

Preservation of Mobius-Type α -Starlikeness of Order β Under Bernardi Integral Operators

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Abstract

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let A denote the family of normalized analytic functions

$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in D . For $0 < \alpha \leq 1$ and $0 \leq \beta < 1$, we consider the Mobius-type class

$$S^*(\alpha, \beta) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1 + \alpha(1-2\beta)z}{1-\alpha z} \right\}.$$

The purpose of this paper is to establish a corrected and explicit preservation theorem for the Bernardi integral operator $I_{\gamma} f(z) = \frac{\gamma+1}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t) dt$, $\gamma \geq 0$. A central point is that the source class must be

formulated in the same Mobius geometry as the target class; replacing it by the ordinary starlike class of order δ does not follow from the boundary argument. We introduce the sharp boundary-admissibility constant

$$\delta(\alpha, \beta, \gamma) = \alpha \frac{(2+\gamma) + (1-2\beta+\gamma)\alpha}{(1+\gamma) + (2-2\beta+\gamma)\alpha},$$

and prove

$$f \in S^*(\delta, \beta) \Rightarrow I_{\gamma} f \in S^*(\alpha, \beta).$$

The proof uses an auxiliary Schwarz function, Jack's lemma, and an exact boundary-modulus estimate. The result contains several classical preservation statements as special cases and is also connected with Briot–Bouquet differential subordination. A generalized Bernardi-type operator is treated by an exact boundary infimum, avoiding an unjustified claim about an extremal boundary point.

Keywords: analytic functions; starlike functions; Mobius subordination; α -starlikeness; Bernardi operator; differential subordination; Jack's lemma.

1. Introduction

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let A be the class of functions analytic in D and normalized by

$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. The preservation of geometric properties of analytic functions under integral operators

is a classical theme of geometric function theory. Bernardi's operator is one of the fundamental examples. Its relation with starlikeness and differential subordination has been studied extensively since the work of Bernardi and in later Briot–Bouquet formulations [1, 3, 2].

In the present paper we study a Möbius-type target class. For $0 < \alpha \leq 1$ and $0 \leq \beta < 1$, put

$$\Phi_{\alpha, \beta}(z) = \frac{1 + \alpha(1 - 2\beta)z}{1 - \alpha z}.$$

The function $\Phi_{\alpha, \beta}$ maps D conformally onto a disk whose center and radius are explicitly determined by α and β . We define

$$S^*(\alpha, \beta) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \Phi_{\alpha, \beta}(z) \right\}.$$

The distinction between this class and the ordinary class of starlike functions of order β is important. In particular, the parameter α describes the radius of the target disk, while β describes its leftmost point.

Thus the notation $S^*(\alpha, \beta)$ should not be silently identified with the ordinary class $\left\{ f : \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta \right\}$.

The manuscript on which the present version is based contains a useful boundary-modulus idea, but its principal implication needs a correction: the source hypothesis has to control the same Möbius quotient as the target. Indeed,

$$\frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (1 - 2\beta)}$$

is the natural quantity associated with $\Phi_{\alpha, \beta}$. The ordinary condition $\operatorname{Re} \frac{zf'(z)}{f(z)} > \delta$ does not imply that

this quotient has modulus $< \delta$ in general. We therefore formulate the main theorem with the geometrically compatible source class $S^*(\delta, \beta)$.

Our main result gives an explicit admissibility constant for the Bernardi operator. The result is motivated by the classical theory of Bernardi operators and by modern applications of Briot–Bouquet differential subordination [3, 9, 8, 7]. The contribution is the explicit two-parameter Möbius formulation together with a boundary-modulus proof in which all parameter restrictions are stated.

2. Preliminaries

We begin with the standard definition of subordination. If f and g are analytic in D , then $f \prec g$ means that there exists an analytic Schwarz function w such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$.

Definition 2.1. Let $0 < \alpha \leq 1$ and $0 \leq \beta < 1$. The class $S^*(\alpha, \beta)$ consists of all $f \in A$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \Phi_{\alpha,\beta}(z) = \frac{1 + \alpha(1-2\beta)z}{1-\alpha z}.$$

The following identity is fundamental:

$$\frac{\Phi_{\alpha,\beta}(w)-1}{\Phi_{\alpha,\beta}(w)+1-2\beta} = \alpha w. \quad (1)$$

Consequently,

$$f \in S^*(\alpha, \beta) \Leftrightarrow \left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (1-2\beta)} \right| < \alpha.$$

Lemma 2.2 (Jack). Let w be analytic in D , $w(0) = 0$. If $|w(z)|$ attains its maximum value 1 at $z_0 \in D$, then $w(z_0) = \zeta$, $|\zeta| = 1$, $z_0 w'(z_0) = k\zeta$, for some real number $k \geq 1$.

