently, for every $s \in \mathbb{C}$ the principal branch defines $(\nabla \mathbf{F})^s$ as a continuous single-valued symbol on all of $\mathbb{T}^m \setminus \mathcal{D}$.

Proof. With $f_j = i \sin t_j$ and $\partial_j \mapsto it_j/h$,

$$\Omega_{jk} = \frac{it_j}{h} i \sin t_k - \frac{it_k}{h} i \sin t_j = -\frac{1}{h} (t_j \sin t_k - t_k \sin t_j),$$

which is real, giving (i). Similarly $S = \sum_j \frac{it_j}{h} i \sin t_j = -\frac{1}{h} \sum_j t_j \sin t_j$, so $-S$ is real; since $u \sin u \geq 0$ for $|u| \leq \pi$, with equality iff $u \in \{0, \pi\}$ in $\mathbb{T}$, we obtain (ii). As $Q \geq 0$, the quantity $\sqrt{Q}$ is real and $\pm i\sqrt{Q}$ purely imaginary, so $\operatorname{Re} z_{\pm} = \operatorname{Re}(-S) = -S$, which is (iii). For (iv), S is real so $S^2 \geq 0$, and $Q \geq 0$, so $\mathcal{N} \geq 0$; off $\mathcal{D}$ it is strictly positive by (ii), hence $\mathcal{N}$ takes values in $(0, \infty)$ and its argument may be taken identically zero, so every winding number vanishes. The final statement follows as in Corollary 7.10: both roots lie in the open right half-plane, where the principal logarithm is single-valued, and (12) extends continuously across $\{Q = 0\}$ by Proposition 3.7(iv). $\square$

This settles, for the operator studied here, the question raised in Remark 4.4, and it inverts the naive expectation. The central difference is the scheme that does exhibit fermion doubling, in the sense of Proposition 4.3; yet it is the scheme whose Clifford norm is real and nonnegative, so that no monodromy can occur and a global fractional calculus exists away from the doubling points. The one-sided schemes have no doubling in their scalar symbol, and are nevertheless the ones for which $(\nabla \mathbf{F})^s$ fails to exist globally. The two phenomena are genuinely independent.

10. Conclusion

10.1 The Continuum Limit

As $h \rightarrow 0$ with ξ fixed, expanding $f_k = ih\xi_k - 1/2 h^2 \xi_k^2 - i/6 h^3 \xi_k^3 + O(h^4)$ and $\partial_j \mapsto i\xi_j$ gives $S(\xi) = -h|\xi|^2 + O(h^2)$, $\Omega_{jk}(\xi) = i/2 h^2 \xi_j \xi_k (\xi_j - \xi_k) + O(h^3)$,

whence $Q = O(h^4)$, $\mathcal{N} = h^2|\xi|^4 + O(h^3)$ and $h^{-1}\nabla\mathbf{F} \rightarrow |\xi|^2$, the symbol of $\nabla^2 = -\Delta$. The bivector part is thus a discretization defect one order below the scalar part, and its leading symbol is antisymmetric in (j, k) and vanishes on the diagonal $\xi_j = \xi_k$. This explains why the obstruction of Sections 4–5 is invisible to formal consistency analysis of one-sided schemes, which probes exactly the regime $h|\xi| \rightarrow 0$ in which the defect disappears.

10.2 Distinction from Fermion Doubling

The results of Sections 4–5 are topological in flavour and therefore invite comparison with the Nielsen–Ninomiya theorem [1, 2, 26], which asserts that a local, hermitian, chirally symmetric lattice Dirac operator must have extra zeros of its symbol in the Brillouin zone. Proposition 4.3 exhibits precisely that pattern for the central difference: the 2^m doubling points $t_j \in \{0, \pi\}$. Theorem 4.1 and Theorem 5.2 are statements of a different kind, in three respects.

The object is different: doubling concerns zeros of a scalar or matrix dispersion relation, and its consequence is a spurious multiplicity of continuum species, whereas Theorem 5.2 concerns the winding of the Clifford norm $\mathcal{N}$ and its consequence is the non-existence of a globally single-valued fractional power. The operator is different: the forward difference Dirac operator is not hermitian, its scalar symbol S has no extra zeros of doubling type, and $\mathcal{N} = S^2 + Q$ vanishes at points where $S \neq 0$, the cancellation being produced by the bivector through $Q = -S^2$. And the obstruction is felt on every domain linking $\mathcal{Z}_h$, not merely at the null points, whereas doubling is a statement about the null points themselves. The Ginsparg–Wilson relation [28] is the standard route around the no-go theorem; it does not address the branch question studied here.

10.3 Open Questions

Four questions are left open. The gap $[\theta_0, \pi/2)$ in Theorem 7.9: the numerical evidence of Appendix A indicates that sectoriality holds for every $\theta < \pi/2$, with the extremal configuration on the antipodal family of Proposition 7.11, but the proof given certifies it only up to $\theta_0 = 1.4044 \dots$. The constants: the lower bound in (21) is not attained, and by Proposition 7.12 the constant $\pi^2 m/4$ in (22) exceeds what the antipodal family forces by a factor $\pi m/\sqrt{2}$. The regularity of Borel realizations: by Remark 6.3 bounded ones exist away from $\{Q = 0\}$, and one would like to know off which closed sets they can be taken continuous. And the extension of the analysis to Bessel-type operators in place of ∂_j , where the analogue of Proposition 3.6 would live in the Hankel–Clifford setting of Prasad and Kumar [33].

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