International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 • Website: www.ijfmr.com • Email: editor@ijfmr.com

Linear Differential Equations

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Abstract

One basic type of differential equations that is essential to mathematics, physics, engineering, and other scientific fields are linear differential equations. These equations are characterized by the linearity of the dependent variable and its derivatives, ensuring that the superposition principle applies.

The general form of linear differential equation of an n-th order is expressed as:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x),$$

Where y is the function of the variable x, and $a_0(x)$, $a_1(x)$,…, $a_n(x)$ are functions of x, and f(x) is a given function (also known as the non-homogeneous term if it's non-zero). When $f(x) = 0$ the equation is termed homogeneous; otherwise, it is non-homogeneous.

Linear differential equations are classified further based on their order and the nature of the coefficients, leading to special cases such as constant coefficient linear equations and systems of linear differential equations. The solutions to these equations can often be represented in terms of exponential, trigonometric, or special functions, depending on the context.

Key methods for solving linear differential equations include separation of variables, integrating factors, characteristic equations, variation of parameters, and the use of Laplace transforms.

This field continues to grow, with modern research delving into the interplay between linear differential equations and areas such as control theory, quantum mechanics, and signal processing, underscoring their enduring significance in science and mathematics.

This abstract highlight the theoretical importance and practical applications of linear differential equations, as well as the analytical and computational techniques used to solve them.

Keywords: Integrating factor, Linear Differential equation in y, Homogeneous and Non-homogeneous Linear differential equations with constant coefficients, higher order linear differential equations with constant coefficients and non constant coefficients, variation of parameters.

INTRODUCTION:

Linear differential equations are an essential branch of mathematics that describe relationships between a function and its derivatives. These equations are used to model and analyse systems where the change in a quantity depends linearly on the quantity itself and possibly on external influences.

Linear differential equations provide a foundation for understanding complex systems, enabling predictions and optimizations in various scientific and engineering domains. They continue to be a critical tool for advancing knowledge and technology.

Linear Differential Equation:

Differential equations are mathematical expressions which include functions and their derivatives. When the dependent variable and its derivatives appear in a linear manner (i.e., without powers, products, or nonlinear functions), the equation is called a **linear differential equation**.

General Form:

For a single dependent variable $y(x)$, the general form of an n-th order linear differential equation is:

$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$ ……….(1)

Can also be written as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

where:

- $y(x)$: Dependent variable (unknown function)
- $y', \ldots, y^{(n)}$: Derivatives of y
- $a_0(x), a_1(x), \ldots : a_n(x)$: Coefficients, which are functions of the independent variable x
- $f(x)$: A given function (source or forcing term)

Linear differential equations can be classified into two categories:

1. **Homogeneous linear differential equations**: In equation (1) if $f(x) = 0$, then the equation is said to be homogeneous.
2. **Non-homogeneous linear differential equations**: In equation (1) if $f(x) \neq 0$, then the equation is said to be non-homogeneous.

Solution Methods:

- **For homogeneous linear equations**, the solution generally involves finding the characteristic equation and solving for the roots. These roots lead to the general solution.
- **For non-homogeneous linear equations**, methods such as undetermined coefficients, variation of parameters, or the Laplace transform are used to find a particular solution, combined with the homogeneous solution to form the general solution.

Linear Differential equation in y (LDE in y)

Linear differential equations expressed in terms of y (the dependent variable) are among the most common forms encountered in mathematics and applications.

A **linear differential equation in y** is one in which y and its derivatives appear linearly. In general, an n-th order linear differential equation in y is written as:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

Linearity:

- Each term involving y or its derivatives appears as a first power.
- No products, powers, or nonlinear functions of y or y' are allowed.

Homogeneous vs. non-homogeneous:

- **Homogeneous**: $f(x) = 0$.

For example: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

- **Non-Homogeneous**: $f(x) \neq 0$.

For example: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$

The Operator D in Linear Differential Equations

The operator D is commonly used in the study of differential equations to simplify notation. It represents the differentiation operator with respect to the independent variable. Specifically:

$$D = \frac{d}{dx}$$

When applied to a function $y(x)$, the operator D acts as:

$$Dy = \frac{dy}{dx}$$

Similarly, higher powers of D correspond to higher-order derivatives:

- $D^2y = \frac{d^2y}{dx^2}$
- $D^3y = \frac{d^3y}{dx^3}$, ...$D^ny = \frac{d^ny}{dx^n}$

Types of Linear Differential Equations in y

1. First-Order Linear Differential Equations

General form:

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are function in x.