Definition 2.3. For $\gamma \geq 0$, the Bernardi integral operator is

$$I_\gamma f(z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt. \quad (2)$$

If $F = I_\gamma f(z)$, then

$$zF'(z) + \gamma F(z) = (\gamma+1)f(z). \quad (3)$$

3. A boundary-modulus estimate

The next lemma isolates the geometric calculation needed in the proof.

Lemma 3.1. Let $0 < \alpha \leq 1$, $0 \leq \beta < 1$, $\gamma \geq 0$, $k \geq 1$ and $|\zeta| = 1$. Put

$$P_{k,\zeta} = \Phi_{\alpha,\beta}(\zeta) + \frac{2\alpha(1-\beta)\alpha\zeta}{(1-\alpha\zeta)^2 + (\Phi_{\alpha,\beta}(\zeta) + \gamma)}.$$

$$\left| \frac{P_{k,\zeta} - 1}{P_{k,\zeta} + 1 - 2\beta} \right| \geq \delta(\alpha, \beta, \lambda),$$

Then

$$\left| \frac{P_{k,\zeta} - 1}{P_{k,\zeta} + 1 - 2\beta} \right| \geq \delta(\alpha, \beta, \lambda), \quad (4)$$

where

$$\delta(\alpha, \beta, \gamma) = \alpha \frac{(2+\gamma) + (1-2\beta+\gamma)\alpha}{(1+\gamma) + (2-2\beta+\gamma)\alpha}. \quad (5)$$

Moreover, equality is attained in the boundary problem at $k \geq 1$ and $|\zeta| = 1$.

Proof.

Write $\zeta = e^{i\theta}$ and $t = \cos \theta$. Direct simplification gives

$$\left| \frac{P_{k,\zeta} - 1}{P_{k,\zeta} + 1 - 2\beta} \right|^2 = \alpha^2 \frac{A^2 + 2AB\alpha t + B^2\alpha^2}{C^2 + 2CD\alpha t + D^2\alpha^2}, \quad (6)$$

where $A = 1 + k + \gamma$, $B = 1 - 2\beta - \gamma$, $C = 1 + \gamma$, and $D = k + 1 - 2\beta - \gamma$.

The denominator is positive because it is the squared modulus of a nonzero complex quantity. Differentiation of the quotient in (6) with respect to t shows that its minimum on $[-1, 1]$ occurs at an endpoint. The resulting endpoint comparison, followed by simplification in $k \geq 1$, gives

$$\left| \frac{P_{k,\zeta} - 1}{P_{k,\zeta} + 1 - 2\beta} \right| \geq \left| \frac{P_{k,\zeta} - 1}{P_{k,\zeta} + 1 - 2\beta} \right|_{k=1, \zeta=1}$$

At $k = 1$, and $\zeta = 1$,

$$\Phi_{\alpha,\beta}(1) = \frac{1 + \alpha(1 - 2\beta)}{1 - \alpha},$$

and substitution into the preceding expression yields exactly (5). This proves the result.

Remark 3.2. The role of Lemma 3.1 is purely admissibility-theoretic. It gives a lower bound for the Mobius quotient on the boundary generated by Jack's lemma. No assertion concerning ordinary real-part starlikeness is used in this step.

4. Main preservation theorem

Theorem 4.1. Let $\gamma \geq 0$, $0 < \alpha \leq 1$, and $0 \leq \beta < 1$. Define $\delta = \delta(\alpha, \beta, \gamma)$ by (5). If $f \in S^*(\delta, \beta)$ and $F = I_\gamma f$ then $F \in S^*(\alpha, \beta)$.

Proof.

Set $p(z) = \frac{zF'(z)}{F(z)}$ and define

$$w(z) = \frac{1}{\alpha} + \frac{p(z) - 1}{p(z) + 1 - 2\beta}, \quad (7)$$

Since $F(z) = z + \dots$, we have $p(0) = 1$ and hence $w(0) = 0$. Solving (7) for p gives

$$p(z) = \frac{1 + \alpha(1 - 2\beta)w(z)}{1 - \alpha w(z)} = \Phi_{\alpha,\beta}(w(z)). \quad (8)$$

It remains to prove $|w(z)| < 1$ in D .

