International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

A Dynamical Analysis of the Labour Market through Mathematical Modelling

Lipika Saikia[1], Manash Kalita[2]

[1]Department of Economics, Gauhati University, Guwahati 781014, Assam, India
[2]Department of Mathematics, Lakhimpur Girls' College, Lakhimpur 787031, Assam, India

Abstract
In this study, we present a three-state linear population model that describes the dynamics of the labour market behaviour. The population has been divided into the categories of employed population (X(t)), unemployed population (Y(t)), and structurally unemployed population (Z(t)). This study discusses the qualitative properties of the model, including the non-negativity, and boundedness of solutions. We also determine its equilibrium point and discuss its local and global asymptotic stability. The impact of different values of the parameters is analyzed by discussing their sensitivity. Numerical simulations are conducted to confirm the main theoretical findings through MATLAB software.

Keywords: Labour market; unemployment; structural unemployment; reskilling.

1 Introduction

The labour market depends mainly on two factors: labour supply and labour demand. Studying labour market behaviour tells us how well the economy is doing and how it affects people's lives 1. Most importantly, the unemployment rate tells us how many people want to work but can not find jobs. Unemployment is known as one of the most important problems worldwide, with significant social as well as psychological impact on both individuals and society 2. Technological change is one of the major challenges of the 21th century, with significant implications for the labour market. Introducing new technologies to a business reduces the need for labour; many workers have lost their jobs 3. The forces that have contributed to the structural displacement of labour throughout the past 100 years are: skill mismatch induced by structural shifts from manufacturing costs, offshoring to countries with lower manufacturing costs, the rise of automation and advance technologies, and the increasingly financialized nature of the global economy 4. Structural unemployment arises through the inability or the failure of labour to adapt to changes in technology and changing demand so that unemployment becomes concentrated for long periods of time among specific occupational groups in particular areas, while elsewhere employers are seeking workers but of a different type with different qualifications to the unemployed 5.

Many educational institutions produce academically qualified graduates without providing them with the skills required by the employment market, which results in increased unemployment. It is essential to develop appropriate skills among young individuals so that they can effectively utilize their potential in industries and public-sector organizations or become self-employed. Investing in skill development and training for young people is vital for strengthening economic growth 6. Recently, researchers are working in the field of labour market economics through the application of mathematical modelling techniques.

Some of the research work involves the state variables: unemployed population, employed population, and available vacancies 781011. Other research work involves the state variables: employed population, structurally unemployed population, and cyclical unemployment population 9.

1.1 Research gap and novelties of the current work

Existing unemployment mathematical models investigated the dynamic behaviour of the labour market, including skill development, time delays, cyclical unemployment, and limited job vacancies. However, they do not introduce structural unemployment within a single dynamical framework. In particular, the interaction between employed workers, unemployed workers, structurally unemployed workers, and reskilling process remains comparatively less explored. Incorporating these distinct states can provide a more realistic representation of the labour-market transitions and underlying dynamics of the labour-market.

This study presents a three-state linear dynamical model consisting of the employed population (X(t)), the unemployed population (Y(t)), and the structurally unemployed population (Z(t)). The proposed system is mathematically analyzed by establishing the non-negativity and boundedness of solutions and determining the equilibrium point and its local and global asymptotic stability. Furthermore, numerical simulations are performed to examine the temporal dynamics of the three labour-market states and to illustrate how variations in the reskilling rate influence structural unemployment and overall labour-market dynamics.

The paper is organized as follows. Section 2 of this paper discusses the formulation of the proposed model along with detailed analysis. In section 3, we carry out a qualitative analysis of the model, which includes the schematic representation of the model (2.1), basic properties such as non–negativity and boundedness of solutions, and an investigation of the equilibrium point with the analysis of both local and global stability. Section 4 presents numerical simulations of the model, offering valuable insights and clear geometric interpretations of the system dynamics. Finally, Section 6 presents the concluding remarks of the study.

2 Model Formulation

We formulate a linear differential equation model showing the dynamics of labour market with the state variables, employed population (X(t)), unemployed population (Y(t)), and structurally unemployed population (Z(t)). The proposed model is given below:

$$\begin{aligned}\frac{dX}{dt} &= \alpha Y - (\beta + \mu + \gamma)X,\\ \frac{dY}{dt} &= \gamma X - (\alpha + \mu + \lambda)Y + \delta Z,\\ \frac{dZ}{dt} &= \lambda Y - (\mu + \delta)Z.\end{aligned} \tag{2.1}$$

With initial conditions,
$X(t) \geq 0, Y(t) \geq 0,$ and $Z(t) \geq 0$.