Working rule to solve First order LDE in y:

Step 1: Compare the given equation with standard form $\frac{dy}{dx} + P(x)y = Q(x)$

Step 2: Note down P , Q values. Find Integrating factor: $\mathbf{I.F} = \mathbf{e^{\int P(x)dx}}$

Step 3: Then General solution will be $\mathbf{y\,(I.F) = \int Q(x)(I.F)dx + c}$

Example: Solve $\frac{dy}{dx} + 5y = e^{-x}$

Sol: Given LDE $\frac{dy}{dx} + 5y = e^{-x}$ -------- (1)

Here $P(x) = 5,\ Q(x) = e^{-x}$

Integrating Factor (I.F) $= e^{\int P(x)dx}$

$= e^{\int 5dx} = e^{5x}$

Now General Solution (G.S) of (1) is

$$\mathbf{y\,(I.F) = \int Q(x)(I.F)dx + c}$$

$$\Rightarrow y\,e^{5x} = \int e^{-x}e^{5x}dx + c$$

$$\Rightarrow y\,e^{5x} = \int e^{4x}dx + c$$

$$\Rightarrow y\,e^{5x} = \frac{e^{5x}}{5} + c \quad (\because \int e^{ax}dx = \frac{e^{ax}}{a} + c)$$

$$\Rightarrow 5y = 1 + 5ce^{-5x}$$

2. Higher-Order Linear Differential Equation (with constant coefficients) General form:

$$P_n \frac{d^n y}{dx^n} + P_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_1 \frac{dy}{dx} + P_0 y = f(x)$$

where $P_0, P_1, \ldots \ldots \ldots . P_n$ are constants or real values and $f(x)$ is a function of x.

Using the D-operator, this can be expressed as:

$$(P_n D^n + P_{n-1} D^{n-1} + \cdots + P_1 D + P_0) y = f(x)$$

The differential equation is thus transformed into an algebraic equation involving the operator D.

Solution of Homogeneous Equations

For a homogeneous equation:

$P_n \frac{d^n y}{dx^n} + P_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_1 \frac{dy}{dx} + P_0 y = 0$ or $(P_n D^n + P_{n-1} D^{n-1} + \cdots + P_1 D + P_0) y = 0$

Here $f(D) = P_n D^n + P_{n-1} D^{n-1} + \cdots + P_1 D + P_0$ is called Auxiliary operator or characteristic operator

and the equation $f(m) = 0$ is called characteristic equation, where m represents the roots of the characteristic equation.

Types of Roots and Their Corresponding Solutions:

1. **Distinct Real Roots**:

If the roots are $m_1, m_2, m_3, \ldots \ldots \ldots . ., m_n$ then the solution is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}$

2. **Repeated Real Roots**:

If a root $m_1$repeated r times and remaining $m_{r+1,} m_{r+2,} \ldots . ., m_n$ are distinct, then the solution is:

$$y = (C_1 + C_2 x + C_3 x^2 + \cdots \ldots . . + C_r x^r) e^{m_1 x} + C_{r+1} e^{m_{r+1} x} + \cdots \ldots \ldots \ldots . + C_n e^{m_n x}$$

3. **Complex Roots**:

If the roots are $a \pm ib$ then the solution is: $y = e^{ax}(C_1 Cosbx + C_2 Sinbx)$

4. Irrational roots:

If the roots are $a \pm \sqrt{b}$ then the solution is: $y = e^{ax}(C_1 Cosh\sqrt{b}x + C_2 Sinh\sqrt{b}x)$

Solution of Non-Homogeneous Equations

For non-homogeneous equations:

$P_n \frac{d^n y}{dx^n} + P_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_1 \frac{dy}{dx} + P_0 y = f(x)$ or $(P_n D^n + P_{n-1} D^{n-1} + \cdots + P_1 D + P_0) y = f(x)$ ……(1) then the solution is the sum of:

1. **The Complementary Solution y_c**:

- The associated homogeneous equation's solution

2. **The Particular Solution y_p**:

- Associated solution that satisfies the non-homogeneous equation.
- Some common methods to find $y_p$include:

❖ **Method of Undetermined Coefficients**: Assumes a specific form for y_p based on $f(x)$.

