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Identities of Generalization Fibonacci Sequences and Other Releted Sequences

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Abstract
In this Paper we Present identities of Generalized Fibonacci sequences. Generalized Fibonacci sequence is defined by recurrence relation$F_k = rF_{k-1} + sF_{k-2}$, $k \geq 2$ with $F_0 = p, F_1 = q$. This was introduced by Gupta, Panwar and Sikhwal. We shall use the Induction method and Binet's formula and give several interesting identities involving them.

1. Introduction

Fibonacci numbers are a popular topic for mathematical enrichment and popularization. The Fibonacci sequence is famous for possessing wonderful and amazing properties. The Fibonacci appear in numerous mathematical problems. Fibonacci composed a number text in which he did important work in number theory and the solution of algebraic equations. The bbok for which he is most famous in the "Liber abaci" published in 1202. In the third section of the book, he posed the equation of rabbit problem which is known as the first mathematical model for population growth. From the statement of rabbit problem, the famous Fibonacci numbers can be derived, 1,1,2,3,5,8,13,21,34,55,89,144,…
This sequence in which each number is the sum of the two preceding numbers has proved extremely fruitful and appears in different areas in Mathematical and Science.
The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and jacobsthal-Lucas sequence are most prominent examples of recursive sequences.

The Fibonacci sequences [6] is defined by the recurrence relation

$$F_k = F_{k-1} + F_{k-2}\ , k \geq 2 \text{ with } F_0 = 0, F_1 = 1 \qquad (1.1)$$

The Lucas sequences [6] is defined by the recurrence relation

$$L_k = F_{k-1} + F_{k-2}\ , k \geq 2 \text{ with } L_0 = 2, L_1 = 1 \qquad (1.2)$$

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.
Kalman and Mena [2]defined generalize Fibonacci sequence by

$$F_n = pF_{n-1} + qF_{n-2}\ , k \geq 2 \text{ with } F_0 = 0, F_1 = 1 \qquad (1.3)$$

Horadam[2] defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_n = H_{n-1} + H_{n-2}\ , n \geq 3 \text{ with } H_1 = r, H_2 = r + s \qquad (1.4)$$

Where r and s are arbitrary integers.
Gupta, Panwar and Sikhwal [7], introduce generalized Fibonacci sequences. They focus only two cases of sequences$\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ which generated by generalized Fibonacci sequences.

They defined related identities of generalized Fibonacci sequences consisting even and odd terms. Also they present connection formulas for generalized Fibonacci sequences, Jacobsthal sequence and Jacobsthal-Lucas sequence. In [8] Gupta and Panwar have present identities involving commen factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. In this paper, we present identities of Generalized Fibonacci sequences.

2. Generalized Fibonacci sequence

Generalized Fibonacci sequence [8] defined as

$F_k = rF_{k-1} + sF_{k-2}, k \geq 2$ with $F_0 = p, F_1 = q$ (2.1)

Where p, q, r ,and s are arbitrary integers.

For different values of p, q, r ,and s many sequences can be determined.

We focus cases of sequences $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$ which generated in (2.1).

If $r = 1, s = p = q = 2$, we get

$X_k = X_{k-1} + 2X_{k-2}$ $for\ k \geq 2$ with $V_0 = 2, V_1 = 2$ (2.2)

The first few terms of $\{X_k\}_{k\geq 0}$ are 2, 2 ,6, 10, 22, 42,…

Its Binet formula is defined by

$X_k = 2\frac{r_1^{k+1} - r_2^{k+1}}{r_1 - r_2}$ (2.3)

If $p = 2, q = 0, r = 1, q = 2$, we get

$Y_k = Y_{k-1} + 2Y_{k-2}$ $for\ k \geq 2$ with $Y_0 = 2, Y_1 = 0$ (2.4)

The first few terms of $\{Y_k\}_{k\geq 0}$ are 2,0,4,4,12,20,…

Its Binet formula is defined by

$Y_k = 4\frac{r_1^{k-1} - r_2^{k-1}}{r_1 - r_2}$ (2.5)

The Jacobsthal sequence [1], is defined the recurrence relation

$J_k = J_{k-1} + 2J_{k-2}, k \geq 2$ with $J_0 = 0, J_1 = 1$ (2.6)

Its Binet formula is defined by

$J_k = \frac{r_1^k - r_2^k}{r_1 - r_2}$ (2.7)

The Jacobsthal-Lucas sequence [1], is defined the recurrence relation

$j_k = J_{k-1} + 2j_{k-2}, k \geq 2$ with $j_0 = 0, j = 1$ (2.8)

Its Binet formula is defined by

$j_k = r_1^k + r_2^k$ (2.9)

Where r_1 and r_2 are the roots of the characteristic equation

$m^2 - m - 2 = 0$ (2.10)

The explicit formula for Generalized Fibonacci sequences is given as

$X_k = \sum_{i=0}^{|\frac{n}{2}|} \quad (n - ii)2^{i+1}$ (2.11)

