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Investigation on the Existence of the Pythagorean Triplets

[Proof of Fermat's last theorem (FLT) & further beyond that]

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Abstract

The famous Pythagorean formulae ($a^n + b^n = c^n$) holds good for some specific positive whole numbers (a, b, c) iff n=2. For n ≥3 no triplet can be found from the universe of natural numbers that can satisfy the same. **This is the famous Fermat's Last Theorem.** In the first part of this article I took an attempt to establish that. So, the triplets can only be found for n=2. Then another natural question arises: "Is there any general rule between those specific triplets?" In the second part of this article I've investigated this underlying rule.

Keywords: Triplet, Pythagorean Formulae, Exponent, Natural Numbers, Fermat's Last Theorem.

1. INTRODUDTION:

The problem of Fermat's Last Theorem, famously proven by Andrew Wiles, represents one of the most celebrated milestones in mathematical history. While modern proofs often rely on highly advanced mathematical structures such as elliptic curves etc., this paper investigates a purely elementary approach. The foundational concepts utilized herein rely only on the fundamental properties of algebraic geometry, demonstrating an alternative, accessible perspective on the core constraints of the theorem.

By the term 'Triplet' here I mean a set of three positive integers a, b, c for a given positive integer n i.e. $[a, b, c]^n$ which satisfy the general Pythagorean relation:

$$a^n + b^n = c^n \quad (1).$$

Where a, b, c & n ϵ N (The set of Natural numbers i.e. {1, 2, 3….}).

Moreover, 'a' can never be equal to 'b' (i.e. a ≠ b). Because if a=b, then:

$$c^n = a^n + a^n = 2\,a^n. \rightarrow c = 2^{1/n}\,a.$$

i.e. 'c' no longer remains Natural, becomes irrational. So let us set without loss of generality,

$$a < b \quad (2).$$

Then a close observation on equation (1) reveals that,

$$b < c \leq a+b \quad (3).$$

The equal sign on the right side of the above inequality holds only in the trivial case when n=1, then $a^1 + b^1 = c^1$ i.e. $a + b = c$.

For n ≥ 2 the inequality (3) strictly becomes,

$$b< c < a+b \quad (4).$$

This implies that,

The lowest value of c, $c_{min} = b+1$ (4a).

& the highest value of c, $c_{max} = a+b-1$ (4b).

Therefore, Range of 'c' should be,

$$c_{range} = c_{max} - c_{min} = (a+b-1) - (b+1) = a-2.$$

To have at least one precise integer value of c, we must have,

$$c_{range} > 0. \Rightarrow a-2 > 0 \text{ i.e. } a > 2.$$

$$\text{Or, } a_{min} = 2+1 = 3 \quad (5).$$

Since, b > a [see equation (2)].

Therefore, $b_{min} = a_{min} + 1 = 3+1 = 4$ (6).

Then in this particular case,

$$c_{min} = b_{min} + 1 = 4+1 = 5 \text{ [see (4a)]}$$

$$\& \quad c_{max} = a_{min} + b_{min} -- 1 = 3+ 4 - 1 = 6$$

i.e. 'c' can theoretically assume the value of 5 or 6. But actual calculation shows that c is equal to 5 for n=2.This yields the well-known Pythagorean triplet $[3, 4, 5]^2$.This is the absolute lowest triplet; no triplet can exist for $n \geq 2$ having a, b, c less than these values.

1.1 Classification of Triplets:-

Triplets are broadly categorized into two types:

1**. Primary Triplets:** Those triplets in which there is no common factor between a, b, and c (i.e., they are coprime), e.g., $[3, 4, 5]^2$, $[5, 12, 13]^2$, etc.