The model assumptions are:

- The model describes dynamics of the labour market incorporating the state variables: the employed population (X(t)), the unemployed population (Y(t)), and the structurally unemployed population (Z(t)).

- Unemployed individuals gain employment at a constant rate α. However, employed individuals transition to the standard unemployed pool at a constant rate γ.
- Employed individuals exit the active labour market through retirement at a proportional rate β. Individuals across all three categories exit the labour system at a uniform natural mortality rate μ.
- Unemployed individuals transition into structural unemployment at a constant rate λ due to skill mismatches or structural shifts.
- Structurally unemployed individuals re–enter the standard unemployed job–seeking pool at a constant rate γ through reskilling or retraining programs.
- The labour market is assumed to be closed; there is no external inflow of new workers or recruitment into the workforce over time.

The conceptual framework of the model (2.1) is illustrated in Figure 1 given below:

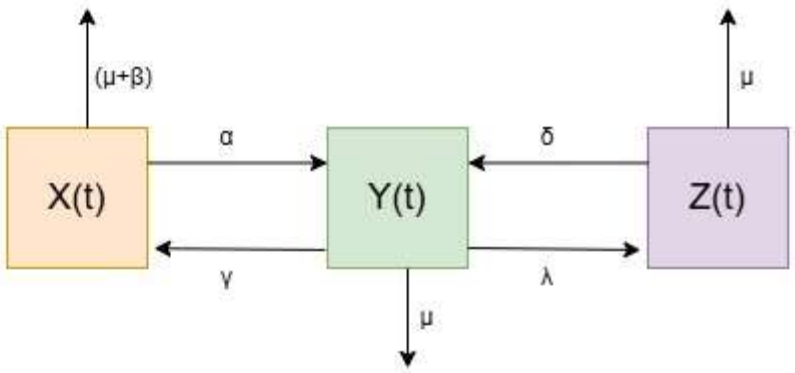

Figure 1 The Schematic diagram for the model (2.1)

Table 1 describes the variables and parameters used in the model (2.1)

Symbol	Definition	Dimension
$X(t)$	individuals who are employed	person
$Y(t)$	unemployed individuals	person
$Z(t)$	structurally unemployed individuals	person
α	rate of transition from unemployed to employed	time $^{-1}$
β	retirement rate	time $^{-1}$
μ	natural mortality	time $^{-1}$
γ	rate of job loss due to unethical work	time $^{-1}$
δ	rate of structurally unemployed workers returning to unemployed through reskilling	time $^{-1}$
λ	rate at which unemployed workers become structurally unemployed	time $^{-1}$

Table 1: The meaning of the parameters and the state variables

3 Qualitative analysis of the model

3.1 Basic Properties of the model (2.1)

3.1.1 Non-negativity and boundedness of solution of the model (2.1)

Proposition 3.1. *The set* $\Omega = \{(X, Y, Z) \in \mathbb{R}_+^3 |\ X \geq 0, Y \geq 0, Z \geq 0\}$. *Is positively invariant for all time* $t \geq 0$ *under the system (2.1).*

Proof. To show that the region $\mathbb{R}_+^3$ is positively invariant, we evaluate the direction of vector fields on the boundary faces of the non-negative octant.

When $X = 0, Y \geq 0, Z \geq 0$,

$$\frac{dX}{dt}|_{X=0} = \alpha Y \geq 0$$

When $Y = 0, X \geq 0, Z \geq 0$,

$$\frac{dY}{dt}|_{Y=0} = \gamma X + \delta Z \geq 0$$

When $Z = 0, X \geq 0, Y \geq 0$,

$$\frac{dZ}{dt}|_{Z=0} = \lambda Y \geq 0$$

Since, all the derivatives are non-negative at each boundary plane. Thus, the solution (X(t), Y(t), Z(t)) of the system (2.1) is positively invariant for all time $t \geq 0$.

Proposition 3.2. *All solutions (X(t), Y(t), Z(t)) of the system (2.1) that initiate in* $\mathbb{R}_+^3$ *are uniformly bounded within the region* Ω

Proof. Let, $N(t) = X(t) + Y(t) + Z(t)$ denote the total population at any time t.