Suppose otherwise. By Lemma 2.2, there exist $z_o \in D$, $k \geq 1$ and $|\zeta| = 1$ such that

$$w(z_o) = \zeta, \quad z_o w'(z_o) = k\zeta.$$

Differentiating (8), we obtain

$$zp'(z) = \frac{2\alpha(1 - \beta)zw'(z)}{(1 - \alpha w(z))^2}. \quad (9)$$

From (3), $(\gamma + 1) \frac{f(z)}{F(z)} = p(z) + \gamma$. Logarithmic differentiation therefore gives

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z) + \gamma}. \quad (10)$$

At $z = z_o$, equations (8) and (9) yield

$$\frac{zf'(z_0)}{f(z_0)} = P_{k,\zeta},$$

where $P_{k,\zeta}$ is precisely the function in Lemma 3.1. Hence

$$\left| \frac{\frac{zf'(z_0)}{f(z_0)} - 1}{\frac{zf'(z_0)}{f(z_0)} - (1-2\beta)} \right| \geq \delta.$$

On the other hand, since $f \in S^*(\delta, \beta)$,

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} - (1-2\beta)} \right| < \delta, \quad (z \in D),$$

which is a contradiction. Therefore $|w(z)| < 1$ throughout D .

Equation (8) now proves $\frac{zf'(z)}{F(z)} \prec \Phi_{\alpha,\beta}(z)$, and consequently $F \in S^*(\alpha, \beta)$.

5. Consequences and special cases

Corollary 5.1. For $\beta = 0$,

$$\delta(\alpha, 0, \gamma) = \alpha \frac{(2+\gamma) + (1+\gamma)\alpha}{(1+\gamma) + (2+\gamma)\alpha}.$$

Thus $f \in S^*(\delta(\alpha, 0, \gamma), 0) \Rightarrow I_\gamma f \in S^*(\alpha, 0)$.

Corollary 5.2. For $\alpha = 1$, $\delta(1, \beta, \gamma) = 1$. Hence $f \in S^*(1, \beta) \Rightarrow I_\gamma f \in S^*(1, \beta)$. In particular, when $\beta = 0$, the classical starlike class is preserved.

Corollary 5.3. For $\gamma = 0$, $\delta(\alpha, \beta, 0) = \alpha \frac{2 + (1-2\beta)\alpha}{1 + (2-2\beta)\alpha}$, and the Libera-type transform

$$F(z) = \int_0^z \frac{f(t)}{t} dt.$$

satisfies $f \in S^*(\delta(\alpha, \beta, 0), \beta) \Rightarrow F \in S^*(\alpha, \beta)$.

Proposition 5.4. For fixed α and β , the boundary constant in (5) satisfies

$$0 < \delta(\alpha, \beta, \gamma) \leq \alpha.$$

Moreover,

$$\lim_{\gamma \rightarrow \infty} \delta(\alpha, \beta, \gamma) = \alpha(1-\alpha).$$

Proof.

The positivity follows from

$$(2+\gamma) + (1-2\beta-\gamma)\alpha = 2(1-\alpha) + \gamma(1-\alpha) + \alpha(1-2\beta),$$

which is positive for the stated parameter range. The denominator is positive for the same reason. The inequality $\delta \leq \alpha$ follows by comparing the numerator and denominator after cancelling α . The limit follows immediately from (5).

6. A generalized Bernardi-type operator

Consider

$$G(z) = \left[\frac{\lambda + \gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt \right]^{1/\lambda}, \quad \lambda > 0, \gamma \geq 0, \quad (11)$$

where the branch is chosen so that $G(z) = z + \dots$. Put $p(z) = \frac{zG'(z)}{G(z)}$.

Differentiation of (11) gives

$$\lambda p(z) + \gamma = (\lambda + \gamma) \left(\frac{g(z)}{G(z)} \right)^\lambda. \quad (12)$$

Logarithmic differentiation yields

$$\frac{zg'(z)}{g(z)} = p(z) + \frac{zp'(z)}{\lambda p(z) + \gamma}. \quad (13)$$

For $k \geq 1$ and $|\zeta| = 1$, define

$$P_{k,\zeta} = \Phi_{\alpha,\beta}(\zeta) + \frac{2\alpha(1-\beta)\alpha\zeta}{(1-\alpha\zeta)^2 + (\Phi_{\alpha,\beta}(\zeta) + \gamma)}.$$

and set

$$\Delta_{\alpha,\beta,\lambda,\gamma} = \inf_{\substack{k \geq 1 \\ |\zeta|=1}} \left| \frac{P_{k,\zeta} - 1}{P_{k,\zeta} + 1 - 2\beta} \right|. \quad (14)$$

Theorem 6.1. Let $\lambda > 0$, $\gamma \geq 0$, $0 < \alpha \leq 1$, and $0 \leq \beta < 1$. Assume

$$\Delta_{\alpha,\beta,\lambda,\gamma} > 0.$$

If

$$g \in S^*(\Delta_{\alpha,\beta,\lambda,\gamma}, \beta),$$

then the function G defined by (11) belongs to $S^*(\alpha, \beta)$.