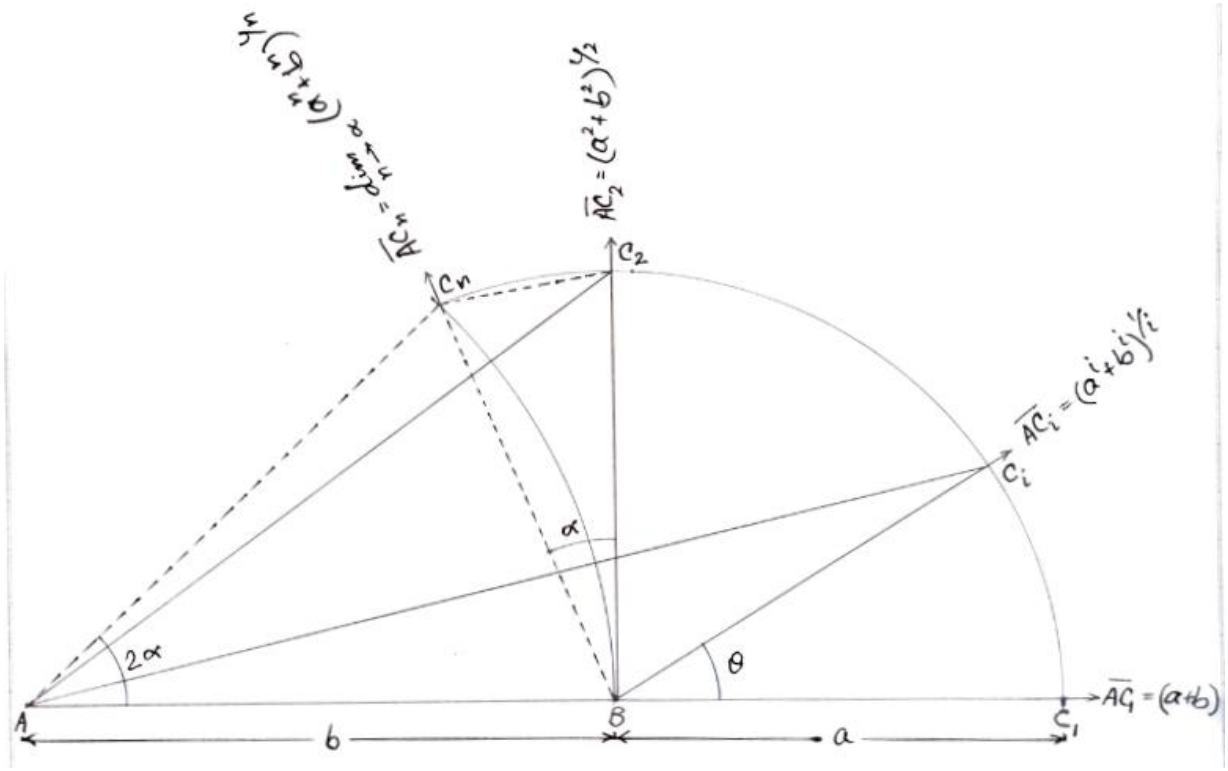
2. **Secondary Triplets:** Those triplets obtained by multiplying a primary triplet through-out by a common integer e.g., $[6, 8, 10]^2 = 2\times [3, 4, 5]^2$, and $[9, 12, 15]^2 = 3\times [3, 4, 5]^2$, etc.

2. INVESTIGATION ON n (Proof of Fermat's last theorem):-

Here I'll investigate the values of n for which the expression, $c^n = a^n + b^n$,holds intact under integer constraints.

2.1 Geometrical Representation of $a^n + b^n = c^n$:

Let us draw an arc of radius b from A which meets the line AB at B & another arc of radius a (a< b) from B that meets the extended AB at C_1 & the previous arc at C_n. Constructing, $BC_2 \perp AC_1$.



For n=1, $AC_1 = AB + BC_1$

$c = a+b$, i.e. $c^1 = a^1 + b^1$ (the trivial linear case).

For n=2, $AC_2^2 = AB^2 + BC_2^2$

$c^2 = a^2 + b^2$, => $c = (a^2+b^2)^{1/2}$ (the classic right triangle case).

For n→∞, $c^n = a^n + b^n = b^n (1+ a^n/b^n) = b^n [1+ (a/b)^n]$.

$\approx b^n$. [since, a<b so, $\lim_{n\to\infty} (a/b)^n \to 0$].

=> $c = b = AC_n$ (see the figure)

Hence, for all values of n ($1 \le n < \infty$), the resulting vector, c, traces along the continuous geometric arc $C_1C_2C_n$.

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*[1]

Here, $AB = AC_n = b$ & $BC_1 = BC_2 = BC_n = a$

Using the law of sine to the isosceles ΔABC_n we get,

$BC_n/\sin 2\alpha = AB/\sin(\pi/2 - \alpha)$

i.e. $a/\sin 2\alpha = b/\cos\alpha$ i.e. $a/2\sin\alpha = b$

So, $\sin\alpha = a/2b$

Now, a<b (our primitive assertion). If a tends b then,

$\lim_{a\to b} \sin\alpha = \lim_{a\to b} a/2b = ½ = \sin 30^o$

This means $\alpha < 30^o$ i.e. $2\alpha < 60^o$.