If $f(x) = e^{ax}$, then

for $f(a) \neq 0$, $\boldsymbol{y_p} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

for $f(a) = 0$, $\boldsymbol{y_p} = \frac{1}{f(D)} e^{ax}$

$= \frac{1}{\varphi(D)} \frac{x^k}{k!} e^{ax}$, where k = the number of times a repeated

$= \frac{1}{\varphi(a)} \frac{x^k}{k!} e^{ax}$, if $\varphi(a) \neq 0$

Example: Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$.

Sol: Given **DE** $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$ .... (1)

Eq (1) in Operator form $(D^2 - 4D + 4)y = e^{2x}$

Here $f(D) = D^2 - 4D + 4$, and $Q(x) = e^{2x}$

Then Auxiliary Equation is $f(m) = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2, 2$ (Repeated roots)

$\therefore\ y_c = (C_1 + C_2x)e^{2x}$

Now $y_p = \frac{1}{f(D)} Q = \frac{1}{D^2-4D+4} e^{2x} = \frac{x^2}{2!} e^{2x}$ [$\because f(D) = f(2) = 0, root\ \ repated\ two\ times$]

Then G.S of (1) is $y = y_c + y_p$

$\Rightarrow\ y = (C_1 + C_2x)e^{2x} + \frac{x^2}{2} e^{2x}$

➢ If $f(x) = Sinbx\ or\ Cosbx$

$\boldsymbol{y_p} = \frac{1}{f(D)} Sinbx = \frac{1}{\varphi(D^2)} Sinbx = \frac{1}{\varphi(-b^2)}\ Sinbx$, if $\varphi(-b^2) \neq 0$

$\boldsymbol{y_p} = \frac{1}{f(D)} Sinbx = \frac{1}{D^2+b^2} Sinbx = -\frac{x}{2b}\ cosbx, if\ \varphi(-b^2) = 0$

$\boldsymbol{y_p} = \frac{1}{f(D)} Cosbx = \frac{1}{D^2+b^2} Cosbx = \frac{x}{2b}\ Sinbx, if\ \varphi(-b^2) = 0$

➢ If $f(x) = x^k$

$$\boldsymbol{y_p} = \frac{1}{f(D)} x^k = \frac{1}{1+\varphi(D)} x^k$$

$$= \left(1+\varphi(D)\right)^{-1} x^k = \left[1 + \varphi(D) + \left(\varphi(D)\right)^2 + \left(\varphi(D)\right)^3 + \cdots \ldots\right] x^k$$

➢ If $f(x) = e^{ax}V$, where V is a function of x

$\boldsymbol{y_p} = \frac{1}{f(D)} e^{ax}V = e^{ax} \frac{1}{f(D+a)} V$

❖ **Variation of Parameters**: A general method that works for any $f(x)$.

Working rule:

1. Reduce the differential equation to the standard form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$
2. Solve the homogeneous equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ to get complementary function

$y_c = c_1u + c_2v$

3. Take Particular integral of (1) as $y_p = Au + Bv$, where A and B are functions of x
4. Find $uv' - u'v$
5. Find A and B using $A = -\int \frac{vR}{uv'-u'v}$, B= $\int \frac{uR}{uv'-u'v}$
6. Write down General Solution of (1): $y = c_1u + c_2v + Au + Bv$

3. Higher-Order Linear Differential Equation (with non-constant coefficients)

General form:

$$\mathrm{a_n(x)}\frac{\mathrm{d^ny}}{\mathrm{dx^n}} + \mathrm{a_{n-1}(x)}\frac{\mathrm{d^{n-1}y}}{\mathrm{dx^{n-1}}} + \cdots + \mathrm{a_1(x)}\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{a_0(x)y = f(x)}$$

- Methods for Solving Higher-Order LDEs with Non-Constant Coefficients

❖ **Power Series Method**

- This method involves expressing the solution $y(x)$ as a power series:
  $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$, where x_0 is the point around which the series is expanded.
- Substitute the series into the differential equation and equate coefficients of like powers of $(x - x_0)$ to find a_n.

This method is particularly effective when the coefficients $a_n(x)$ are polynomials.

Frobenius Method

- Used when the solution has a singularity at $x = x_0$ (regular singular points).
- Assumes a solution of the form: $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{n+r}$, where r is determined from the indicial equation.