$Y_k = \sum_{i=0}^{|\frac{n}{2}|} \quad (n - ii)2^{i+1} - \sum_{i=0}^{|\frac{n-1}{2}|} \quad (n - i - 1i)2^{i+1}$ (2.12)

3. Main Results

Theorem 1: For $m \geq 1$, following equality holds:

$X_i X_{j+1} + 2X_{i-1}X_j = 2X_{i+j+1}$ (3.1)

Proof: By induction, for $i = 1$

$$X_1 X_{j+1} + 2X_0 X_j = 2X_{j+1} + 4X_j$$
$$= 2(X_{j+1} + 2X_j)$$

$= 2X_{j+2}$, which is also true for $i = 1$,

Let us suppose formula is true until $i - 1$,

$$X_{i-1}X_{j+1} + 2X_{i-2}X_j = 2X_{i+j}$$

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Then

$$2X_{i+j+1} = 2(X_{i+j} + 2X_{i+j-1})$$
$$= X_{i-1}X_{j+1} + 2X_{i-2}X_j + 2(X_{i-2}X_{j+1} + 2X_{i-3}X_j)$$
$$X_{j+1}(X_{i-1} + 2X_{i-2}) + 2X_j(X_{i-2} + 2X_{i-3})$$
$$2X_{i+j+1} = X_iX_{j+1} + 2X_{i-1}X_j)$$

Hence Proof.

Corollary 1.1:

(i) $X_iX_{j+1} + 2X_{i-1}X_j = 4J_{i+j}$ (3.2)

(ii) $X_iX_{j+1} + 2X_{i-1}X_j = 4Y_{i+j+1}$ (3.3)

Theorem 2: For $m \geq 1$, following equality holds:

$Y_iY_{j+1} + 2Y_{i-1}Y_j = 4Y_{i+j-1}$ (3.4)

Proof: By Binet's formula

$$Y_iY_{j+1} + 2Y_{i-1}Y_j = 4Y_{i+j-1}$$
$$= \frac{16}{9}(r_1^{i-1} - r_2^{i-1})(r_1^j - r_2^j) + \frac{32}{9}(r_1^{i-2} - r_2^{i-2})(r_1^{j-1} - r_2^{j-1})$$
$$= \frac{16}{9}\{(r_1^{i+j-1} + r_1^{i+j-3}) + (r_2^{i+j-1} + 2r_2^{i+j-3}) - (r_1^{i-1}r_2^j + 2r_1^{i-2}r_2^{j-1}) - (r_1^jr_2^{i-1} + 2r_1^{j-1}r_2^{i-2})\}$$
$$= \frac{16}{9}\{(r_1^{i+j-1} + r_1^{i+j-2}) + (r_2^{i+j-1} - 2r_2^{i+j-2})\}$$
$$= \frac{16}{9}\left\{r_1^{i+j-1}\left(1 + \frac{1}{r_1}\right) + r_2^{i+j-1}\left(1 - \frac{2}{r_2}\right)\right\}$$
$$= \frac{16}{9}\{r_1^{i+j} - r_2^{i+j}\}$$
$$= 4Y_{i+j-1}$$

Hence Proof.

Corollary 2.1:

$(i) Y_iX_{j+1} + 2Y_{i-1}Y_j = 16J_{i+j}$ (3.5)

(ii) $Y_iY_{j+1} + 2X_{i-1}X_j = 8Y_{i+j-1}$ (3.6)

Lemma 2.2: If j is a natural number and $i = j + 1$, then

$\frac{r_1^jr_2^i - r_1^ir_2^j}{r_1 - r_2} = (-1)^i r_1$ (3.7)

Theorem 3: If j is a natural number and $i = j + 1$, then following equality hold

$Y_iX_{j-1} - Y_{i-1}Y_j = (-1)^i r_1^{j+2}$ (3.8)

Proof: By Binet's formula

$$Y_iX_{j-1} - Y_{i-1}Y_j = \frac{16}{9}\{(r_1^{i-1} - r_2^{i-1})(r_1^{j-2} - r_2^{j-2}) - (r_1^{i-2} - r_2^{i-2})(r_1^{j-1} - r_2^{j-1})\}$$
$$= \frac{16}{9}\{r_1^ir_2^j(r_1^{-2}r_2^{-1} - r_1^{-1}r_2^{-2}) - r_1^jr_2^i(r_1^{-1}r_2^{-2} - r_1^{-2}r_2^{-1})\}$$
$$= \frac{16}{9}\left\{(r_1^jr_2^i - r_1^ir_2^j)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right\}$$
$$= 4\left(\frac{r_1^jr_2^i - r_1^ir_2^j}{r_1 - r_2}\right)$$

$$= (-1)^i r_1^{j+2}$$

Hence Proof.

Theorem 4: If j is a natural number and $i = j + 1$, then following equality hold

$$X_i X_{j-1} - X_{i-1} X_j = (-1)^i r_1^{j+2} \qquad (3.9)$$

4. Conclusion

In this paper we have derived some identities of generalized Fibonacci sequences through Binet's formula and Induction method.

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