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A Generalized an Extended Fractional Mellin Transform and Parsevals Identity

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ABSTRACT

Integral transform is one of the techniques in the function transformation methods. Integral transforms have been interesting tools for solving different problems arising in applied mathematics, mathematical physics and engineering science for at least two centuries. We have studied an extended fractional Mellin transform in the generalized sense. For this testing space E and its dual space E* are considered. We have investigated inversion formula by using inversion of classical Mellin transform and prove Parseval's Identity for an extended fractional Mellin transform. Also discussed some results of this transform.

Keywords: Testing space, extended fractional Mellin transform, Fourier transform, inversion of classical Mellin transform, Mellin transform, Parseval's Identity

INTRODUCTION:

Integral transform is one of the techniques in the function transformation methods. Integral transforms have been interesting tools for solving different problems arising in applied mathematics, mathematical physics and engineering science for at least two centuries.

Fractional Mellin transform is a generalization of the ordinary Mellin transform. The basic theoretical properties of Mellin-type fractional integrals, known as generalizations of the Hadamard-type fractional integrals [5]. Fractional Mellin transform becomes used in visual navigation since it can control the range of rotation and scaling. Fractional Mellin analysis was developed in which the so-called Hadamard- type integrals, which represent the appropriate extensions of the classical Riemann-Liouville and Weyl fractional integrals [5]. Fractional Mellin transform used mainly in image encryption.

In this paper we have discussed on inversion formula by using inversion of classical Mellin transform and prove Parseval's Identity for an extended fractional Mellin transform.

Fractional Mellin Transform

A classical theory of the Mellin transform is extended to a generalized function space which is a dual of testing function space developed by A. H. Zemanian [74].

One dimensional fractional Mellin transform , with parameter θ of f(u) is defined as,

$$\mathrm{FRMT}[f(u)] = F_\theta(r) = \int_{-\infty}^{\infty} f(u) K_\theta(u,r)\,du,$$

where the kernel $K_\theta(u,r) = u^{\frac{2\pi i r}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(r^2+\log^2 u)}$, $0 < \theta \le \frac{\pi}{2}$.

It has many applications in areas of image encryption, Spectrum analysis etc. [81, 58].

1.1 The testing function space $E(R^n)$

An infinitely differentiable complex valued function ∅ on R^n belongs to $E(R^n)$ if for each compact set $I \subset S_a$,

where,

$S_a = \{u : u \in R^n, |u| \le a, a > 0\}$

$$\gamma_{E_m}(\emptyset) = \sup_{u \in I} |D_u^m \emptyset(u)| < \infty$$

`Thus, $E(R^n)$ will denote the space of all $\emptyset \in E(R^n)$ with support contained in S_a. Moreover, we say that f is an extended fractional Mellin transformable, if it is member of E^*, the dual space of E.

1.2 Theorem

To prove for each $r \in R^n$ and $0 < \theta \le \frac{\pi}{2}$ the function $K_\theta(u, r)$ is belongs to $E(R^n)$ as function of u,

where $K_\theta(u,r) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, u^{-ir\operatorname{cosec}\theta - 1}$

1.3 Definitions

1.3.1 Definition of generalized fractional Fourier transform (FRFT)

The distributional fractional Fourier transform of $f(x) \in E^*(R^n)$, $0 < \theta \le \frac{\pi}{2}$ is defined by,

$FRFT\{f(x)\} = F_\theta(p) = \langle f(x), K_\theta(x,p)\rangle$,

where $K_\theta(x,p) = \sqrt{\frac{1-i\cot\theta}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+p^2)\cos\theta - 2(xp)]}$

where right hand side is meaningful i.e. , $K_\theta(x,p) \in E$ and $f \in E^*$.

1.3.2 Definition of generalized fractional Mellin transform (FRMT)

A classical theory of the Mellin transform is extended to a generalized function space which is a dual of testing function space developed by A. H. Zemanian [74].

The distributional fractional Mellin transform of $f(u) \in E^*(R^n)$, $0 < \alpha \le \frac{\pi}{2}$ is defined by,

$FRMT\{f(u)\} = F_\alpha(r) = \langle f(u), K_\alpha(u,r)\rangle$,

One dimensional fractional Mellin transform , with parameter θ of f(u) is defined as,

$$FRMT[f(u)] = F_\alpha(r) = \int_{-\infty}^{\infty} f(u) K_\alpha(u,r)\,du,$$

where the kernel$K_\alpha(u,r) = u^{\frac{2\pi i r}{\sin\theta} - 1} e^{\frac{\pi i}{\tan\alpha}(r^2 + \log^2 u)}$, $0 < \alpha \le \frac{\pi}{2}$.