Taking the time derivative of $N(t)$ along the trajectories of the system (2.1) gives:

$$\frac{dN}{dt} = \frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt}$$

Now substituting the values of $\frac{dX}{dt}, \frac{dY}{dt}$, and $\frac{dZ}{dt}$, we get,

$$\frac{dN}{dt} = -\beta X - \mu N$$

$$\Rightarrow N(t) \leq -\mu N(t), \quad (since, \quad -\beta X \leq 0)$$

$$\Rightarrow N(t) \leq N(0)e^{-\mu t}$$

Taking the limit superior $t \to \infty, N(t) \leq N(0)$. Therefore, all solutions starting in $\mathbb{R}_+^3$ are uniformly bounded within the region Ω.

3.2 Stability of the equilibrium point of the model (2.1)

Let $T^* = (X^*, Y^*, Z^*)$ be the equilibrium point of the system (2.1), where $X = X^*, Y = Y^*, Z = Z^*$. Now, by substituting these into the right-hand-side of the equations of (2.1), we get,

$$\alpha Y^* - (\beta + \mu + \gamma)X^* = 0 \quad (3.1)$$

$$\gamma X^* - (\alpha + \mu + \lambda)Y^* + \delta Z^* = 0 \quad (3.2)$$

$$\lambda Y^* - (\mu + \delta)Z^* = 0 \quad (3.3)$$

We calculate the unique economically valid equilibrium point for the system (2.1) is $(X^*, Y^*, Z^*) = (0,0,0)$.

3.2.1 Local Stability

The characteristic equation with respect to the equilibrium point $T^* = (X^*, Y^*, Z^*)$ of the model (2.1) is obtained by computing $\det[\rho I_3 - B] = 0$, where ρ is the eigen value and I_3 is the identity matrix of order 3, and

$$B = \begin{pmatrix} -(\beta + \mu + \gamma & \alpha & 0 \\ \gamma & -(\alpha + \mu + \lambda) & \delta \\ 0 & \lambda & -(\mu + \delta) \end{pmatrix}$$

The required characteristic equation is,

$$\rho^3 + (d + b + c)\rho^2 + (cd - \lambda\delta + bd + bc)\rho + bcd - \lambda b\delta + \alpha\gamma\lambda = 0, \quad (3.4)$$

where, $b = \beta + \mu + \gamma$, $c = \alpha + \mu + \lambda$, and $d = \mu + \delta$.

Since, for a polynomial of degree 3, $\rho^3 + a_1\rho^2 + a_2\rho + a_3 = 0$ all roots of the polynomial have negative real parts if it satisfies $a_1 > 0, a_2 > 0, a_3 > 0$, and $a_1a_2 > a_3$. Accordingly, we have examined that the characteristic polynomial (3.4) satisfies the conditions $a_1 > 0, a_2 > 0, a_3 > 0$, and $a_1a_2 > a_3$, where $a_1 = d + b + c$, $a_2 = cd - \lambda\delta + bd + bc$, and $a_3 = bcd - \lambda b\delta + \alpha\gamma\lambda$. Therefore, the eigenvalues are strictly negative real numbers, which confirms that the equilibrium point $T^* = (X^*, Y^*, Z^*)$ is locally asymptotically stable.

3.2.2 Global Stability

To prove global stability, we construct a linear Lyapunov candidate function,

$$V(X,Y,Z) = X + Y + Z$$

Since, $X, Y, X \geq 0$, $V(X,Y,Z) \geq 0$ with $V(0,0,0) = 0$.

Taking the derivative along trajectories of the system (2.1):

$$\frac{dV}{dt} = \frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt} = -\beta X - \mu(X + Y + Z)$$

Since all parameters are positive and $X, Y, X \geq 0$,

$$\frac{dV}{dt} \leq 0 \; for\ all\ (X,Y,Z) \in \mathbb{R}^3_+$$

Furthermore, $\frac{dV}{dt} = 0$ if and only if $X = 0, Y = 0, Z = 0$.

It is seen that the time derivative,

$$\frac{dV}{dx} = \begin{cases} 0, & \text{for } (X,Y,Z) = (X*, Y*, Z*) \\ < 0, & \text{for} (X,Y,Z) \neq (X*, Y*, Z*) \end{cases}$$

By LaSalle's Invariance Principle, the unique equilibrium point of the system (2.1) is globally asymptotically stable.

