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Common Fixed-Point Theorem of An Infinite Sequence of Mappings in Hilbert Space

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Abstract:
The aim of this present paper is to obtain a common fixed point for an infinite sequence of mappings on Hilbert space. Our purpose here is to generalize the previous result [1].

Key words and phrases: Common fixed point, Infinite Sequence of mappings, Hilbert space

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Introduction:-

In 1991, Koparde and Waghmode have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\|T_i x - T_j y\|^2 \le a\left(\|x - T_i x\|^2 + \|y - T_j y\|^2\right) \quad \text{(A)}$$

$$\text{for all } x, y \in S \text{ and } x \ne y; 0 \le a < 1/2$$

Letter in 1998, Pandhare and Waghmode, have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\|T_i x - T_j y\|^2 \le a\|x - T_i x\|^2 + b\left(\|x - T_i x\|^2 + \|y - T_j y\|^2\right) \quad \text{(B)}$$

$$\text{for all } x, y \in S \text{ and } x \ne y; 0 \le a, 0 \le b < 1 \text{ and } a + 2b < 1.$$

This result is generalizes by T.Veerapandi & S. Anil Kumar in 1999 & the new condition is

$$\|T_i x - T_j y\|^2 \le a\|x - y\|^2 + b\left(\|x - T_i x\|^2 + \|y - T_j y\|^2\right) + \frac{c}{2}\left(\|x - T_i x\|^2 + \|y - T_j y\|^2\right) \quad \text{(C)}$$

$$\text{for all } x, y \in S \text{ and } x \ne y; 0 \le a, 0 \le b < 1 \text{ and } a + 2b < 1$$

In 2005, V.H. Badshah and G. Meena, have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\|T_i x - T_j y\| \le \alpha \frac{\|x - T_i x\| . \|y - T_j y\|}{\|x - y\|} + \beta\|x - y\| \quad \text{(D)}$$

$$\text{for all } x, y \in S \text{ and } x \ne y; \alpha \ge 0, \beta \ge 0 \ \& \ \alpha + \beta < 1$$

In 2014, A. K. Sharma, V. H. Badshah and V. K.Gupta have proved common fixed point theorem for the sequence $\{T_n\}_{n=1}^{\infty}$ of mappings satisfying the condition

$$\|T_i x - T_j y\| \le \alpha \frac{\|x - T_i x\|^2 + \|y - T_j y\|^2}{\|x - T_i x\| + \|y - T_j y\|} + \beta\|x - y\| \quad \text{(E)}$$

$$\text{for all } x, y \in S \text{ and } x \ne y; \alpha \ge 0, \beta \ge 0 \ \& \ 2\alpha + \beta < 1$$

Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point.

Main Result

We proved fixed point theorem for the infinite sequence $\{T_n\}_{n=1}^{\infty}$ to generalize our previous results.

Theorem. Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be an infinite sequence of mappings satisfying the following condition

$$\|T_i x - T_j y\| \le \left(\alpha + \beta \frac{\|y - T_i x\|}{1 + \|x - y\|} \right) \|y - T_j y\| \qquad \text{(F)}$$

for all x, y in S and $x \neq y$ also $\alpha \ge 0,\ 1 \ge \beta \ge 0$ and $\alpha + 2\beta < 1$.

Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point

Proof. Let S be a closed subset of a Hilbert space H and $\{T_n\}_{n=1}^{\infty} : S \to S$ be an infinite sequence of mappings. Let $x_0 \in S$ be any arbitrary point in S.

Define a sequence $\{x_n\}_{n=1}^{\infty}$ in S by

$$x_{n+1} = T_{n+1} x_n ,\ for\ n = 0,1,2, ...$$

For any integer $n \ge 1$.

$$\|x_{n+1} - x_n\| = \|T_{n+1} x_n - T_n x_{n-1}\|$$

$$\le \left(\alpha + \beta \frac{\|x_{n-1} - T_{n+1} x_n\|}{\|x_n - x_{n-1}\|} \right) \|x_{n-1} - T_n x_{n-1}\|$$

$$\le \left(\alpha + \beta \frac{\|x_{n-1} - x_{n+1}\|}{\|x_n - x_{n-1}\|} \right) \|x_{n-1} - x_n\|$$

$$\le \alpha \|x_{n-1} - x_n\| + \beta \|x_{n-1} - x_{n+1}\|$$

$$\le \alpha \|x_{n-1} - x_n\| + \beta \|x_{n-1} - x_n\| + \beta \|x_n - x_{n+1}\|$$

$$i.e. \quad \|x_{n+1} - x_n\| \le \alpha \|x_{n-1} - x_n\| + \beta \|x_{n-1} - x_n\| + \beta \|x_n - x_{n+1}\|$$