It has many applications in areas of image encryption, Spectrum analysis etc. [81, 58].

where right hand side is meaningful i.e. , $K_\alpha(u,r) \in E$ and $f \in E^*$.

1.3.3 Definition of generalized an extended fractional Mellin transform

An extended fractional Mellin transform of order α [3] is defined as

$$M_\alpha(r) = EFRMT[f(u)] = \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_0^\infty f(u) K_\alpha(u,r)\,du$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_0^\infty f(u) u^{-ir\operatorname{cosec}\alpha - 1}\,du, \qquad \text{where } K_\alpha(u,r) = u^{-ir\operatorname{cosec}\alpha - 1}$$

$$M_\alpha(r) = EFRMT[f(u)] = A\int_0^\infty f(u) u^{-ir\operatorname{cosec}\alpha - 1}\,du \qquad \text{where } A = \sqrt{\frac{1-j\cot\alpha}{2\pi}}$$

where right hand side is meaningful i.e. , $K_\alpha(u,r) \in E$ and $f \in E^*$.

1.4 Properties of an extended fractional Mellin transform

If $M_{\theta_1}(r) = EFRMT[f_1(u)](r)$ $M_{\theta_2}(r) = EFRMT[f_2(u)](r)$

S.N.	Property	Formula
1	**Linearity Property**	$EFRMT[A_1 f_1(u) + A_2 f_2(u)](r) = A_1 EFRMT[f_1(u)](r) + A_2 EFRMT[f_2(u)](r)$
2	**Differential Property**	$EFRMT[f'(u)](r) = -r\left(r + \frac{\sin\theta}{i}\right) EFRMT\left(\frac{1}{u} f(u)\right)(r)$

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3	**Scaling Property**	$\mathrm{EFRMT}[f(au)](r) = a^{u^{\frac{ir}{\sin\theta}}}\mathrm{EFRMT}[f(u)](r)$
4	**Modulation Property-I**	$\mathrm{EFRMT}[f(u)\cos au](r) = \frac{A}{2}\{\mathrm{EFRMT}[e^{iau}f(u)](r) + \mathrm{EFRMT}[e^{-iau}f(u)](r)\}$
5	**Modulation Property-II**	$\mathrm{EFRMT}[f(u)\sin au](r) = \frac{A}{2}\{\mathrm{EFRMT}[e^{iau}f(u)](r) - \mathrm{EFRMT}[e^{-iau}f(u)](r)\}$

2. An inversion formula for an extended fractional Mellin transform

If an extended fractional Mellin transform of order α [3] is defined as

$$M_\alpha(r) = \mathrm{EFRMT}[f(u)](r) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_0^\infty f(u)K_\alpha(u,r)du$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_0^\infty f(u)u^{-ir\operatorname{cosec}\alpha-1}du, \quad \text{where } K_\alpha(u,r) = u^{-ir\operatorname{cosec}\alpha-1}$$

then by an inversion it is possible to recover $f(u)$ by means of the inversion formula

$$f(u) = \frac{1}{2\pi}\int_0^\infty M_\alpha(r)\tilde{K}_\alpha(u,r)dr$$

where, $\tilde{K}_\alpha(u,r) = \frac{-i}{2\pi\sin\alpha}u^{\frac{ir}{\sin\alpha}}$

Proof: We have

$$M_\alpha(r) = \mathrm{EFRMT}[f(u)] = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_0^\infty f(u)K_\alpha(u,r)du$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_0^\infty f(u)u^{-ir\operatorname{cosec}\alpha-1}du, \quad \text{where } K_\alpha(u,r) = u^{-ir\operatorname{cosec}\alpha-1}$$

$$M_\alpha(r) = \{M[f(u)]\}(\xi) = G(\xi) \tag{1}$$

where $\xi = \frac{-ir}{\sin\alpha} \Rightarrow r = i\,\xi\sin\alpha, \quad d\xi = \frac{-i}{\sin\alpha}dr$

$$M_\alpha(i\xi\sin\alpha) = \{M[f(u)]\}(\xi) \tag{2}$$

$$G(\xi) = M_\alpha(i\xi\sin\alpha) = \{M[f(u)]\}(\xi)$$

The right hand side is Fractional Mellin transform of $f(u)$ with argument ξ. Invoking fractional Mellin inversion we can write,

$$f(u) = \frac{1}{2\pi}\int_0^\infty G(\xi)u^{-\xi}d\xi \tag{3}$$

Now putting value of $f(u)$

Putting the value of $G(\xi), \xi, d\xi$.

