

# Near-Cordial Labeling of $K_2$ -Corona Graphs

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## Abstract:

Let  $H$  be a finite simple graph of order  $n$  and size  $q$ . We study a natural binary-labeling construction on the  $K_2$ -corona  $H \odot K_2$ . Starting from a friendly labeling  $g$  of  $H$ , each vertex  $v$  of  $H$  is kept with label  $g(v)$ , while the two new vertices in the attached copy of  $K_2$  receive the opposite label  $1 - g(v)$ . For this canonical opposite-pair extension we derive exact formulas for the vertex and edge discrepancies. These formulas yield sufficient conditions for cordiality and upper bounds for the closeness measure  $D_2$ . The results are presented as a framework for studying  $K_2$ -corona graphs; they are not claimed here as a complete characterization or as a verified novel theorem against the entire literature.

**Keywords:** cordial labeling; friendly labeling;  $K_2$ -corona; binary labeling; vertex discrepancy; edge discrepancy; graph labeling.

## 1. Introduction

A binary labeling of a graph  $G$  is a function  $f : V(G) \rightarrow \{0, 1\}$ . An edge  $xy$  receives the induced label  $|f(x) - f(y)|$ . A labeling is cordial when the numbers of 0- and 1-labeled vertices differ by at most 1 and the numbers of 0- and 1-labeled edges also differ by at most 1. A labeling satisfying the vertex condition is called friendly.

Recent work introduced two measures of closeness to cordiality. For a labeling  $f$ , define  $\Delta_v(f) = |v_0(f) - v_1(f)|$  and  $\Delta_e(f) = |e_0(f) - e_1(f)|$ . Then  $D_1(G) = \min_f \{\Delta_v(f) + \Delta_e(f)\}$ , while  $D_2(G)$  is the minimum of  $\Delta_e(f)$  over friendly labelings. In particular,  $D_1(G) \leq 1$  or  $D_2(G) \leq 1$  implies that  $G$  is cordial.

The present work studies the corona transformation  $H \odot K_2$ . The main idea is to start with a friendly labeling of  $H$  and extend it in a prescribed way to every attached  $K_2$ . The resulting discrepancies can be written exactly in terms of the order and size of  $H$  and the number of crossing edges of the base labeling. Scope. We do not claim that the criterion below is a complete characterization of cordiality for  $H \odot K_2$ . Nor do we claim definitive novelty until a systematic search of MathSciNet, zbMATH Open, Scopus/Web of Science, and the current graph-labeling survey has been completed.

## 2. Preliminaries

**Definition 2.1** (Binary vertex labeling). A binary vertex labeling of a finite simple graph  $G$  is a map  $f : V(G) \rightarrow \{0, 1\}$ .

**Definition 2.2** (Induced edge labeling). For  $uv \in E(G)$ , the induced edge label is  $f^*(uv) = |f(u) - f(v)|$ . Thus  $f^*(uv) = 0$  when the endpoints have equal labels and 1 otherwise.

**Definition 2.3** (Discrepancies). Put  $v_i(f)$  = number of vertices with label  $i$  and  $e_i(f)$  = number of edges with label  $i$ . Define  $\Delta_v(f) = |v_0(f) - v_1(f)|$  and  $\Delta_e(f) = |e_0(f) - e_1(f)|$ . A labeling is friendly if  $\Delta_v(f) \leq 1$  and cordial if  $\Delta_v(f) \leq 1$  and  $\Delta_e(f) \leq 1$ .

**Definition 2.4 ( $K_2$ -corona).** For every vertex  $v_i$  of  $H$ , take a disjoint copy of  $K_2$  with vertices  $x_i$  and  $y_i$ , and join  $v_i$  to both  $x_i$  and  $y_i$ . The resulting graph is denoted  $H \odot K_2$ .

If  $H$  has  $n$  vertices and  $q$  edges, then  $H \odot K_2$  has  $3n$  vertices and  $q + 3n$  edges.

### 3. Canonical Opposite-Pair Extension

Let  $H$  have vertex set  $V(H) = \{v_1, \dots, v_n\}$ . Let  $g$  be a friendly labeling of  $H$ . For each  $v_i$ , let  $x_i$  and  $y_i$  be the two vertices of the attached  $K_2$ . Define  $f$  on  $H \odot K_2$  by

$$f(v_i) = g(v_i), \quad f(x_i) = f(y_i) = 1 - g(v_i), \quad i = 1, \dots, n.$$

Thus, the two new vertices attached to  $v_i$  receive the label opposite to  $v_i$ . The edge  $x_i y_i$  always receives label 0, while both corona edges  $v_i x_i$  and  $v_i y_i$  always receive label 1.

### 4. Exact Vertex Discrepancy

Let  $a(g)$  be the number of vertices of  $H$  labeled 0 by  $g$ . Then  $n - a(g)$  vertices are labeled 1. In  $H \odot K_2$ , every base vertex labeled 0 contributes two additional 1-labeled vertices, and every base vertex labeled 1 contributes two additional 0-labeled vertices.