$$\Rightarrow (1 - \beta) \|x_{n+1} - x_n\| \le (\alpha + \beta) \|x_n - x_{n-1}\|$$

$$\Rightarrow \|x_{n+1} - x_n\| \le \frac{(\alpha + \beta)}{1 - \beta} \|x_n - x_{n-1}\|$$

$$If\ \ k = \frac{\alpha + \beta}{1 - \beta}\ \ then\ k < 1.$$

$$\|x_{n+1} - x_n\| \le k \|x_n - x_{n-1}\|$$

$$\le k \|x_n - x_{n-1}\| \le k^2 \|x_{n-1} - x_{n-2}\| \le k^3 \|x_{n-2} - x_{n-3}\| \le ... \le k^n \|x_1 - x_0\|$$

$$i.e.\ \ \|x_{n+1} - x_n\| \le k^n \|x_1 - x_0\| \quad for\ all\ \ n \ge 1\ \ is\ \ integer.$$

Now for any positive integer $m \ge n \ge 1$

$$\|x_n - x_m\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{m-1} - x_m\|$$

$$\le k^n \|x_1 - x_0\| + k^{n+1} \|x_1 - x_0\| + ... + k^{m-1} \|x_1 - x_0\|$$

$$\le k^n \|x_1 - x_0\| \left(1 + k + \ldots + k^{m-n-1}\right)$$

$$i.e. \|x_n - x_m\| \le \left(\frac{k^n}{1-k}\right)\|x_1 - x_0\| \to 0 \text{ as } n \to \infty \ (k<1)$$

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since S is a closed subset of a Hilbert space H, so $\{x_n\}_{n=1}^{\infty}$ converges to a point u in S.

Now we will show that u is common fixed point of infinite sequence $\{T_n\}_{n=1}^{\infty}$ of mappings from S into S.

Suppose that $T_n u \ne u$ for all n.

Consider for any positive integer m ($\ne n$)

$$\|u - T_m u\| \le \|u - x_n\| + \|x_n - T_m u\|$$
$$= \|x_n - T_m u\|$$
$$= \|T_n x_{n-1} - T_m u\|$$
$$\le \left(\alpha + \beta \frac{\|u - T_n x_{n-1}\|}{\|x_{n-1} - u\|}\right)\|u - T_m u\|$$
$$\le \left(\alpha + \beta \frac{\|u - x_n\|}{\|x_{n-1} - u\|}\right)\|u - T_m u\|$$
$$\le \alpha\|u - T_m u\| + \beta \frac{\|u - x_n\|}{\|x_{n-1} - u\|}\|u - T_m u\|$$

$$i.e. \quad \|u - T_m u\| \le \alpha\|u - T_m u\| + \beta \frac{\|u - x_n\|}{\|x_{n-1} - u\|}\|u - T_m u\|$$

$$\Rightarrow \|u - T_m u\| \le \frac{\beta}{1-\alpha}\frac{\|u - x_n\|}{\|x_{n-1} - u\|}\|u - T_m u\| \quad \to 0 \text{ as } n \to \infty$$

So $\|u - T_m u\| \le 0.$

Hence $u = T_m u$ *and so* $u = T_n u$ *for all* $n.$

Hence u is a common fixed point of infinite sequence $\{T_n\}_{n=1}^{\infty}$ of mappings.

Uniqueness

Suppose that there is $u \ne v$ *such that* $T_n v = v$ *for all* $n.$

Consider $\|u - v\| = \|T_n u - T_n v\|$

$$\le \left(\alpha + \beta \frac{\|v - T_n u\|}{\|u - v\|}\right)\|v - T_n v\| \le 0$$

`` $i.e.$ $\|u - v\| \le 0$

$$\Rightarrow \|u - v\| = 0$$

Thus $u = v.$

Hence fixed point is unique.

Example. Let X = [0, 1], with Euclidean metric d .Then {X, d} is a Hilbert space with the norm defined by $d(x,y)=\|x-y\|$.

Let $\{x_n\}_{n=1}^{\infty}=\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ be the sequence in X and let $\{T_n\}_{n=1}^{\infty}$ be the infinite sequence of mappings such that

$$x_{n+1}=T_{n+1}x_n\,,\ for\,n\ =\ 0,\ 1,\ 2,\ldots$$

Taking $x=\frac{1}{2^n}$ *and* $y=\frac{1}{2^{n-1}};x\neq y.Also\ i\ =\ n+1\ \ and\ \ j=n.$

Then from (F) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in *X*, which is converges in *X* also it has a common point in *X*.

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