By the law of sine to the isosceles ΔBC_2C_n we get,

$C_2C_n/\sin\alpha = BC_2 / \sin(\pi/2 - \alpha/2)$

$C_2C_n/\sin\alpha = a/\cos\alpha/2$

$C_2C_n/2\sin\alpha/2 = a$

$C_2C_n = 2a \sin\alpha/2$;

[1] * This portion may be skipped over.

$C_2C_n < 2a \sin 15^o$, $C_2C_n < 2a.(\sqrt{3}-1)/2\sqrt{2}$, $C_2C_n < (\sqrt{3}-1)\,a/\sqrt{2}$.

Let us find the value of c for an arbitrary value i of n. In that case let BC_i makes an angle θ ($0^o \le \theta < 120^o$) with BC_1 (the reference baseline), then by the law of cosines:

$$BC_i^2 = a^2 + b^2 + 2ab\cos\theta \;\rightarrow\; c^2 = a^2 + b^2 + 2ab\cos\theta$$

If this expression mapped consistently for an arbitrary power n, raising both sides to the power of n/2 gives:

$$c^n = (a^2 + b^2 + 2ab\cos\theta)^{n/2} \qquad (7)$$

Though a and b are strictly natural numbers by our baseline consideration, the geometric factor $\cos\theta$ is continuous and dynamic (sometime integral, sometime fractional, sometime irrational). This general expression reduces to the unique Pythagorean form $c^n = a^n + b^n$ only under highly constrained conditions:

• When $\theta = 0 \Rightarrow \cos\theta = 1$:

Then, $c^n = (a^2 + b^2 + 2ab)^{n/2} = (a + b)^n \Rightarrow c = a + b$, which satisfies the equation only when n = 1.

• When $\theta = \pi/2 \Rightarrow \cos\theta = 0$:

Then, $c^n = (a^2 + b^2)^{n/2} \Rightarrow c^2 = a^2 + b^2$, which satisfies the equation exactly when n = 2.

• When $n \to \infty$:

We see that, $\theta = \pi/2+\alpha \Rightarrow \cos\theta = -\sin\alpha = -a/2b$. Substituting this into the limit expression:

$\lim_{n\to\infty} c^n = \lim_{n\to\infty} (a^2 + b^2 + 2ab\,(-a/2b))^{n/2} = \lim_{n\to\infty} (b^2)^{n/2} = \lim_{n\to\infty} b^n$

$= \lim_{n\to\infty} b^n\,(1 + (a/b)^n) = \lim_{n\to\infty} (a^n + b^n)$. [Since, $\lim_{n\to\infty} (a/b)^n = 0$ (as b>a)]

Hence, $\lim_{n\to\infty} c^n = \lim_{n\to\infty} (a^n + b^n)$.

Consequently, the mathematical expression can retain its pristine structural integer form only for n = 1 (corresponding to $\theta = 0$) and n = 2 (corresponding to $\theta = \pi/2$). No other value of θ ($0^o \le \theta < 120^o$) can result an integral value of c that can satisfy the equation (1). **Hence, no finite integer exponent n > 2 can exist for which $c^n = a^n + b^n$ holds true for natural numbers.** This completes the conceptual framework regarding **Fermat's Last Theorem**.

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*[2]

Eliminating c from eqn. (1) & eqn. (7) we get,

$$(a^n + b^n)^2 = (c^n)^2 = (c^2)^n = (a^2 + b^2 + 2ab\cos\theta)^n$$

$$\Rightarrow a^{2n} + b^{2n} + 2a^nb^n = [(a^2 + b^2) + 2ab\cos\theta]^n$$

$$= \sum_{i=0}^{n} {}^nC_i\,(a^2 + b^2)^{n-i}\,(2ab\cos\theta)^i$$

$$= (a^2 + b^2)^n + \sum_{i=1}^{n} {}^nC_i\,(a^2 + b^2)^{n-i}\,(2ab\cos\theta)^i$$

$$= \sum_{i=0}^{n} {}^nC_i\,(a^2)^{n-i}\,(b^2)^i + \sum_{i=1}^{n} {}^nC_i\,(a^2 + b^2)^{n-i}\,(2ab\cos\theta)^i$$

$$\Rightarrow \mathbf{2a^nb^n = \sum_{i=1}^{n-1} {}^nC_i\,(a^2)^{n-i}\,(b^2)^i + \sum_{i=1}^{n} {}^nC_i\,(a^2 + b^2)^{n-i}\,(2ab\cos\theta)^i.}$$

$$\mathbf{= \sum_{i=1}^{n-1} {}^nC_i\,[(a^2)^{n-i}\,(b^2)^i + (a^2 + b^2)^{n-i}\,(2ab\cos\theta)^i] + (2ab\cos\theta)^n.}$$