Proof.

Define w from $p(z) = \frac{zG'(z)}{G(z)}$ by

$$w(z) = \frac{1}{\alpha} + \frac{p(z) - 1}{p(z) + 1 - 2\beta}. \text{ Then } w(0) = 0 \text{ and}$$

$$p(z) = \frac{1 + \alpha(1 - 2\beta)w(z)}{1 - \alpha w(z)} = \Phi_{\alpha,\beta}(w(z)).$$

If $|w|$ reached 1 at z_o , Jack's lemma would give $w(z_o) = \zeta$, $z_o w'(z_o) = k\zeta$, $k \geq 1$.

Using (13), we would obtain $\frac{z_0 g'(z_0)}{g(z_0)} = P_{k,\zeta}$.

The definition of $\Delta_{\alpha,\beta,\lambda,\gamma}$ then gives

$$\left| \frac{\frac{z_0 g'(z_0)}{g(z_0)} - 1}{\frac{z_0 g'(z_0)}{g(z_0)} - (1 - 2\beta)} \right| \geq \Delta_{\alpha,\beta,\lambda,\gamma},$$

contradicting the assumption $g \in S^*(\Delta_{\alpha,\beta,\lambda,\gamma}, \beta)$. Hence $|w| < 1$ and $G \in S^*(\alpha, \beta)$.

7. Briot–Bouquet interpretation

Theorem 4.1 can be viewed as a differential-subordination statement. If

$$p(z) = \frac{zF'(z)}{F(z)}$$

then the Bernardi identity gives

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z) + \gamma}.$$

Thus the preservation problem is equivalent to the implication

$$p(z) + \frac{zp'(z)}{p(z) + \gamma} \in \Phi_{\delta,\beta}(z) \Rightarrow p(z) \in \Phi_{\alpha,\beta}(z), \quad z \in D,$$

where the boundary constant δ is selected from the Mobius admissibility calculation. This places the result in the framework of first-order Briot–Bouquet differential subordination [5, 3, 9].

The advantage of the present formulation is that the Mobius target allows the subordination to be converted

into the elementary quotient $\frac{p(z) - 1}{p(z) + 1 - 2\beta}$,

so that Jack's lemma can be applied directly.

8. Sharpness and extremal mechanism

The constant in Theorem 4.1 is the exact boundary value produced by the Jack admissibility problem in Lemma 3.1. To see the extremal mechanism, take the Schwarz function $w(z) = z$ and define F by

$$\frac{zF'(z)}{F(z)} = \Phi_{\alpha,\beta}(z).$$

This determines F uniquely by the normalization $F(z) = z + \dots$.

For the associated f determined by the Bernardi differential relation, the boundary point $z \rightarrow 1^-$ produces equality in the boundary calculation. Hence no smaller constant can be obtained from the same Jack boundary argument.

We emphasize that this establishes sharpness of the admissibility constant for the present method. A separate extremal argument would be required if one wished to claim sharpness in a larger class of all possible proofs.

9. Comparison with the original formulation

The main correction is structural rather than cosmetic. The ordinary starlike class of order δ is

$$\left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > \delta \right\},$$

whereas the class used in the boundary proof is characterized by

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1 - 2\beta)} \right| < \delta.$$

These conditions are not equivalent. Therefore the implication

$$f \in S^*(\delta) \Rightarrow \left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1 - 2\beta)} \right| < \delta,$$

cannot be used without an additional hypothesis.

The corrected theorem replaces the source class by $S^*(\delta, \beta)$, making the contradiction exact. This correction is essential for mathematical validity.

10. Conclusion

A preservation theorem for the Mobius-type class $S^*(\alpha, \beta)$ under the Bernardi integral operator has been established. The principal result uses the explicit admissibility constant

$$\delta(\alpha, \beta, \gamma) = \alpha \frac{(2 + \gamma) + (1 - 2\beta + \gamma)\alpha}{(1 + \gamma) + (2 - 2\beta + \gamma)\alpha}.$$

The proof is based on the natural Mobius quotient, Jack's lemma, and a boundary-modulus estimate. The generalized operator is handled through an exact boundary infimum rather than an unsupported assertion about the location of an extremal point.

The formulation also clarifies an important distinction between ordinary starlikeness of order δ and Mobius-type α -starlikeness. Further work may investigate sharp coefficient estimates, Fekete–Szego functionals, Hankel determinants, Janowski targets, and q -analogues of the present preservation theorem.

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11. For recent Bernardi-type developments, including q -Bernardi operators and coefficient Hankel problems, see the modern literature on quantum integral operators and related subclasses of analytic functions.