Therefore, the total numbers are

$$v_0(f) = a(g) + 2[n - a(g)] = 2n - a(g),$$

$$v_1(f) = [n - a(g)] + 2a(g) = n + a(g).$$

**Theorem 4.1.** For the canonical opposite-pair extension,

$$\Delta_v(f) = |n - 2a(g)|.$$

Hence the extension is friendly exactly when  $|n - 2a(g)| \leq 1$ . In particular, friendliness of  $g$  alone does not automatically imply friendliness of  $f$ .

### 5. Exact Edge Discrepancy

Let  $c(g)$  denote the number of edges of  $H$  whose endpoints receive different labels under  $g$ . These are precisely the base edges labeled 1. Hence  $q - c(g)$  base edges receive label 0.

For every  $v_i$ , the edge  $x_i y_i$  has label 0, and the two edges  $v_i x_i$  and  $v_i y_i$  have label 1. Across all  $n$  attached copies this contributes  $n$  edges labeled 0 and  $2n$  edges labeled 1.

Consequently,

$$e_0(f) = [q - c(g)] + n,$$

$$e_1(f) = c(g) + 2n.$$

**Theorem 5.1.** For the canonical opposite-pair extension,

$$\Delta_e(f) = |e_0(f) - e_1(f)| = |q - n - 2c(g)|.$$

### 6. A Cordiality Criterion

**Theorem 6.1.** Let  $H$  be a finite simple graph of order  $n$  and size  $q$ . Suppose  $H$  has a friendly labeling  $g$  such that both

$$|n - 2a(g)| \leq 1 \quad \text{and} \quad |q - n - 2c(g)| \leq 1.$$

Then  $H \odot K_2$  is cordial.

**Proof.** By Theorem 4.1, the first inequality gives  $\Delta_v(f) \leq 1$ . By Theorem 5.1, the second gives  $\Delta_e(f) \leq 1$ . Hence  $f$  is cordial. ■

**Remark.** This is a sufficient condition arising from one explicit construction. Failure of the condition does not prove that  $H \odot K_2$  is non-cordial, because another labeling of the corona graph may still be cor

dial.

## 7. A Parameter for the Canonical Construction

Define

$$\beta_2(H) = \min \{ |q - n - 2c(g)| : g \text{ is a friendly labeling of } H \text{ and } |n - 2a(g)| \leq 1 \}.$$

Whenever the set in this definition is nonempty, the canonical construction gives

$$D_2(H \odot K_2) \leq \beta_2(H).$$

If  $\beta_2(H) \leq 1$ , then  $H \odot K_2$  is cordial by the preceding criterion.

## 8. Alternative Upper Bound for $D_1$

Without requiring the extended labeling to be friendly, the same construction gives the valid bound

$$D_1(H \odot K_2) \leq \min_g \{ |n - 2a(g)| + |q - n - 2c(g)| \},$$

where the minimum is over all binary labelings  $g$  of  $H$ . This follows directly from the two exact discrepancy formulas.

## 9. Example: $H = K_2$

Let  $H = K_2$ . Take one endpoint labeled 0 and the other labeled 1. Then  $n = 2$ ,  $q = 1$ ,  $a(g) = 1$ , and  $c(g) = 1$ . The extension has

$$\Delta_v = |2 - 2(1)| = 0, \quad \Delta_e = |1 - 2 - 2(1)| = 3.$$

Thus this particular canonical extension is not cordial. This example illustrates why the exact formulas are useful: they prevent an incorrect conclusion based only on the friendliness of the original graph.

## 10. Important Scope and Research Direction

The results above establish exact discrepancy formulas for a specific, explicitly defined extension. They do not constitute a complete characterization of cordial  $K_2$ -corona graphs. They also do not, by themselves, establish that a broad family such as all paths or all trees has cordial  $K_2$ -coronas.

For a submission-quality research paper, the next mathematical step should be to determine which friendly labelings of  $H$  realize the required values of  $a(g)$  and  $c(g)$ , and to compare the resulting criterion with independent cordial labelings of  $H \odot K_2$ . A stronger paper could then target exact values of  $D_1$  or  $D_2$  for a nontrivial family of base graphs, or a new characterization theorem for a restricted class of  $H$ .

## 11. Conclusion

We introduced and analyzed the canonical opposite-pair extension from a friendly labeling of  $H$  to its  $K_2$ -corona. The construction admits exact formulas for both vertex and edge discrepancies. These formulas give transparent sufficient conditions for cordiality and upper bounds for closeness-to-cordiality measures. The framework is suitable as a starting point for a deeper study of  $K_2$ -corona graphs, but further family-specific theorems and literature verification are required before claiming publication-level novelty.

## References

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