Reduction of Order

- Used to find a second solution when one solution $y_1(x)$ is already known.
- Assumes the second solution has the form: $y_1(x) = v(x)y_1(x)$, where $v(x)$ is a function to be determined.

Special Functions

For certain types of equations, solutions are expressed in terms of special functions like:

- **Legendre polynomials** for Legendre's equation.
- **Bessel functions** for Bessel's equation.
- **Hermite polynomials** for Hermite's equation.

Green's Function (Non-Homogeneous Case)

If the homogeneous solutions are known, Green's function can be used to construct the solution to the non-homogeneous equation.

Applications:

Linear differential equations model various real-world phenomena, such as electrical circuits, mechanical vibrations, heat conduction, population dynamics, and fluid flow, among others. They are foundational in both pure and applied mathematics due to their simplicity and the variety of methods available to solve them.

1. Engineering Applications

a. Electrical Circuits

- Used in analysing **RLC circuits** (Resistor, Inductor, Capacitor circuits).
- The behaviour of current and voltage in circuits is modelled by linear differential equations.

- Example: $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = V(t)$, where L is inductance, R is resistance, C is capacitance, i is current, and V(t) is the voltage source.

b. Mechanical Vibrations

- Used in studying systems with springs and dampers.
- Governs the motion of masses attached to springs: $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$, where m is mass, c is damping coefficient, k is the spring constant, F(t) is an external force.

2. Physics Applications

a. Motion of Projectiles

- Linear differential equations describe the motion of projectiles under gravity and other forces such as air resistance.

b. Quantum Mechanics

- The **Schrödinger equation**, which is fundamental in quantum mechanics, is a second-order differential equation: $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$, where ψ is the wave function, $\hbar$ is Planck's constant, and $\hat{H}$ is the Hamiltonian operator.

3. Economics

a. Growth Models

- Linear differential equations model population and economic growth (e.g., Malthusian growth model): $\frac{dP}{dt} = kP$, where P is the population or capital, and k is the growth rate.

b. Dynamic Systems

- Models for pricing, interest rates, and investment decisions often involve linear systems of equations.

4. Biology and Medicine

a. Population Dynamics

- The logistic growth model: $\frac{dN}{dt} = rN(1 - \frac{N}{K})$, where N is the population size, r is the growth rate, and K is the carrying capacity.

b. Spread of Diseases

- Epidemiological models, exemplified by the SIR (Susceptible-Infectious-Recovered) model, are defined by the following equations:
  $\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI - \gamma I, \frac{dR}{dt} = \gamma I$, where S, I, R represent different populations, and β, γ are constants.

5. Environmental Science

a. Heat Transfer

- The **heat conduction equation** (Fourier's law): $\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$, where T is temperature and k is thermal conductivity.

b. Fluid Dynamics

- Navier-Stokes equations in fluid dynamics involve differential equations to describe the motion of fluid substances.

6. Astronomy and Astrophysics

a. Orbital Mechanics

- Linear differential equations model the motion of planets and satellites under gravitational forces.

b. Stellar Evolution

- Used in equations governing the structure and evolution of stars.

7. Chemistry

a. Reaction Kinetics

- The rates of chemical reactions are modelled using linear or nonlinear differential equations.
- Example: $\frac{d[A]}{dt} = -k[A]$, where [A] is the concentration of a reactant and k is the reaction rate constant.

b. Diffusion Processes

- Fick's laws of diffusion describe how substances move through space.

8. Control Systems

- Linear differential equations form the basis of control theory used in automation and robotics.
- Example: Modelling feedback systems in cruise control or temperature regulation.

9. Signal Processing

- Used to design and analyse filters (e.g., low-pass and high-pass filters).
- Explains how input and output signals relate to one another.

10. Social Sciences

a. Modelling Social Behaviour

- Opinion dynamics, traffic flow, and crime modelling often use systems of linear differential equations.

b. Demography

- Models' population changes with respect to birth and death rates.

Conclusion:

Linear differential equations form a significant area of study in mathematics due to their wide applicability and well-developed solution techniques. Understanding the principles of linearity allows for the analysis of complex systems across different scientific fields.

Linear differential equations not only serve as a theoretical cornerstone in mathematics but also play a pivotal role in addressing real-world challenges. Their systematic study equips scientists, engineers, and mathematicians with tools to understand and solve complex dynamic systems effectively. Mastery of LDEs opens the door to deeper insights into both natural and man-made phenomena.

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