$$f(u) = \frac{\sqrt{1-i\cot\alpha}}{(2\pi)^{3/2}}\int_0^\infty M_\alpha(i\xi\sin\alpha)u^{ir/\sin\alpha}\left(\frac{-i}{\sin\alpha}\right)dr$$

$$f(u) = \int_0^\infty M_\alpha(r)\tilde{K}_\alpha(u,r)dr \qquad where, \quad \tilde{K}_\alpha(u,r) = \frac{\sqrt{1-i\cot\alpha}}{i(2\pi)^{3/2}\sin\alpha}u^{ir/\sin\alpha}$$

3. Parseval's Identity

i) $M_\alpha(r) = \mathrm{EFRMT}[f(u)](r)$ *and* $G_\alpha(r) = \mathrm{EFRMT}[g(u)](r)$ then $\int_0^\infty f(u)\overline{g(u)}du = \frac{i\sqrt{1+i\cot\alpha}}{(2\pi)^{3/2}\sin\alpha}\int_0^\infty \overline{G_\alpha(r)}\ F_\alpha\{uf(u)\}dr$

Proof:

$$M_\alpha(r) = EFRMT[f(u)](r) = = \sqrt{\frac{1-icot\alpha}{2\pi}}\int_0^\infty f(u)u^{-ircosec\alpha-1}du,$$

By using inversion formula for an extended fractional Mellin transform

$$g(u) = \frac{\sqrt{1-icot\alpha}}{i(2\pi)^{3/2}sin\alpha}\int_0^\infty G_\alpha(r)u^{ir/_{sin\alpha}}dr$$

Taking complex conjugate of the above term

$$\overline{g(u)} = \frac{-\sqrt{1+icot\alpha}}{i(2\pi)^{3/2}sin\alpha}\int_0^\infty \overline{G_\alpha(r)}u^{-ir/_{sin\alpha}}dr$$

Consider $\int_0^\infty f(u)\overline{g(u)}du$

$$= \int_0^\infty f(u)du\left\{\frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}\int_0^\infty \overline{G_\alpha(r)}u^{-ir/_{sin\alpha}}dr\right\}$$

$$= \int_0^\infty \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}\overline{G_\alpha(r)}\,dr\left\{\int_0^\infty uu^{-1}u^{-ir/_{sin\alpha}}f(u)du\right\}$$

$$= \int_0^\infty \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}\overline{G_\alpha(r)}\,dr\left\{\int_0^\infty uu^{-1}u^{-ir/_{sin\alpha}}f(u)du\right\}$$

$$= \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}\int_0^\infty \overline{G_\alpha(r)}\,dr\left\{\int_0^\infty uu^{-ircosec\alpha-1}f(u)du\right\}$$

$$= \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}\int_0^\infty \overline{G_\alpha(r)}\;F_\alpha\{uf(u)\}dr$$

$$\int_0^\infty f(u)\overline{g(u)}du = \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}\int_0^\infty \overline{G_\alpha(r)}\;F_\alpha\{uf(u)\}dr \qquad (4)$$

ii) To prove

$$\int_0^\infty |f(u)|^2 = A\int_0^\infty |F_\alpha(r)|^2\,dr \qquad \text{where}\quad A = \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}$$

Proof: let $f(u) = g(u)$

$$F_\alpha(r) = G_\alpha(r)$$
$$\overline{F_\alpha(r)} = \overline{G_\alpha(r)}$$

By using equation (4)

$$\int_0^\infty |f(u)|^2 = \int_0^\infty f(u)\overline{f(u)}\;du = \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}\int_0^\infty F_\alpha(r)\overline{F_\alpha(r)}dr$$

$$= A\int_0^\infty |F_\alpha(r)|^2\,dr \qquad \text{where}\quad A = \frac{i\sqrt{1+icot\alpha}}{(2\pi)^{3/2}sin\alpha}$$

Conclusion:

In this paper we have discussed the generalized form of an extended fractional Mellin transform. We have investigated inversion formula and prove Parseval's Identity for an extended fractional Mellin transform.

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