4 Numerical Simulation

In this section, we will verify the analytical results by means of numerical simulations of the model (2.1). For this purpose, we consider the values of the parameters given in Table 2, which are taken from different sources listed.

Parameters	Values	Justification/Source
α	0.40	Assumed benchmark for job finding rate
β	0.015	Based on average career duration(˜40 years)
μ	0.01	Standard demographic mortality assumption
γ	0.05	Assumed parameter for baseline illustration

δ	0.08	Reflects duration required for structural retraining
λ	0.10	Assumed structural transition rate

Table 2: Parameter values taken into consideration for numerical simulations

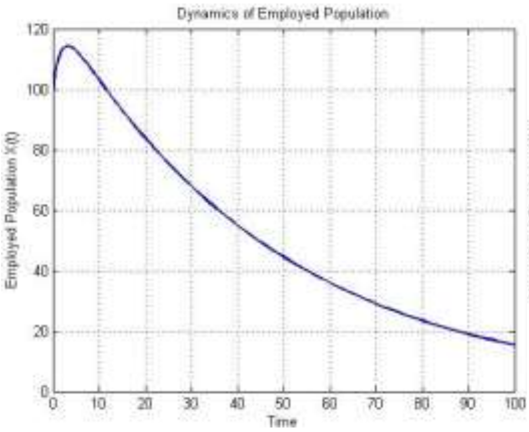

Figure 2(a)

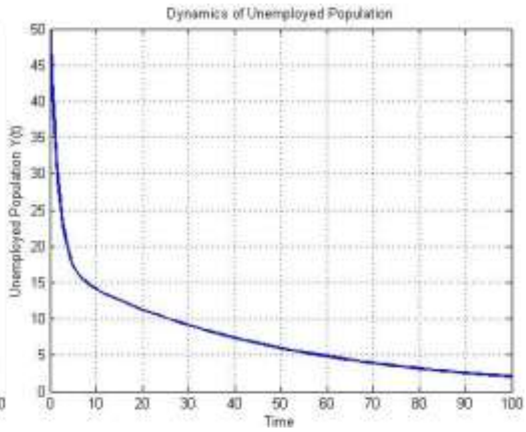

Figure 2(b)

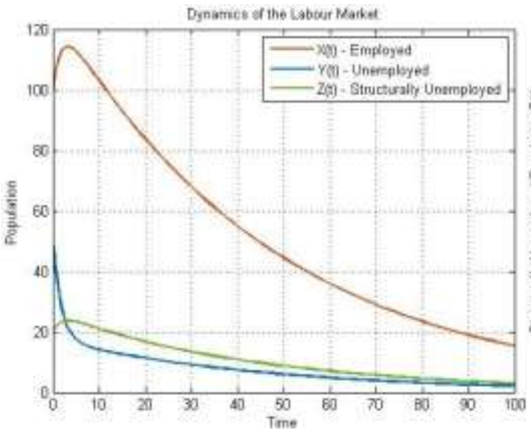

Figure 2(c)

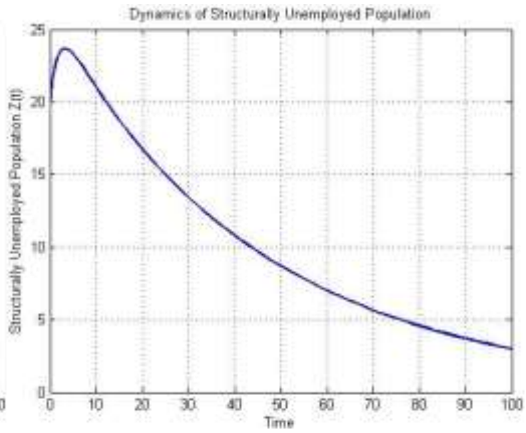

Figure 2(d)

***Figure 2:** Numerical simulations showing time-dependent dynamics of labour market categories: (a) employed population (X(t)), (b) unemployed population (Y(t)), (c) structurally unemployed population (Z(t)), and (d) combined dynamics of the labour market.*

Figure 2 presents a numerical simulation demonstrating the time–dependent dynamics of the labour market over the time horizon $t = 0 \; to \; t = 100$. Between $t = 0 \; and \; t = 10$, the labour market undergoes rapid initial friction resolution and skill re–allocation. Unemployed workers (Y) are rapidly absorbed into active employment (X), whereas structural displacement (Z) increases sharply to its peak. A major economic shock or technological transition triggers rapid hiring while simultaneously making existing skills obsolete. While standard unemployed individuals (Y) are able to find jobs, a substantial part of the labour force becomes structurally unemployed (Z) due to immediate skill mismatches with the demands of the labour market. The employed population (X) continues to be the dominant labour category throughout this period.