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3. INVESTIGATION ON a & b (The General Rule for Triplets):-

Since n = 1 is a trivial case and $n \ge 3$ has been shown to yield no integer solutions, so we drop the exponent suffix (on the right-hand top of the square bracket) from now on to represent the triplets and focus entirely on n = 2 (i.e. $a^2 + b^2 = c^2$). Suppose the hypotenuse c is exactly k units ahead of the base side b, where $k \in N$:

[2] This portion may be skipped over.

$$c = b + k \quad (8)$$

Squaring both sides yields: $c^2 = (b + k)^2 = b^2 + 2bk + k^2$

Substituting this into the Pythagorean Theorem ($c^2 - b^2 = a^2$):

$$a^2 = 2bk + k^2 \quad (9)$$

Two distinct systematic cases arise based on the parity of k:

Case I: If k is an odd integer:

Then 'a' is parameterized as:

$$a = k\,(2m + 1), \text{ where } m = 1, 2, 3, \ldots \text{ etc.} \quad (10)$$

Substituting this expression for 'a' back into equation (9) we get,

$$[k\,(2m + 1)]^2 = 2bk + k^2 \Rightarrow k^2(4m^2 + 4m + 1) = 2bk + k^2$$

$$4k^2m^2 + 4k^2m + k^2 = 2bk + k^2 \Rightarrow 4k^2m\,(m + 1) = 2bk$$

Solving for b and subsequently c:

$$b = 2km\,(m + 1) \quad (11)$$

$$c = 2km\,(m + 1) + k \quad (12)$$

Thus, for odd step values, the infinite family of triplets is given by:

$$\mathbf{[k\,(2m + 1),\ 2km\,(m + 1),\ 2km\,(m + 1) + k]} \text{ for } k = 1, 3, 5 \ldots \text{etc.} \quad (13)$$

Case II: If k is an even integer:

Then 'a' is parameterized as:

$$a = k\,(m + 1), \text{ where } m = 1, 2, 3 \ldots \text{ etc.} \quad (14)$$

Substituting this expression into equation (9):

$$[k\,(m + 1)]^2 = 2bk + k^2 \Rightarrow k^2\,(m^2 + 2m + 1) = 2bk + k^2$$

$$k^2m^2 + 2k^2m + k^2 = 2bk + k^2 \Rightarrow k^2m\,(m + 2) = 2bk$$

Solving for b and subsequently c:

$$b = km\,(m + 2)/2 \quad (15)$$

$$c = km\,(m + 2)/2 + k \quad (16)$$

Thus, for even step values, the infinite family of triplets is given by:

$$\mathbf{[k\,(m + 1),\ km\,(m + 2)/2,\ km\,(m + 2)/2 + k]} \text{ for } k = 2, 4, 6 \ldots \text{etc.} \quad (17)$$

3.1 Illustrative Numerical Examples:

If k is an odd integer:

The triplets are [k (2m + 1), 2km (m + 1), 2km (m + 1) + k], for k = 1, 3, 5 . . .

For k=1:

m=1 => [3,4,5], m=2 => [5,12,13], etc……

For k=3:

m=1 => [9,12,15], m=2 => [15,36,39], etc……

Similarly, for k = 5, 7, 9 …etc., etc.

If k is an even integer:

The triplets are [k (m + 1), km (m + 2)/2, km (m + 2)/2 + k], for k = 2, 4, 6 . . .

For k=2:

m=1 => [4,3,5], m=2 => [6,8,10], etc……

For k=4:

m=1 => [8, 6, 10], m=2 => [12, 16, 20], etc……

Similarly, for k = 6, 8, 10 …etc., etc.

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4. CONCLUSION: -

Pierre de Fermat (1601–1665) famously claimed that he had found a "truly marvelous proof "of the theorem, but that proof remains a mystery still to this day. Sir Andrew Wiles (1953–) successfully found a definitive proof in 1994, but it spans over a hundred pages, utilizing advanced modern mathematical structures that are highly complex to digest. The analytical geometric and algebraic path presented in this paper offers a direct framework designed to be highly accessible. Furthermore, the derived parameterization rules provide an elegant, closed-form mechanism to easily generate all possible Pythagorean triplets across the infinite integer domain.

Finally, only God knows whether Fermat himself meant this particular approach as his"truly marvelous proof" or not….?