Over the time horizon, $t = 10 \; to \; t = 100$, all three populations enter a sustained decline and gradually stabilize near low equilibrium levels. The graph clearly shows that structural unemployment remains higher than standard unemployment $(Z(t) > Y(t))$. The persistence of $Z(t)$ relative to $Y(t)$ indicates that long-term labour inefficiency is driven primarily by structural barriers and skill gaps rather than temporary job-matching delays. The overall decline in $Z(t)$ reflects gradual retraining, reskilling, or labour force exits (e.g., retirement). The simultaneous long-term decline across all categories suggests broader demographic shifts or systemic market contraction, with the entire mathematical model stabilizing into a steady state over time.

The graph $2(d)$ indicates that the employed population (X) consistently dominates the labour market at all points throughout the simulation horizon. The structurally unemployed population (Z) surpasses standard unemployment (Y) shortly after $t \approx 5$ and remains higher for the rest of the timeline. Once the short–term job–matching phase passes, structural unemployment becomes a larger problem than simple unemployment$(Z(t) > Y(t))$, which highlights that long–term labour market inefficiency in this system is driven primarily by skill mismatches rather than a lack of job–seeking activity. Also, graph $2(d)$ demonstrates that the dynamic differential equations describing the labour market are stable. The entire system clears its initial volatility and gradually converges toward a low-level, steady-state equilibrium.

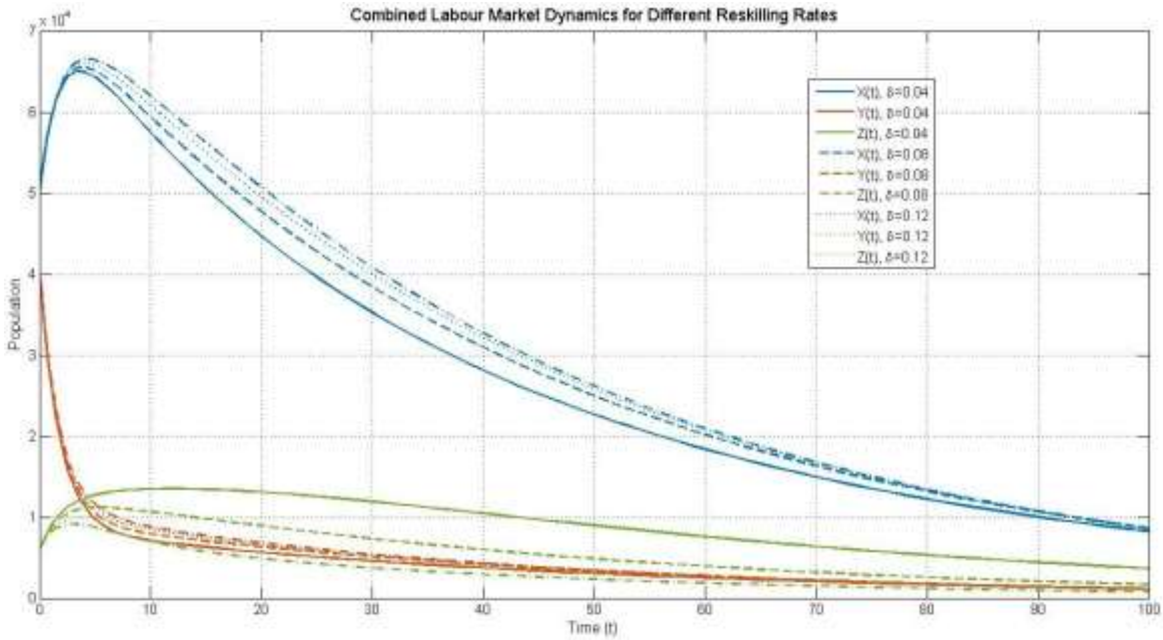

Figure 3: Combined labour market dynamics illustrating the time-dependent trajectories of the employed (X(t)), unemployed (Y(t)), and structurally unemployed (Z(t)) populations under varying reskilling rate parameter values.

Figure 3 represents the dynamic behaviour of the labour market over time, when varying the values of the reskilling rate δ. The employed workers population experiences an initial short–term surge before undergoing a long-run decline that gradually flattens out toward equilibrium. Initially, the unemployed worker population starts at a high level, then declines sharply within the first time period $(t < 10)$ and remains at a relatively low level thereafter. The structurally unemployed population increases slightly early on as workers transition due to skill mismatches, reaching a peak around $t = 15$ *to* $t = 20$, before gradually declining toward a stable low-level baseline. The graph also indicates that higher values of δ correspond to faster worker retraining and more skill adaptation.

High initial short-term unemployment rapidly drops as workers are absorbed into employment, resulting in a temporary increase in total employment. Over time, skill mismatches and technological changes shifts some workers into structural unemployment, which increases gradually and persists longer than short–term unemployment. A higher reskilling rate helps displaced workers transition back into the workforce faster, sustaining higher employment and keeping structural lower compared to slower reskilling. Eventually, labour inflows and outflows reach a balance, causing all three groups to stabilize at a steady–state equilibrium.

5 Conclusion

This paper analyzes a three–state linear model consisting of three compartments: employed population, unemployed population, and structurally unemployed population. We establish that the non–negative orthant is positively invariant, and all solutions remain bounded. The model admits a unique economically meaningful equilibrium at $(X^*, Y^*, Z^*) = (0,0,0)$. Both local and global asymptotic stability is verified, and they are found to be stable. Numerical simulations show an initial rapid reallocation phase followed by monotonic convergence of the population states toward the equilibrium at the origin. Simulations show that structural unemployment remains consistently higher than ordinary unemployment, emphasizing the persistent impact of skill mismatches. Moreover, a higher reskilling rate speeds up the transition of structurally unemployed workers into employment, thereby maintaining higher overall workforce levels.

References

1. Chatterjee P, Dev A. Labour Market Dynamics and Worker Flows in India: Impact of Covid-19. *Indian Journal of Labour Economics* 2023; 66:299–327. https://doi.org/10.1007/s41027-022-00420-7.
2. Ashi H, Al-Maalwi RM, Al-Sheikh S. Study of the unemployment problem by mathematical modeling: Predictions and controls. *The Journal of Mathematical Sociology* 2022; 46:301–313. https://doi.org/10.1080/0022250X.2021.1931173.
3. Lima Y, Barbosa CE, dos Santos HS, de Souza JM. Understanding Technological Unemployment: A Review of Causes, Consequences, and Solutions. *Societies* 2021; 11(2):50. https://doi.org/10.3390/soc11020050.
4. Yue Shun, B. W. Structural unemployment and structural injustice. *Inquiry* 2025; 1–34. https://doi.org/10.1080/0020174X.2025.2501616.
5. Standing G. The notion of structural unemployment. *International Labour Review* 1983; 122(2):137–153.
6. Misra A.K., Singh A.K. Singh P.K. Modeling the Role of Skill Development to Control Unemployment. *Differential Equations and Dynamical Systems* 2022; 30:397–409. https://doi.org/10.1007/s12591-017-0405-3.
7. Misra AK, Singh AK. A Delay Mathematical Model for the Control of Unemployment. emphDiffer Equ Dyn Syst 2013; 21:291–307. https://doi.org/10.1007/s12591-012-0153-3.

8. Kaslik, E., Neamt, u, M., Vesa, L. F. Global stability analysis of a five-dimensional unemployment model with distributed delay. *Mathematics* 2021; 9(23): 3037. https://doi.org/10.3390/math9233037.
9. El Yahyaoui, M., Amine, S. Mathematical model of unemployment with a cyclical component. *International Journal of Dynamics and Control* 2023; 11(3):1115–1122. https://doi.org/10.1007/s40435-022-01044-x.
10. Al-Sheikh, S., Al-Maalwi, R., Ashi, H. A. A mathematical model of unemployment with the effect of limited jobs. *Comptes Rendus. Math ématique* 2021; 359(3): 283–290. https://doi.org/10.5802/crmath.164.
11. Petaratip T, Niamsup P. Stability analysis of an unemployment model with time delay. *AIMS Mathematics* 2021; 6(7):7421-7440. https://doi.org/10.3934/math.2021434.